A Fixed Point Theorem with Implicit Relation

in Fuzzy Metric Space

R. K. Saini, B. Singh and A. Jain

Department of Mathematics
D.A.V. College, Muzaffarnagar, India
rksaini03@yahoo.com

Abstract. The purpose of this paper is to prove a common fixed point theorem with implicit relations in fuzzy metric space by studying the relationship between the continuity and reciprocal continuity. Our result generalize and fuzzify the result in [21, 15, 29].

Mathematics Subject Classification: Primary 54H25, Secondary 47H10

Keywords: Fixed Point Theorem, Fuzzy Metric Space, Reciprocal Continuity, Implicit Relation

1. INTRODUCTION

In recent years several authors have generalized commuting condition of mapping introduced by Jungck, Sessa initiated the tradition of improving commutative condition in fixed point theorems by introducing the notion of weakly commuting, weak\(\ast\) commuting and reciprocal continuity of mappings. Pathak [23] defines weak\(\ast\) commuting and weak\(\ast\ast\) commuting mapping in metric space and prove some theorem. Popa [25] proved theorem for weakly compatible non-continuous mapping using implicit relations. It was extended by Imdad et.al. [14] using coincidence commuting property. Singh and Jain [32] also extend the relation of Popa [25] and [26] in fuzzy metric space. Very recently Jain et al. [15] prove a theorem for three mappings using implicit relation.

Zadeh [34] introduce the concept of fuzzy sets, which laid the foundation of fuzzy mathematics. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [19], and George and Veeramani [11] modified the notion of fuzzy metric spaces with the help of continuous t-norms.
Recently, many authors have proved fixed point theorems involving fuzzy sets \([1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 17, 18, 21, 22, 29]\). Vasuki [33] investigated some fixed point Pant [22] introduced the notion of reciprocal continuity of mappings in metric spaces. Balasubramaniam et al. [2] proved the problem on the existence of a contractive definition which generates a fixed point but does not force the mapping to be continuous at the fixed point.

2. PRELIMINERIES

For the terminologies and basic properties of fuzzy metric, we begin with some following definitions:

**Definition 2.1**[27]: A binary operation \(*: [0, 1] \times [0, 1] \rightarrow [0, 1]\) is a continuous \(t\)-norm if \(((0, 1), \ast)\) is an abelian topological monoid and \(*\) is satisfying the following conditions:

(a) \(a \ast 1 = a\) for all \(a \in [0, 1]\);
(b) \(a \ast b \leq c \ast d\) whenever \(a \leq c\) and \(b \leq d\), and \(a, b, c, d \in [0, 1]\).

**Definition 2.2**[11]: A 3-tuple \((X, M, \ast)\) is called a fuzzy metric space if \(X\) is an arbitrary set, \(*\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X \times (0, \infty)\) satisfying the following conditions:

for all \(x, y, z \in X\), and \(s, t > 0\),

(FM -1) \(M(x, y, t) > 0\);
(FM -2) \(M(x, y, t) = 1\) if and only if \(x = y\);
(FM -3) \(M(x, y, t) = M(y, x, t)\);
(FM -4) \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)\);
(FM -5) \(M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]\) is continuous.

Then \(M\) is called a fuzzy metric on \(X\). The function \(M(x, y, t)\) denote the degree of nearness between \(x\) and \(y\) with respect to \(t\).

**Example 2.1**: In a fuzzy metric space \((X, M, \ast)\), if \(a \ast a \geq a\) for all \(a \in [0, 1]\) then \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0, 1]\).

**Lemma 2.1**[12]: For \(x, y \in X\), \(M(x, y, \cdot)\) is non-decreasing on \((0, \infty)\).

**Example 2.2**[11]: Let \(X = \mathbb{N}\). Define \(a \ast b = \max\{0, a + b - 1\}\) for all \(a, b \in [0, 1]\) and let \(M\) be fuzzy set on \(X \times (0, \infty)\) as follows:

\[
M(x, y, t) = \begin{cases} x & \text{if } x \leq y, \\ y & \text{if } y \leq x, \\ \frac{t}{t + d(x, y)} & \text{otherwise,} \end{cases}
\]

for all \(x, y \in X\) and \(t > 0\). Then \((X, M, \ast)\) is a fuzzy metric space. Note that, in the above example, there exists no metric \(d\) on \(X\) satisfying

\[
M(x, y, t) = \frac{t}{t + d(x, y)}
\]
where $M(x, y, t)$ is as defined in above example. Also note the above function $M$ is not a fuzzy metric with the t-norm defined as $a \ast b = \min\{a, b\}$. 

**Definition 2.3** [11]: Let $(X, M, \ast)$ be a fuzzy metric space. Then

(a) A sequence $\{x_n\}$ in $X$ is said to be convergent in $X$ if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

(b) A sequence $\{x_n\}$ in $X$ is said to be Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$, such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all $n, m \geq n_0$.

(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

**Definition 2.4** [9]: Two self mappings $A$ and $S$ of a fuzzy metric space $(X, M, \ast)$ are called compatible if $\lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} ASx_n = \lim_{n \to \infty} SAx_n = x$, for some $x$ in $X$.

**Definition 2.5**[33]: Two self mappings $A$ and $S$ of a fuzzy metric space $(X, M, \ast)$ are called weakly commuting if $M(ASx, SAx, t) = M(Ax, Sx, t)$, for all $x$ in $X$ and $t > 0$.

**Definition 2.6**[33]: Two self mappings $A$ and $S$ of a fuzzy metric space $(X, M, \ast)$ are called point-wise $R$–weakly commuting if there exists $R > 0$ such that $M(ASx, SAx, t) \leq M(Ax, Sx, t/R)$, for all $x$ in $X$ and $t > 0$.

**Remark 2.1**: Clearly pointwise $R$–weakly commutativity implies weak commutativity only when $R \leq 1$.

**Definition 2.7**: Two self mappings $A$ and $S$ of a fuzzy metric space $(X, M, \ast)$ is called weakly commuting if $A(X) \subseteq S(X)$ and for any $x$ in $X$.

$$M(A^2S^2x, S^2A^2x, t) \leq M(A^2x, S^2x, t)$$

**Remark 2.2**: If $A$ and $S$ are idempotent maps i.e $A^2 = A$ and $S^2 = S$. Then weak commutative reduced to weak commuting pair $(A, S)$

$$M(AS^2x, SA^2x, t) \leq M(ASx, SAx, t) \leq M(A^2Sx, S^2Ax, t) \leq M(A^2x, S^2x, t)$$

However, point wise $R$–weakly commuting mappings need not to be compatible as shown in the example.

**Example 2.3**: Let $X = [2, 20)$ with the usual metric $d$ and define $M(x, y, t) = \frac{t}{t + |x - y|}$ for all $x, y \in X$ and $t > 0$. Clearly $(X, M, \ast)$ is a complete fuzzy metric space where $\ast$ is defined by $a \ast b = ab$ for all $a, b \in [0, 1]$. Let $A$ and $S$ be self mappings of $X$ defined as
\[ Ax = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 8, & 2 < x \leq 5 \end{cases} \quad \text{and} \quad Sx = \begin{cases} 12 + x, & 2 < x \leq 5 \\ x - 3 & x > 5 \end{cases} \]

It can be verified that \( A \) and \( S \) are point wise \( R \)-weakly commuting mappings but not compatible. Also, neither \( A \) nor \( S \) is continuous, not even at their coincidence points.

**Definition 2.8 [1]:** Two self maps \( A \) and \( S \) which are idempotent maps i.e. \( A^2 = A \) and \( S^2 = S \), of a fuzzy metric space \((X, M, \ast)\) are called reciprocally continuous on \( X \) if
\[
\lim_{n \to \infty} A^2 S^2 x_n = A^2 x \quad \text{and} \quad \lim_{n \to \infty} S^2 A^2 x_n = S^2 x,
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} A^2 x_n = \lim_{n \to \infty} S^2 x_n = x, \quad \text{for some } x \in X.
\]
That is
\[
M(A^2 S^2 x_n, S^2 A^2 x_n, t) \geq M(A^2 S x_n, S^2 A x_n, t) \geq M(AS x_n, SA x_n, t) \geq M(A^2 S^2 x_n, S^2 A^2 x_n, t)
\]
where \( x_n \) is sequence in \( X \) such that
\[
\lim_{n \to \infty} M(A^2 x_n, S^2 x_n, t) = M(A^2 x, S^2 x, t), \quad \text{for all } t > 0.
\]
Thus if two self mapping are weak commuting then they are reciprocally continuous as well.

**Example 2.5:** Let \((X, M, \ast)\) be the Fuzzy metric space with
\[
M(x, y, t) = \left[ \frac{\exp(|x-y|)}{t} \right]^{-1} \quad \text{for all } x, y \in X \text{ and } t > 0. \text{ Let } X = [0, 1]. \text{ Define}
\]
\( A \) and \( T \) by \( Ax = x/(x+2) \) and \( Tx = x/2 \) for all \( x \in X \), where \( Ax = [0, 1/3] \) and \( Tx = [0, 1/2] \), then \([0, 1/3] \subset [0, 1/2] \).

\[
M(A^2 T^2 x, T^2 A^2 x, t) \geq M(A^2 x, T^2 A x, t) \geq M(A^2 T x, T^2 A^2 x, t)
\]

\[
M(A^2 T x, T^2 A x, t) = \left[ \frac{\exp \left( \frac{x}{3x+16} - \frac{x}{12x+16} \right)}{t} \right]^{-1} \geq \left[ \frac{\exp \left( \frac{x}{3x+8} - \frac{x}{4x+8} \right)}{t} \right]^{-1}
\]

\[
M(A^2 T x, T^2 A x, t) = \left[ \frac{\exp \left( \frac{x}{3x+8} - \frac{x}{4x+8} \right)}{t} \right]^{-1} \geq \left[ \frac{\exp \left( \frac{x^2}{(3x+8)(4x+8)} \right)}{t} \right]^{-1}
\]

\[
M(A^2 T x, T^2 A x, t) \geq \left[ \frac{\exp \left( \frac{x^2}{(x+8)(6x+8)} \right)}{t} \right]^{-1} \geq \left[ \frac{\exp \left( \frac{5x^2}{(x+8)(6x+8)} \right)}{t} \right]^{-1}
\]

\[
M(A T^2 x, T A^2 x, t) \geq M(A T^2 x, TA^2 x, t)
\]
Fixed point theorem

\[ M( AT^2 x, TA^2 x, t) = \left( \exp \left( \frac{x}{x+8} - \frac{x}{6x+8} \right) \right)^{\frac{1}{t}} \geq \exp \left( \frac{5x^2}{(x+8)(6x+8)} \right) \]

\[ \geq \exp \left( \frac{x}{x+4} - \frac{x}{2x+4} \right) \geq M( ATx, TAx, t) \]

\[ M( ATx, TAx, t) = \left( \exp \left( \frac{x}{x+4} - \frac{x}{2x+4} \right) \right)^{\frac{1}{t}} \geq \exp \left( \frac{x^2}{(x+4)(2x+4)} \right) \]

Hence \( (A, T) \) is weak commuting map, which implies the reciprocally continuity of \( A \) and \( T \).

**Lemma 2.2[29]:** Let \((X, M, *)\) be a fuzzy metric space. If there exists \( k \in (0, 1) \) such that \( M(x, y, kt) \geq M(x, y, t) \), then \( x = y \).

**Lemma 2.3[9]:** Let \( \{y_n\} \) be a sequence in a fuzzy metric space \((X, M, *)\) with the condition \((FM-6)\). If there exists \( k \in (0, 1) \) such that \( M(y_n, y_{n+1}, kt) \geq M(y_n, y_{n+1}, t) \) for all \( t > 0 \) and \( n \in \mathbb{N} \), then \( \{y_n\} \) is a Cauchy sequence in \( X \).

The following theorem was proved by Balasubramaniam et al. [2]:

**Theorem 2.1:** Let \((A, S)\) and \((B, T)\) be point-wise R-weakly commuting pairs of self mappings of complete fuzzy metric space \((X, M, *)\) such that

(i) \( AX \subseteq TX, \ BX \subseteq SX \),

(ii) \( M(Ax, By, t) \geq M(x, y, ht); 0 < h < 1, x, y \in X \) and \( t > 0 \).

Suppose that \((A, S)\) and \((B, T)\) is compatible pairs of reciprocally continuous mappings. Then \( A, B, S \) and \( T \) have a unique common fixed point.

The following theorem was proved by Pant and Jha [22]:

**Theorem 2.2:** Let \((A, S)\) and \((B, T)\) be point-wise R-weakly commuting pairs of self mappings of complete fuzzy metric space \((X, M, *)\) such that

(i) \( AX \subseteq TX, \ BX \subseteq SX \),

(ii) \( M(Ax, By, t) \geq M(x, y, ht); 0 < h < 1, x, y \in X \) and \( t > 0 \).
Let \((A, S)\) and \((B, T)\) be compatible mappings. If any of the mappings in
compatible pairs \((A, S)\) and \((B, T)\) is continuous then \(A, B, S\) and \(T\) have a unique
common fixed point.

Remark 2.3: In [22], Pant and Jha proved that theorem 2.2 is an analogue of the
theorem 2.1 by obtaining a connection between continuity and reciprocal
continuity in fuzzy metric space.

3. IMPLICIT RELATIONS

Let \(\phi\) be the set of all real continuous functions \(F: [0, 1]^6 \to \mathbb{R}\) is continuous
function such that

\((F_1)\): \(F\) is non-increasing in the fifth and sixth variable

\((F_2)\): If for some \(k \in (0, 1)\), we have

\((F_a)\): \(F\{u(kt), v(t), v(t), u(t), u(t) * v(t/2), 1\} \geq 1\), \text{ or }

\((F_b)\): \(F\{u(kt), v(t), v(t), u(t), u(t) * v(t/2)\} \geq 1\),

for any fixed \(t > 0\) and any non-decreasing function \(u, v: (0, \infty) \to [0, 1]\),
with \(0 < u(t), v(t) \leq 1\) then there exists \(q \in (0, 1)\) with \(u(qt) \geq v(t) * u(t)\)

\((F_3)\): If for some \(k \in (0, 1)\), we have

\(F\{u(kt), u(t), 1, 1, u(t), u(t)\} \geq 1\), \text{ or } \(F\{u(kt), 1, u(t), 1, u(t), 1\} \geq 1\),

\text{ or } \(F\{u(kt), 1, 1, u(t), 1, u(t)\} \geq 1\),

for some \(t > 0\) and any non-decreasing function \(u: (0, \infty) \to [0, 1]\), then
\(u(kt) \geq u(t)\).

Example 3.1: Let \(F(u_1, \ldots, u_6) = \min\{u_1, \min\{u_2, u_3, u_4, u_5, u_6\}\}\) and \(a * b = \min\{a, b\}\).

Let \(t > 0, u(t) > 0, v(t) \leq 1, k \in (0, 1/2)\), where \(u, v: [0, \infty) \to [0, 1]\) are non-decreasing functions.

Now, suppose that \(F\{u(kt), v(t), v(t), u(t), u(t) * v(t/2), 1\} \geq 1\),
i.e. in \((F_a)\),

\[
F\{u(kt), v(t), v(t), u(t), 1, u(t/2) * v(t/2)\} = \frac{u_1}{\min\{v(t), v(t), u(t), u(t), u(t) * v(t/2)\}} \geq 1.
\]

Thus \(u(qt) \geq v(t) * u(t)\), if \(q = 2k \in (0, 1)\).

Finally in \((F_3)\), suppose that \(t > 0\) is fixed, \(u: [0, \infty) \to [0, 1]\) is a non-decreasing function and

\[
F\{u(kt), v(t), v(t), u(t), 1, u(t/2) * v(t/2)\} \geq 1.
\]
Fixed point theorem

2491

\[ F\{u(kt), u(t), 1, 1, u(t), u(t)\} = \frac{u(kt)}{u(t)} \geq 1, \]

or

\[ F\{u(kt), 1, u(t), 1, u(t), u(t)\} = \frac{u(kt)}{u(t)} \geq 1, \]

for some \( k \in (0, 1). \)

Then we have \( u(kt) \geq u(t) \) and thus \( F \in \phi. \)

Example 3.2: Let \( F(u_1, \ldots, u_6) = \frac{u^2}{\min\{u^2 + u_3, u^2 + u_4, u^2 + u_5\}} \) and

\[ a \ast b = \min\{a, b\}. \]

Let \( t > 0, u(t) > 0, v(t) \leq 1, k \in (0, 1/2), \) where \( u, v : [0, \infty) \rightarrow [0, 1] \) are non-decreasing functions. Now suppose that

\[ F\{u(kt), v(t), v(t), u(t), 1, u(t) \ast_2 v(t)\} \geq 1, \]

i.e. in \( (F_a), \)

\[ F\{u(kt), v(t), v(t), u(t), 1, u(t) \ast_2 v(t)\} = \frac{u^2(kt)}{\min\{v(t) \ast v(t), v(t) \ast u(t), v(t) \ast u(t)\}} \]

max\{1, u(t) \ast_2 v(t)\} \geq 1.

Thus \( u(qt) \geq v(t) \ast u(t) \) if \( q = 2k \in (0, 1). \) For \( (F_b) \) a similar argument works.

Finally in \( (F_3), \) suppose that \( t > 0, \) a fixed, \( u : [0, \infty) \rightarrow [0, 1] \) is a non-decreasing function and

\[ F\{u(kt), u(t), 1, 1, u(t), u(t)\} = \frac{u(kt)}{u(t)} \geq 1, \]

or

\[ F\{u(kt), 1, u(t), 1, u(t), 1\} = \frac{u(kt)}{u(t)} \geq 1, \]

or

\[ F\{u(kt), 1, 1, u(t), 1, u(t)\} = \frac{u(kt)}{u(t)} \geq 1, \]

for some \( k \in (0, 1). \)

Then we have \( u(kt) \geq u(t) \) and thus \( F \in \phi. \)

Lemma 3.1: Let \( (X, M, *) \) be a complete fuzzy metric space with \( t \ast t \geq t \) for all \( t \in [0, 1] \) and the condition \( (FM-6). \) Let \( (A, S) \) and \( (B, T) \) be point wise \( R \)-weakly commuting pairs of self mappings of \( X \) such that

(i) \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \)

(ii) \( \exists k \in (0, 1), \) such that

\[ F\{M(A^2x, B^2y, kt), M(S^2x, T^2y, t), M(T^2y, B^2y, t), M(S^2x, A^2x, t), M(S^2x, B^2y, t), M(T^2y, A^2x, t)\} \geq 1 \]

for all \( x, y \in X, t > 0. \) Then the continuity of one mapping in the compatible pair \( (A, S) \) or \( (B, T) \) implies the reciprocal continuity.

4. MAIN RESULT

Theorem 4.1: Let \( (X, M, *) \) be a complete fuzzy metric space with \( t \ast t \geq t \) for all \( t \in [0, 1] \) and the condition \( (FM-6). \) Let \( (A, S) \) and \( (B, T) \) be point wise \( R \)-weakly commuting pairs of self maps on \( X \) satisfying

(i) \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \)
(II) \( (A, S) \) and \( (B, T) \) are compatible pairs and one of the mapping in each pair is continuous;

(III) there exists \( k \in (0, 1) \) such that
\[
F\{M(A^2x, B^2y, kt), M(S^2x, T^2y, t), M(T^2y, B^2y, t), M(S^2x, A^2x, t),
= M(S^2x, B^2y, t), M(T^2y, A^2x, t) \geq 1
\]
for all \( x, y \in X, t > 0 \) and where \( F \in (\), then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

Proof: Let \( x_0 \) be any point in \( X \), then by (I) there exists points \( x_1, x_2 \in X \) such that \( A^2x_0 = T^2x_1 = y_0 \) and \( B^2x_1 = S^2x_2 = y_1 \), where \( A, B, S \) and \( T \) are idempotent maps then we define sequence \( \{x_n\} \) and \( \{y_n\} \) such that \( y_1 = A^2x_2 = T^2x_{n+1} \) and \( y_{2n+1} = B^2x_{2n+2} = S^2x_{2n+2}, \) for \( n = 0, 1, 2, \ldots \).

When \( y_n \triangleright y_{n+1}, \) for all \( n = 0, 1, 2, \ldots, \) we put \( x = y_{2n} \) and \( y = y_{2n+1} \), in (III), we have
\[
F\{M(A^2x_{2n}, B^2x_{2n+1}, kt), M(S^2x_{2n}, T^2x_{2n+1}, t), M(T^2x_{2n+1}, B^2x_{2n+2}, t), M(S^2x_{2n}, A^2x_{2n+1}, t),
\]
\[
M(S^2x, B^2y, t), M(T^2y, A^2x, t) \geq 1
\]
\[
\text{or} \quad F\{M(y_{2n}, y_{2n+1}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+2}, t),
\]
\[
M(y_{2n+1}, y_{2n+2}, t)/2, M(y_{2n+1}, y_{2n+2}, t) \geq 1
\]
\[
\text{Using condition (F_5) in (F_2), we have}
\]
\[
M(y_{2n}, y_{2n+1}, q^t) \geq M(y_{2n}, y_{2n+1}, t) \geq M(y_{2n}, y_{2n+1}, t)
\]
\[
(4.1)
\]
\[
\text{which implies a * b = min \{a, b\}} \quad \text{that} \quad M(y_{2n}, y_{2n+1}, q^t) \geq M(y_{2n}, y_{2n+1}, t).
\]
\[
(4.2)
\]
\[
\text{In general, we have for m = 1, 2, \ldots, and t > 0,}
\]
\[
M(y_{2n}, y_{2n+1}, q^t) \geq M(y_{2n}, y_{2n+1}, t)
\]
\[
(4.3)
\]
\[
\text{We shall prove that} \quad \{y_n\} \text{is a Cauchy sequence for which}
\]
\[
M(y_{2n}, y_{2n+1}, q^t) \geq M(y_{2n}, y_{2n+1}, q^t) \geq \ldots \geq M(y_{2n}, q^t) \rightarrow 1
\]
as \( n \rightarrow \infty \). It follows that \( M(y_{2n}, y_{2n+1}, t) \rightarrow 0 \) as \( n \rightarrow \infty \), for all \( n \in N \), i.e. result holds for \( m = 1 \).

By induction hypothesis suppose that the result holds for \( m = r \), i.e.
\[
M(y_{2n}, y_{2n+1}, q^t) \geq M(y_{2n}, y_{2n+1}, t) \rightarrow 1
\]
\[
(4.4)
\]
Now \( M(y_{2n}, y_{2n+1}, t) \geq M(y_{2n}, y_{2n+1}, t/2) \geq M(y_{2n}, y_{2n+1}, t/2) \rightarrow 1 \rightarrow 1 \).

Thus the result holds for \( m = r + 1 \) and so by induction it holds for all \( n \in N \). Hence \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, then there exists a point \( z \) in \( X \) such that \( y_n \rightarrow z \) as \( n \rightarrow \infty \). Moreover its subsequences \( \{A^2x_n\}, \{T^2x_{2n+1}\}, \{B^2x_{2n+1}\} \) and \( \{S^2x_{2n+2}\} \) also converges to \( z \), i.e. \( y_{2n} = A^2x_{2n} = T^2x_{2n+1} \) \( z \), and \( y_{2n+1} = B^2x_{2n+1} = S^2x_{2n+2} \rightarrow z \).

Since \( A \) and \( S \) are compatible and reciprocally continuous mappings, then
\[
A^2S^2x_n \rightarrow A^2z \quad \text{and} \quad S^2A^2x_{2n} \rightarrow S^2z.
\]

By compatibility of \( A^2 \) and \( S^2 \), it yields \( \lim_{n \rightarrow \infty} M(A^2S^2x_n, S^2A^2x_{2n}) = 1 \), i.e. \( M(A^2z, S^2z) = 1 \rightarrow A^2z = S^2z \). Since \( A(X) \subseteq T(X) \), then there exists a point \( u \in X \) such that \( A^2z = T^2u \).

Now using (III), we have
\[
F\{M(A^2z, B^2u, kt), M(S^2z, T^2u, t), M(T^2u, B^2u, t), M(S^2z, A^2z, t),
\]
\[
M(S^2z, B^2y, t), M(T^2y, A^2z, t) \geq 1
\]
2493

\[ M(S^2z, B^2u, t), M(T^2u, A^2z, t) \leq 1 \]

i.e. \[ \{ M(A^2z, B^2u, kt), M(A^2z, A^2z, t), M(A^2z, B^2u, t), M(A^2z, A^2z, t),
M(A^2z, B^2u, t), M(A^2z, A^2z, t) \} \leq 1 \]

or \[ \{ M(A^2z, B^2u, kt), 1, M(A^2z, B^2u, t), 1, M(A^2z, B^2u, t), 1 \} \leq 1 \]

From (F.3), we have \[ M(A^2z, B^2u, kt) \geq M(A^2z, B^2u, t), \] which implies by Lemma 2.2, \[ A^2z = B^2u. \] Thus \[ A^2z = S^2z = B^2u = T^2u \] (4.4)

Now again using the R-weakly conditions of mappings there exists \( R > 0 \) such that
\[ M(A^2S^2z, S^2A^2z, t) \geq M(AS^2z, SA^2z, t) \geq M(ASz, SAz, t) \]
\[ M(A^2z, S^2z, t) \geq M(A^2z, S^2z, t/R) \]

i.e. \[ A^2z = S^2z, \] and \[ A^2Az = A^2Sz = S^2Az = S^2z. \]

Similarly since B and T are pointwise R-weakly commuting mappings, we have
\[ M(B^2T^2z, T^2B^2z, t) \geq M(BT^2z, TB^2z, t) \]
\[ M(BTz, TBz, t) \geq M(B^2z, T^2z, t) \]

which implies \[ B^2z = T^2z. \] Again putting \( x = x_{2n}, \) and \( y = z, \) in (III), we get
\[ \{ M(A^2x_{2n}, B^2z, kt), M(S^2x_{2n}, T^2z, t), M(T^2z, B^2z, t), M(S^2x_{2n}, A^2x_{2n}, t),
M(S^2x_{2n}, B^2z, t), M(T^2z, A^2x_{2n}, t) \} \leq 1 \]

taking \( n \to \infty, \) \[ \{ M(A^2z, B^2z, kt), M(S^2z, T^2z, t), M(T^2z, B^2z, t),
M(S^2z, A^2z, t), M(S^2z, B^2z, t), M(T^2z, A^2z, t) \} \leq 1 \]

i.e. \[ \{ M(A^2z, B^2z, kt), M(A^2z, B^2z, t), 1, 1, M(A^2z, B^2z, t), M(A^2z, B^2z, t) \leq 1, \]

From (F.3), we have,
\[ M(A^2z, B^2z, kt) \geq M(A^2z, B^2z, t), \]
i.e. from Lemma 2.2, \[ A^2z = B^2z. \] Similarly, \[ B^2z = S^2z. \]

Thus \[ A^2z = S^2z = B^2z = T^2z. \] (4.5)

Again we suppose that \( z = A^2z \) is a common fixed point of \( A, B, S \) and \( T. \) For this we have by (III), putting \( x = x_{2n}, y = z \)
\[ \{ M(A^2x_{2n}, B^2z, kt), M(S^2x_{2n}, T^2z, t), M(T^2z, B^2z, t), M(S^2x_{2n}, A^2x_{2n}, t),
M(S^2x_{2n}, B^2z, t), M(T^2z, A^2x_{2n}, t) \} \leq 1 \]

taking \( n \to \infty, \)
\[ \{ M(z, B^2z, kt), M(T^2z, z, t), M(T^2z, B^2z, t), M(z, z, t),
M(z, B^2z, t), M(T^2z, z, t) \} \leq 1 \]

i.e. \[ \{ M(z, B^2z, kt), M(B^2z, z, t), 1, 1, M(z, B^2z, t), M(B^2z, z, t) \leq 1 \]

From (F.3), we have, \[ M(z, B^2z, kt) \geq M(B^2z, z, t) \]
i.e. from Lemma 2.2, \( z = B^2z. \) It follows from (4.5) that \( z = A^2z = S^2z = B^2z = T^2z \) is a common fixed point of \( A, B, S \) and \( T. \) For uniqueness, we put \( x = Az, y = w (\neq z) \) in (III), we get easily

**Corollary 4.1:** Let \( (X, M, *) \) be a complete fuzzy metric space and let \((A, S)\) and \((B, T)\) be self mapping of \( X \) satisfying:

(I)* \[ AX \leq TX \text{ and } BX \leq SX; \]

(II) \((A, S)\) and \((B, T)\) are compatible pairs and one of the mapping in each pair is continuous;

(III) there exists \( k \in (0, 1) \) such that
\[ \{ M(Ax, By, kt), M(Sx, Ax, t), M(Ty, Sx, t), M(Ty, By, t), M(Sx, By, 2t),
M(Ax, Ty, t) \} \leq 1 \]
for all \( x, y \in X, t > 0 \), then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \), where

\[
F(t_1 \ldots t_6) : [0, 1]^6 \to R \subseteq F^*.
\]

**Proof:** From the definition \( F(t_1 \ldots t_6) : [0, 1]^6 \to R \subseteq F^* \), and \( t^* t \geq t \), we have

\[
F(M(Ax, By, kt), M(Sx, Ax, t), M(Ty, Sx, t), M(Ty, By, t), M(Sx, Ty, t) \geq 1
\]
and hence from theorem 4.1; \( A, B, S \) and \( T \) have a unique common fixed point.

**Corollary 4.2:** Let \( (X, M, *) \) be a complete fuzzy metric space and let \( (A, S) \) and \( (B, T) \) be self mappings of \( X \) satisfying (I), (II) of corollary 4.1 and (III)** there exists \( k \in (0, 1) \) such that

\[
F(M(Ax, By, kt), M(Sx, Ax, t), M(Ty, Sx, t), M(Ty, By, t), M(Ax, Ty, t)) \geq 1
\]
for all \( x, y \in X, t > 0 \), then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof:** We have the following example \( F\{t_1 \ldots t_5\} = 18t_1 - 16t_2 + 8t_3 - 10t_4 + t_5 + 1 \)
then \( F\{t_1 \ldots t_4\} > F\{t_1 \ldots t_5\} \)
i.e. \( F(M(Ax, By, kt), M(Sx, Ax, t), M(Ty, Sx, t), M(Ty, By, t)) \geq 1 \)

The result proved easily.

**REFERENCES**

[13]. O. Hadzie, Common fixed point theorems for families of mapping in complete metric space, Math. Japon. 29 (1984), 127 - 134.
[14]. M. Imdad, S. Kumar and M. S. Khan, Remarks on some fixed point theorems satisfying an implicit relations, Radovi Mathematics, 11 (2002), 135 – 143.
[23]. H. K. Pathak, Weak commuting mapping and fixed point, IJPAM, 17 (1986), 201 – 211.


[34]. L. A. Zadeh, Fuzzy sets, Inform and Control, 8 (1965), 338 - 353.

Received: January, 2010