SYNOPSIS

Given a subset \( K \) of \( \mathbb{R}^n \) and a mapping \( F \) from \( \mathbb{R}^n \) into itself, the variational inequality problem is to find a vector \( \bar{x} \in K \) such that

\[
\langle F(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \text{for all} \quad x \in K.
\] (VI)

In the important case where the set \( K \) is a closed convex cone, the variational inequality problem is equivalent to the complementarity problem: Find a vector \( x \in \mathbb{R}^n \) so that

\[
x \in K, \quad F(x) \in K^*, \quad \langle F(x), x \rangle = 0.
\] (CP)

The theory as well as the applications of both the variational inequality and the complementarity problems has been well documented in the literature. Various extensions of these two problems have recently been introduced and studied by many authors.

The present research was partly motivated by numerous recent works in which the variational inequality and the complementarity problems involving multifunctions arising from diverse applications, are discussed. We concern ourselves to a class of problems where the multivaluedness enters in the form of the subdifferential of a certain sublinear functional. It is observed that this type of problems arises in the study of a class of quasidifferentiable mathematical programming problems in which the objective function contains a differentiable function and a support function.
The whole work is divided into five chapters.

In Chapter 0, some notations used in the text are explained and geometric preliminaries which are relevant to our work are provided.

Chapter I deals with the Complementarity Problems for Multifunctions in the following form: Given a continuous map \( F : \mathbb{R}^n \to \mathbb{R}^n \) and a lower semicontinuous positively homogeneous convex function \( h : \mathbb{R}^n \to \mathbb{R} \), find \( x \in \mathbb{R}^n \), \( x^* \in \partial h(x) \) (the subdifferential of \( h \) at \( x \)) such that

\[
\begin{align*}
x &\geq 0, \\
F(x) + x^* &\geq 0, \\
\langle F(x) + x^*, x \rangle & = 0.
\end{align*}
\]  

\( (I^*) \)

It includes two sections. Section 1 deals with the case when \( F \) is a linear map of the form \( F(x) = Mx + r \) with \( M \in \mathbb{R}^{n \times n} \) and \( r \in \mathbb{R}^n \). Existence results under generalized sets of conditions involving \( M \) are established in Subsection 1.4 and some of these results are illustrated through simple examples in Subsection 1.6. In Subsection 1.5, an application to quasidifferentiable quadratic programming is presented. Section 2 is devoted to the case when \( F \) is nonlinear. We obtain existence results under certain monotonicity and coercivity condition on the map \( F \), and apply these results to a quasidifferentiable convex programming problem. In Subsection 2.5 of this section, we also give results on bounds for solutions of monotone complementarity problems.
In Chapter II, we discuss a class of complementarity problems for multifunctions over cone domains. It is described as follows: Let $K$ be a closed, convex cone in $\mathbb{R}^n$, and let $K^*$ be the polar of $K$. Given a continuous map $F : K \rightarrow \mathbb{R}^n$ and a lower semicontinuous positively homogeneous convex function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ find $x \in \mathbb{R}^n$ and $x^* \in \partial h(x)$ such that

$$x \in K, \quad F(x) + x^* \in K^*,$$

$$\langle F(x) + x^*, x \rangle = 0. \quad (II^{**})$$

This chapter too is divided into two sections. Section 3 deals with the case when $F$ is linear and Section 4 deals with the case when $F$ is nonlinear. The results of this chapter extend to cone domains most of the existence results for $(I^{**})$ given in Chapter I.

Chapter III is devoted to the study of an interesting generalization of (VI) to multifunctions. This problem denoted as the Variational-like Inequality Problem, may be stated as follows: Let $K$ and $C$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. Given two maps $M : K \times C \rightarrow \mathbb{R}^n$ and $\gamma : K \times K \rightarrow \mathbb{R}^n$, and a point-to-set map $V : K \rightarrow C$, find $\bar{x} \in K$, $\bar{x}^* \in V(\bar{x})$ such that

$$\langle M(\bar{x}, \bar{x}^*), \gamma(x, \bar{x}) \rangle \geq 0 \quad \text{for all } x \in K. \quad (VLI)$$

When $K$ is a closed, convex cone, we have the following complementarity problem related to (VLI). Find $x \in \mathbb{R}^n$, $x^* \in V(x)$ such that

$$(ix)$$
We establish some existence theorems for the above two problems. Two classes of new mappings, \( \gamma \)-convex and \( \gamma \)-monotone, are introduced and existence results under these mappings are given. Finally, application of these existence results to two specific problems: (a) a nonlinear programming problem, and (b) a saddle point problem, both associated with a function \( L(x,y) \) of two vector arguments, defined over a product set \( X \times D \), \( X \subseteq \mathbb{R}^n \), \( D \subseteq \mathbb{R}^p \), is shown.

The last chapter, Chapter IV, aims at developing existence results for a more general class of quasidifferentiable programming problems in which the objective function contains a differentiable function and a lower semicontinuous sublinear function. The proofs of these results do not use the theory of the complementarity problem. On the other hand, a new existence result for the complementarity problem (I) is obtained as a byproduct.

\[ x \in K, \quad M(x,x^*) \in K^x, \]
\[ \langle M(x,x^*), x \rangle = 0. \]
0.1. Notations and Definitions

(1) \( \mathbb{R}^n \) is the \( n \)-dimensional real vector space, \( \mathbb{R}^{mxn} \) is the space of \( mxn \) real matrices,
\( \mathbb{R}_+^n = \{ x \in \mathbb{R}^n : x_i \geq 0 \ , \ i = 1,2,\ldots,n \} \) is the non-negative orthant of \( \mathbb{R}^n \),
\( \mathbb{R} \) is the set of real numbers,
\( \mathbb{R}_+ \) is the set of nonnegative numbers in \( \mathbb{R} \).

(2) For any matrix \( A \in \mathbb{R}^{mxn} \),
\( A^T \) denotes the transpose of \( A \),
\( A_i \) or \( (A)_i \) denotes the \( i \)-th row of \( A \).

(3) For any \( x = (x_i) \) , \( y = (y_i) \) in \( \mathbb{R}^n \),
\( \langle x,y \rangle = y^T x \) denotes the usual inner product of \( x \) and \( y \);
\( \| x \|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \) is the \( l_p \) norm of \( x \),
\( \| x \| \) is any norm of \( x \).

(4) For nonempty sets \( S, K \) in \( \mathbb{R}^n \),
\( \text{int } S \) denotes the interior of \( S \),
\( S + K = \{ s + k : s \in S \ , \ k \in K \} \),
\( S \times K = \{ (s,k) : s \in S \ , \ k \in K \} \).
(5) $S^*$ denotes the dual (also polar) cone of a nonempty set $S \subseteq \mathbb{R}^n$ defined by
$$S^* = \{ y \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } x \in S \}.$$

(6) For any $x, y \in \mathbb{R}^n$,
$$x \geq y \text{ means } x_i \text{ is either greater than or equal to } y_i \text{ for all } i, \ i = 1, 2, \ldots, n; \ x > y \text{ means } x_i \text{ is greater than } y_i \text{ for all } i, \ i = 1, 2, \ldots, n.$$

(7) For a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$,
$$\nabla f(x)$$
denotes the gradient of $f$ with respect to $x$.
For a continuously differentiable mapping $g : \mathbb{R}^n \to \mathbb{R}^m$,
$$J_g(x)$$
denotes the Jacobian matrix of $g$ evaluated at $x$ whose $i, j$-th component is $\frac{\partial g_j(x)}{\partial x_i}$.

(8) Let $H : \mathbb{R}^n \to \mathbb{R}$. The directional derivative of $H$ at $x$ in the direction of $z$, denoted by $H'(x; z)$, is given by
$$H'(x; z) = \lim_{\lambda \to 0^+} \frac{(H(x + \lambda z) - H(x))}{\lambda}.$$
where $\lambda \to 0^+$ means $\lambda$ approaches zero through positive values. For $H : \mathbb{R}^n \to \mathbb{R}$, differentiable at $x$,
$$H'(x; z) = \langle \nabla H(x), z \rangle.$$

(9) Let $H : \mathbb{R}^n \to \mathbb{R}$ be a convex function. A vector $x^* \in \mathbb{R}^n$ is said to be subgradient of $H$ at $x \in \mathbb{R}^n$ if
$$H(y) - H(x) \geq \langle x^*, y - x \rangle \text{ for all } y \in \mathbb{R}^n.$$
The set of all subgradients of $H$ at $x$, denoted by $\partial H(x)$, is called the sub-differential of $H$ at $x$.

0.2 Convex Cones and Some of Its Properties

(1) A nonempty subset $S$ in $\mathbb{R}^n$ is a closed, convex cone if, and only if,
   (a) $S$ is closed, and
   (b) $\alpha x + \beta y \in S$ for $x, y \in S$ and $\alpha, \beta \geq 0$.

(2) Let $S$ and $K$ be nonempty sets in $\mathbb{R}^n$. Then
   (a) $S^*$, the dual cone of $S$, is a closed, convex cone,
   (b) $S$ is a closed, convex cone if, and only if, $S = S^{**}$,
   (c) For a closed, convex cones $S$ and $K$, $S \times K$ is a closed, convex cone,
      $(S \times K)^* = S^* \times K^*$,
      $\text{int}(S \times K)^* = \text{int} S^* \times \text{int} K^*$.

(3) A nonempty set $S$ in $\mathbb{R}^n$ is a pointed cone if it is a cone, and $S \cap (-S) = \{0\}$.

(4) Let $S$ be a closed, convex cone in $\mathbb{R}^n$. Then $\text{int} S^*$ is nonempty if $S$ is pointed, in which case,
    $\text{int} S^* = \{x \in S^*: 0 \nmid y \in S \Rightarrow \langle y, x \rangle > 0\}$. 