Duality and Existence Theory
for Nondifferentiable Programming

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Abstract. Necessary and sufficient conditions of optimality are given for a nonlinear nondifferentiable program, where the constraints are defined via closed convex cones and their polars. These results are then used to obtain an existence theorem for the corresponding stationary point problem, under some convexity and regularity conditions on the functions involved, which also guarantees an optimal solution to the programming problem. Furthermore, a dual problem is defined, and a strong duality theorem is obtained under the assumption that the constraint sets of the primal and dual problems are nonempty.

Key Words. Optimality conditions, directional derivatives, subdifferentials, constraint qualifications, stationary point problems, duality theorems.

1. Introduction

Let $P$ and $Q$ be closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The polar of $Q$, denoted by $Q^*$, is the set of all $u$ in $\mathbb{R}^m$ such that $(u, x) \geq 0$ for all $x \in Q$, where $(u, x)$ denotes the inner product of $u$ and $x$. We consider the problem

\[ (P): \begin{array}{ll}
\text{minimize} & F(x) = f(x) + h(x), \\
\text{subject to} & x \in X, \\
& X = \{ x : x \in P, g(x) \in Q^* \}, \
& h(x) = \max_{e \in \mathcal{N}} x^T e,
\end{array} \]

where $\mathcal{N}$ is the set of all normal vectors to $Q^*$.

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\[ S \text{ is a compact convex subset of } \mathbb{R}^n, \text{ and } f: \mathbb{R}^n \rightarrow \mathbb{R} \text{ and } g: \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ are differentiable maps. It follows from the definition and known facts (see Ref. 1) that } h(x) \text{ is a lower semicontinuous convex function whose subdifferential } \partial h(x) \text{ is given by} \]
\[ \partial h(x) = \{ y \in S : x^Ty = h(x) \}. \]

Its directional derivative at \( x \) in the direction of \( d \), which we denote by \( h'(x; d) \), is given by
\[ h'(x; d) = \max_{y \in \partial h(x)} d^Ty. \]

Note that, since \( \partial h(x) \) is compact, the maximum in (4) is attained.

We obtain necessary Kuhn-Tucker conditions for a minimum of (P) under the Slater constraint qualification. These conditions are shown to be also sufficient if \( f \) is convex. It is noted that our results are intimately related to those of Mond (Ref. 2), Mond and Schechter (Ref. 3), and Schechter (Ref. 4), and they can be taken as a generalization of their results to cone domains.

While applying a computational method such as the proximal point algorithm (Ref. 5) to solve a convex program, one feels that it is useful if the existence of an optimal solution is ensured before the actual computation is taken up. With this point in view, the question of the existence of an optimal solution to (P) is studied by means of the stationary point problem

\[(\text{STP}): \text{ find } x \in \mathbb{R}^n, u \in \mathbb{R}^m \text{ such that} \]
\[ (F - u^Tg)'(x, d) \geq 0, \forall d \in P - x, \quad (5a) \]
\[ g(x) \in Q^*, x \in P, u \in Q, u^Tg(x) = 0. \quad (5b) \]

Assuming a constraint qualification for (P) and the convexity of \( f \), we prove that (STP) has a solution, and hence that (P) attains an optimal solution.

Further, based upon our optimality conditions, the following problem is shown to be a dual problem of (P).

\[(D): \text{ maximize } G(x, u, y) = f(x) - u^Tg(x) - x^T(\nabla f(x) - A(x)u), \]
subject to \( \nabla f(x) - A(x)u + y \in P^* \), \[ u \in Q, y \in S. \]

where \( A(x) \) is the Jacobian matrix of \( g \), given by \( A(x) = \partial g/\partial x \). A strong duality theorem is proved which asserts that, if (P) and (D) have feasible solutions, they have optimal solutions.
2. Optimality Conditions

The following theorem will be the main tool for deriving necessary conditions of optimality. It can be obtained from Theorem 7.1.2 of Ref. 6 by taking $K$ as the positive cone in place of $-K$.

**Theorem 2.1.** Let $X^0$ be a nonempty convex set in $\mathbb{R}^n$. Let $\alpha: X^0 \to \mathbb{R}$ be convex, and let $\beta: X^0 \to \mathbb{R}^m$ be $K$-concave on $X^0$, where $K$ is a closed convex cone. If the system

$$\alpha(x) < 0, \quad \beta(x) \in K^*,$$

has no solution in $X^0$, then there exist $p > 0$, $q \in K^*$, $(p, q) \neq 0$, such that

$$p\alpha(x) - q^T\beta(x) \geq 0, \quad \text{for all } x \in X^0.$$

Consider the following condition involving $g$ and $Q$.

**Condition 2.1. Slater Condition for Cone Domain.** The function $g$ is $Q^*$-concave on $P$. $Q^*$ has nonempty interior and there exists an $x \in P$ such that $g(x) \in \text{int } Q^*$.

**Theorem 2.2.** Suppose that Condition 2.1 is satisfied. If $x^0$ is optimal for (P), then there exists $u^0 \in \mathbb{R}^m$ such that (5a) and (5b) hold.

**Proof.** Since $x^0$ is optimal for (P), the system

$$F'(x^0, x - x^0) < 0, \quad x \in X,$$

has no solution. To show this, let $x$ be a solution to the system. It then follows from the definition of the directional derivative that

$$F(x^0 + \lambda(x - x^0)) < F(x^0),$$

for $\lambda$ positive and sufficiently small. But by the $Q^*$-concavity (for the definition, see Ref. 6) of $g$, the feasible set $X$ is convex; therefore, the point $\lambda x + (1 - \lambda)x^0$ is in $X$ for each $\lambda \in [0, 1]$. Thus, we get a contradiction to the fact that $x^0$ is an optimal point. Further, notice that

$$F'(x^0, x - x^0) = (\nabla f(x^0) + y^0, x - x^0),$$

(6)

for some $y^0 \in \partial h(x^0)$. Obviously, it is a linear function, and hence a convex function of $x$. Now, applying Theorem 2.1, we have $p \geq 0$, $q \in Q$, $p$ and $q$ not both zero, such that

$$pF'(x^0, x - x^0) - q^Tg(x) \geq 0,$$

(7)

for all $x \in P$. If $p = 0$, then

$$q^Tg(x) \leq 0, \quad \text{for all } x \in P,$$
and, in particular, for \( x = \hat{x} \). So we have

\[
q^T g(\hat{x}) = 0, \quad \text{for } 0 \neq q \in Q \quad \text{and} \quad g(\hat{x}) \in \text{int} \ Q^*,
\]

which is impossible. Thus, we can conclude that \( p > 0 \). Dividing (7) by \( p \), we obtain

\[
F'(x^0, x - x^0) - (u^0, g(x)) > 0,
\]

with \( u^0 = (q/p) \in Q \). Letting \( x = x^0 \) in (8), we get

\[
(u^0, g(x^0)) = 0.
\]

But \( g(x^0) \in Q^* \), since \( x^0 \) is feasible for (P), and \( u^0 \in Q \), and hence

\[
(u^0, g(x^0)) = 0.
\]

Therefore,

\[
(u^0, g(x)) = 0.
\]

Now, let \( x \) be any arbitrary but fixed point in \( P \). Since \( P \) is convex and \( x^0 \in P \),

\[
x^0 + \lambda (x - x^0) \in P, \quad \text{for each } \lambda \in [0, 1].
\]

From (8), we then have

\[
\lambda F'(x^0, x - x^0) - (u^0, g(x^0 + \lambda (x - x^0)) - g(x^0)) = 0, \quad \text{for every } \lambda \in [0, 1].
\]

Dividing by \( \lambda \) and letting \( \lambda \to 0 \), we obtain

\[
(F - u^0) g'(x^0, x - x^0) \geq 0.
\]

Since \( x \) is arbitrary, (9) holds for all \( x \in P \). This completes the proof. □

The next theorem gives sufficient condition for \( x^0 \) to be an optimal solution of (P).

**Theorem 2.3.** Let \( f \) be convex, and let \( g \) be \( Q^* \)-concave over \( P \). If there exist \( (x^0, u^0) \) satisfying (5a) and (5b), then \( x^0 \) is optimal for (P).

**Proof.** Let \( x \) be feasible for (P). We will show that \( F(x) \geq F(x^0) \). By (4), there exists \( y^0 \in \partial h(x^0) \) such that

\[
h'(x^0, x - x^0) = (y^0, x - x^0).
\]

Then, from (5a), we have

\[
\langle \nabla f(x^0) - A(x^0) u^0 + y^0, x - x^0 \rangle \geq 0.
\]
By (5b) and the concavity of \( g \), we have
\[
\langle A(x^0)u^0, x - x^0 \rangle \geq \langle u^0, g(x) - g(x^0) \rangle \geq 0.
\]
Since \( y^0 \in \partial h(x^0) \) and \( f \) is convex, it follows that
\[
F(x) - F(x^0) \geq f(x) - f(x^0) + \langle y^0, x - x^0 \rangle
\]
\[
\geq \langle \nabla f(x^0) + y^0, x - x^0 \rangle.
\]
Now, combining the above inequalities, we get
\[
F(x) - F(x^0) \geq 0.
\]

3. Existence Results

**Theorem 3.1.** Let \( P \) be pointed, and \( p \in \text{int} \; P^* \). Suppose that Condition 2.1 holds. If there is an \( x \in X \) and a constant \( r \geq 0 \) such that
\[
\min\{(x - \tilde{x})^T \eta : \eta \in \nabla f(x) + \partial h(x)\} \leq 0,
\]
for all \( x \in X \) with \( p^T (x - \tilde{x}) \geq r \), then there exists a solution to (STP), and

hence an optimal solution to (P), if \( f \) is convex.

**Proof.** For a real \( \alpha > 0 \), consider the set
\[
X(\alpha) = \{x : x \in X, p^T (x - \tilde{x}) \leq \alpha \}.
\]
It is clear that \( X(\alpha) \) is nonempty, compact, and convex. The lower semicontinuous function \( F(x) \) does attain its minimum on the compact set \( X(\alpha) \).

Let \( x^* \) be a minimal point. By Theorem 2.2, there exist \( u'' \in \mathbb{R}^m \) and \( \xi'' \in \mathbb{R} \) such that
\[
(F - u''g)'(x^*, d) + \xi'' p^T d \geq 0, \quad \forall d \in P - x^*,
\]
\[
g(x^*) \in Q^*; x^* \in P; u'' \in Q; u''g(x^*) = 0.
\]
\[
\xi'' \geq 0, \quad \{\alpha - p^T (x^* - \tilde{x})\} \xi'' = 0.
\]

If \( \xi'' = 0 \), then \( (x^*, u'') \) constitutes a solution of (STP). We assume, therefore, that \( \xi'' > 0 \) for every \( \alpha > 0 \). Now, choose an \( \alpha \) (\( \alpha' \), say), such that \( \alpha' > r \), and then denote \( (x^*, u^*) \) for \( \alpha = \alpha' \) by \( (x', u') \). From (11c), we get
\[
p^T (x' - \tilde{x}) = \alpha' - r, \quad \text{with} \; x' \in X(\alpha') \subset X.
\]

Hence, it follows from (10) that
\[
0 \leq \min\{(x' - \tilde{x})^T \eta : \eta \in \nabla f(x') + \partial h(x')\}
\]
\[
= -\max\{(\tilde{x} - x')^T \eta : \eta \in \nabla f(x') + \partial h(x')\}
\]
\[
= F^*(x', \tilde{x} - x').
\]
Since \( \bar{x} \) is feasible for (P), proceeding as in the proof of Theorem 2.3, we get from (11a), (11b), and the concavity of \( g \) that

\[
F'(x', \bar{x} - x') + \xi' p^T (\bar{x} - x') \geq 0,
\]

which, by (12), implies that

\[
0 \leq \xi' p^T (\bar{x} - x') = -\xi' \alpha < 0,
\]

a contradiction. This contradiction establishes the fact that (STP) has a solution. Hence, it follows from Theorem 2.3 that (P) has an optimal solution if \( f \) is convex. \( \square \)

**Remark 3.1.** It can be easily checked that, under the convexity assumption of \( f \) and \( g \), the condition (10) is not only sufficient for existence but also necessary.

**Theorem 3.2.** Let \( P \) be pointed, and let Condition 2.1 hold. If \( f \) is convex and there is an \( x \in X \) such that \( \partial F(\bar{x}) \cap \text{int} \ P^* \) is nonempty, then (STP) has a solution, and hence (P) has an optimal solution.

**Proof.** Let \( x \in X \). Since \( F \) is quasidifferentiable, it follows that

\[
\xi F(x) - F(x) - \langle \xi, x - \bar{x} \rangle \geq 0, \quad \forall \xi \in \partial F(\bar{x}),
\]

\[
F(\bar{x}) - F(x) - \langle \eta, \bar{x} - x \rangle \geq 0, \quad \forall \eta \in \partial F(x).
\]

Adding these two inequalities, we have

\[
(\eta, x - \bar{x}) \geq (\xi, x - \bar{x}), \tag{13}
\]

for all \( \eta \in \partial F(x) \) and \( \xi \in \partial F(\bar{x}) \). Note that

\[
\partial F(x) = \nabla f(x) + \partial h(x);
\]

see Ref. 7, Theorem 23.8. Now, choose \( \xi \in \partial F(\hat{x}) \), such that \( \hat{x} \in \text{int} \ P^* \), and write \( p = \hat{x} \). Then, from (13), we get

\[
(x - \hat{x})^T \eta = 0, \quad \forall \eta \in \nabla f(x) + \partial h(x),
\]

if \( p^T (x - \hat{x}) > 0 \). This shows that the condition (10) of Theorem 3.1 can be satisfied. Hence, the proof is complete. \( \square \)

**4. Duality Theory**

It will be assumed henceforth that \( f \) is convex and \( g \) is \( Q \)-concave over \( P \).

An argument similar to that used in the proof of Theorem 2.3 gives the following theorem.
Theorem 4.1. If \( x \) is feasible for (P) and \( (x^*, u, y) \) is feasible for (D), then
\[
F(x) \geq G(x^*, u, y).
\]
As an immediate consequence of this and Theorem 2.2, we have the following theorem.

Theorem 4.2. If \( x^0 \) is optimal for (P) and Condition 2.1 holds, then there exist \( u^0 \in Q, y^0 \in S \) such that \( (x^0, u^0, y^0) \) yields an optimal solution of (D). Moreover, the extremal values for (P) and (D) are equal.

In addition, we can have a strong duality theorem in the following form.

Theorem 4.3. Let \( P \) and \( Q \) be pointed. If there exists \( x \in P \) such that \( \partial F(x) \cap \text{int } P^* \) is nonempty and \( g(x) \in \text{int } Q^* \), then each of the problems (P) and (D) has an optimal solution with equal extremal values.

Proof. Applying Theorem 3.2, we get \( x^0 \in R^n, u^0 \in R^m \) such that \( (x^0, u^0) \) satisfies (5a) and (5b). Then, from (5b), it follows that \( x^0 \) is feasible for (P). From (5a) and (4), we have
\[
\langle \nabla f(x^0) - A(x^0)u^0, x - x^0 \rangle \leq 0, \quad \forall x \in P, \tag{14}
\]
where \( y^0 \in \partial h(x^0) \subseteq S \). But it is easy to show that (14) holds if and only if \( (x^0, u^0, y^0) \) satisfies
\[
\nabla f(x^0) - A(x^0)u^0 + y^0 \in P^*, \tag{15A}
\]
\[
(x^0, \nabla f(x^0) - A(x^0)u^0 + y^0) = 0. \tag{15b}
\]
Now, it is clear from (5a) that \( (x^0, u^0, y^0) \) is feasible for (D). Moreover, by (5b) and (15b), we have
\[
G(x^0, u^0, y^0) = f(x^0) + (x^0, y^0) - (u^0, g(x^0)) - (x^0, \nabla f(x^0) - A(x^0)u^0) - f(x^0) + (x^0, y^0) + h(x^0) = F(x^0).
\]
Then, the desired result follows from Theorem 4.1.

Remark 4.1. We observe that, in the statement of Theorem 4.3, \( x \) is feasible for (P), and \( (x, 0, y) \) is feasible for (D), for some \( y \in \partial h(x) \subseteq S \). Thus, a strong duality theorem between (P) and (D) is proved under only the feasibility assumption.

References

ON THE MINIMUM OF A CLASS OF NONDIFFERENTIABLE FUNCTIONS

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A class of nondifferentiable functions that is relevant to nondifferentiable mathematical programming is introduced. A sufficient condition is given under which each function in the class attains its infimum on a polyhedral convex set if it is bounded below there. Quadratic functions, as a special case, are shown to satisfy this sufficient condition.

1. Introduction

It is well known\(^1\) that a quadratic (not necessarily convex) function \(Q\) bounded below on a polyhedral convex set \(X \subset \mathbb{R}^n\) attains its infimum there. Perold\(^5\) defines a class of continuous functions \(G\) which, in particular, contains all quadratic functions\(^3\) and shows that each function in \(G\) attains its infimum on \(X\) if and only if it is bounded below on \(X\).

In this paper we consider the class of functions \(F: \mathbb{R}^n \to \mathbb{R}\) given by

\[
F(x) = \frac{1}{2} x^T Dx + h(x)
\]

where \(D\) is an \(n \times n\) symmetric matrix and \(h\) is lower semicontinuous sublinear function. Note that \(h: X \to \mathbb{R}\) is sublinear if and only if it satisfies

**Property I:** \(h(x + y) \leq h(x) + h(y)\) for all \(x, y \in X\)

**Property II:** \(h(\lambda x) = \lambda h(x)\) for all \(x \in X\) and all \(\lambda \geq 0\).

Clearly, quadratic functions are contained in the above class. We give sufficient conditions under which the function \(F\) attains its infimum on \(X\). An existence theorem
to a generalized complementarity problem is obtained by using this result. The theorems given here are particularly relevant to the nondifferentiable optimization problems, such as those in Mond. Mond and Schechter and Schechter, where the objective function is the sum of a differentiable function and a lower semicontinuous positively homogeneous convex function. The latter function generally appears in any of the following forms:

(a) \((x^T B x)^{1/2}\) with \(B\) as an \(n \times n\) symmetric positive semi-definite matrix.
(b) \(\|Sx\|_p\) (\(p > 1\)) \(S\) is a \(k \times n\) matrix and the \(p\) norm is given by

\[
\|y\|_p = \left( \sum_{i=1}^{k} |y_i|^p \right)^{1/p}.
\]
(c) \(\max (x^T v | v \in C)\), where \(C\) is any compact convex subset of \(\mathbb{R}^n\).

It can be easily checked that the functions (a), (b) and (c) are sublinear.

2. Main Results

Let \(X\) be a nonempty polyhedral convex set of the form

\[ X = \{ x \in \mathbb{R}^n : Ax \leq b \} \]

where \(A \in \mathbb{R}^{m \times n}\). For real \(a > 0\), consider the set

\[ X(a) = \{ x : x \in X, \|x\| \leq a \} \]

where \(\|x\|\) denotes Euclidean norm of \(x\). Since \(X\) is nonempty, there exists \(a_0\) such that the sets \(X(a)\), for \(a_0 \leq a < \infty\), are nonempty and compact. The lower semicontinous function \(F\) attains its infimum on each such compact set. Let \(y\) be a minimal point of \(F\) on \(X(a)\), and let

\[ f(a) = \inf_{x \in X(a)} F(x). \]

Then

\[ F(y) = f(a). \]

Because \(a < a'\) implies \(X(a) \subseteq X(a')\), the function \(f(a)\) is monotone nonincreasing. Thus \(f(a)\) is bounded above for \(a_0 \leq a < \infty\).

Lemma 1—Let \(F\) be bounded below on \(X\). Let \(\{a_k\}\) be a sequence of positive reals with \(a_k \to \infty\). If, for every \(a \in \{a_k\}\), there is a minimal point \(x_a\) of \(F\) on \(X(a)\) such that \(\|x_a\| = a\), then there exists a vector \(t\) \(t \neq 0\) satisfying
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\[ At \leq 0, \quad t^T \, Dt = 0, \quad h(t) \leq 0. \tag{3} \]

Moreover, there is a fixed \( k \) and a scalar \( \delta > 0 \) such that

\[ x^* - \lambda t \in X, \quad \|x^* - \lambda t\| < \|x^*\| \text{ for all } 0 < \lambda < \delta. \tag{4} \]

**Proof:** Let \( x^k \) be a minimal point of \( F \) over \( X'(x_k) \) with \( \|x^k\| = a_k \), and let \( t_k = x^k/a_k \). Then \( \|t_k\| = 1 \) for all \( k \). Since \( (t^k) \) lies in the unit sphere which is compact, there is a subsequence that converges to some vector \( t \) in the unit sphere.

Let this subsequence be one with \( k \in \{a_k\}_k \in \mathbb{R} \), where \( \Gamma \) is an index set. Thus we have

\[ t = \lim_{k \in \Gamma} t_k = \lim_{k \in \Gamma} (x^k/a_k), \quad \|t\| = 1. \]

Since \( x^k \in X, At^k = (1/a_k) \, Ax^k \leq (b/a_k) \) for all \( k \in \Gamma \), which implies

\[ At \leq 0. \]

Further, we have

\[ f(a_k) = a_k^2 (t^k^T \, Dt^k) + a_k \, h(t^k) \text{ for all } k \in \Gamma. \tag{5} \]

Since \( F \) is bounded below on \( X \), it follows from (2) that \( f(a_k) \) is bounded below for all \( k \in \Gamma \). It was seen that \( f(a_k) \) is also bounded above. Hence, it follows from (5) and the lower semicontinuity of \( h \) that

\[ t^k^T \, Dt = 0, \quad h(t^k) \leq \lim_{k \in \Gamma} h(t^k) = 0. \]

Thus, (3) is established.

Let \( I = \{1, 2, \ldots, m\} \), and let \( J = \{j : (At)_j < 0\} \).

Then we have

\[ (At)_j < 0 \text{ for all } j \in J. \]

\[ (At)_j = 0 \text{ for all } j \in I \backslash J. \tag{6} \]

Choose an index \( \eta \in \Gamma \) large enough such that

\[ \begin{cases} 
(At^\eta)_j < \frac{1}{2} \cdot (At)_j < 0 & \text{for all } j \in J, \\
\frac{a_k}{2} \cdot (At)_j \leq b_j & \text{for all } j \in J, 
\end{cases} \]
Then, for all $k \in \Gamma$ with $a_k \geq a_\epsilon$, we have

$$\langle A t^k \rangle_j = \frac{a_k}{2} \langle A t \rangle_j \leq b_j \text{ for all } j \in J.$$  \hfill (7)

From (6) and (7) and the fact that $x^k \in X$, it follows that there is a $\delta_1 > 0$ such that

$$x^k - \lambda t \in X \text{ for all } 0 > \lambda > \delta_1 \text{ and all } k \in \Gamma.$$  \hfill (8)

Now, choose a fixed $k \in \Gamma$ with $a_k \geq a_\epsilon$ so large that $t^T t^k > 0$. Then, because of $t^T x^k > 0$ and $\|t\| = 1$, we can find a $\delta_2 > 0$ such that

$$\|x^k - \lambda t\| > \|x^k\| \text{ for all } 0 > \lambda > \delta_2.$$  \hfill (9)

Setting $\delta = \min(\delta_1, \delta_2)$, (4) is obtained from (8) and (9).

**Lemma 2**—Let $y$ be a minimal point of $F$ over $X(a)$, $a \geq a_\epsilon$. Then the set

$$C_a = \{y : y \in X(a), F(y) = f(a)\}$$

is compact.

**Proof:** Clearly, $C_a$ is a nonempty subset of the compact set $X(a)$. Since every closed subset of a compact set is compact, we have only to show that $C_a$ is closed. Let $y^0$ be a limit point of $C_a$, and let $\{y^i\}$ be a sequence in $C_a$ that converges to $y^0$. Now for each $y^i \in \{y^i\}$, $F(y^i) = f(a)$, and since $F$ is lower semicontinuous,

$$F(y^0) \leq \lim_{i \to \infty} F(y^i) \leq f(a).$$

But $\{y^i\} \subset X(a)$ which is closed; and therefore $y^0 \in X(a)$ and $F(y^0) \geq f(a)$. This shows that $y^0 \in C_a$, and hence $C_a$ is closed.

**Theorem 1**—Let $F$ be bounded below over $X$. If

$$A t \leq 0$$

$$t^T D t = 0$$

$$h(t) \leq 0$$

then $F$ attains its infimum on $X$.

**Proof:** By Lemma 2, the set $\{y : y \in X(a), F(y) = f(a)\}$ is nonempty and compact, for any $a \geq a_\epsilon$ and the continuous function $\|y\|$ assumes its minimum on the set. Therefore, we can have, for every $a$, at least one vector $x^a$ satisfying

$$x^a \in X(a), F(x^a) = f(a) \quad \text{...(11)}$$

$$x^a = \min \{\|y\| : y \in X(a), F(y) = f(a)\}.$$  \hfill (12)
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We now distinguish between two cases.

Case 1 — \( \|x^a\| < a \) for arbitrarily large values of \( a \). This means that there exists \( \bar{a} \) such that \( \|x^a\| < a \) for all \( a > \bar{a} \). Let \( \bar{f} = \inf \{ F(x) \mid x \in X \} \). Since \( F \) is bounded below on \( X \), \( \bar{f} \) is finite. From (1), we then have

\[
\bar{f} = \lim_{a \to \infty} f(a), \quad f(a) \geq \bar{f} \text{ for all } a \geq a_0.
\]

Assume that \( f(a) > \bar{f} \) for all \( a > \bar{a} \). Since \( f(a) \) is monotone nonincreasing and tends to \( \bar{f} \), we can find \( \bar{a} < a_1 < a_2 \) such that \( F(a_1) > f(a_2) \). For brevity, denote the vector \( x^a \) for \( a = a_k \) by \( x^k \). Because of \( \bar{a} < a_1 \), we have \( \|x^\| < a_2 \). Because of \( f(a_2) > f(a_3) \), we have \( a_2 < \|x^\| \). Now, choose \( a_0 = \|x^\| \), and then:

\[
\bar{a} < a_1 < a_2 < a_3.
\]

But \( \bar{a} < a_3 \) implies \( \|x^\| < a_3 = \|x^\| \), and \( a_2 < a_3 \) implies \( F(x^\) \leq F(x^) \). If we have \( F(x^k) = F(x^k) \), then \( \|x^\| < \|x^\| \) would contradict the definition of \( x^\) as given by (12), and if we have \( F(x^k) < F(x^k) \), \( \|x^\| = a_3 \) would contradict the definition of \( x^\) as given by (11). Hence, we can conclude that

\[
\bar{f} = f(a) = F(x^a) \text{ for some } a.
\]

Case 2 — \( \|x^a\| = a \) for arbitrarily large values of \( a \). This implies that the set of points \( x^a \), \( a \geq a_3 \), is unbounded. Then we can find a sequence \( \{a_k\} \) with \( a_k \to \infty \), \( \|x^\| = a_k \) for all \( k \), where \( x^k \) corresponds to \( a_k \). Thus we have a case to which Lemma 1 is applicable. Consequently, we get a vector \( t \), \( t \neq 0 \), satisfying

\[
At \leq 0, \quad t^TDt = 0, \quad h(t) \leq 0.
\]

Since \( x^k \in X \), it follows from (13) and the sublinearity of \( h \) that

\[
x^k + \lambda t \in X
\]

\[
F(x^k + \lambda t) \leq F(x^k) + \lambda (t^TDx^k + h(t))
\]

for all \( k \) and all \( \lambda \geq 0 \). This together with the boundeness of \( F \) below on \( X \) implies

\[
t^TDx^k + h(t) \geq 0 \text{ for all } k.
\]

From (10) and (13), it follows that

\[
h(t) + h(-t) = 0.
\]

Finally, from (13) — (15) and the sublinearity of \( h \), it follows that

\[
F(x^k - \lambda t) \leq F(x^k) - \lambda (t^TDx^k + h(t)) + \lambda (h(t) + h(-t))
\]

\[
\leq F(x^k)
\]
for all $k$ and all $\lambda \geq 0$. But by conclusion (4) of Lemma 1, we have also a scalar $\delta$ such that

$$x^* - \lambda t \in X, \|x^* - \lambda t\| < \|x^k\|$$

for some fixed $k$ and all $0 < \lambda < \delta$. Now, if we set $y = x^* - \lambda t$ for $0 < \lambda < \delta$, then we have from (16) that

$$y \in X(x_k), \|y\| < \|x^k\|$$

$$F(y) < F(x^k).$$

This contradicts the definition of $x^*$ as given by (12). Hence, we can conclude that $\|x^k\| = \alpha$ cannot hold for arbitrarily large values of $\alpha$. This completes the proof of the theorem.

The existence result for quadratic functions is recovered in the following corollary.

**Corollary 1**—Let $F(x) = \frac{1}{2} x^T D x + h(x)$, where $h(x)$ is any of the following functions: (i) $c^T x$, (ii) $(x^T B x)^{1/2}$ with $B$ positive semidefinite and (iii) $\|S x\|_p (p > 1)$ with $S \in \mathbb{R}^{k \times n}$. If $F$ is bounded below on $X$, then $F$ attains its infimum there.

**Proof:** Note that, for the functions (ii) and (iii), $h(-x) = h(x)$ and $h(x) \geq 0$. Since condition (10) of Theorem 1 is automatically satisfied when $h$ assumes any of the forms (i), (ii) and (iii), the conclusion of the corollary follows from Theorem 1.

Since $h$ is a lower semicontinuous convex function, its subdifferential $\partial h(x)$ is nonempty, compact and convex (Rockafellar). Consider the complementarity problem (CP): Find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \ z \geq 0$$

$$z \in D x + \partial h(x), \ x^T z = 0.$$ 

Generalized complementarity problem of this form were studied by Saigal and others; and may be of interest in non-differentiable programming.

We give the following existence theorem for (CP).

**Theorem 2**—Let $F$ be bounded below on $x \geq 0$. If

$$t \geq 0$$

$$t^T D t = 0$$

$$h(t) \leq 0$$

then $h(t) + h(-t) = 0$. 

\[ \]