CHAPTER I
1.1. CONVERGENCE OF RANDOM VARIABLES:

Let \( \{X_n, n \geq 1\} \) be a sequence of random variables defined on a probability space \((\Omega, \mathcal{F}, P)\). \( \{X_n\} \) is said to converge to a random variable \( X \) if \( \{X_n(\omega)\} \) converges to \( X(\omega) < \infty \) as \( n \to \infty \) for all \( \omega \in \Omega \). Here \( \{X_n\} \) is said to converge to \( X \) everywhere. If \( \{X_n(\omega)\} \) converges to \( X(\omega) \) only for \( \omega \in C \), then \( C \) is called the set of convergence of \( \{X_n\} \). We know that \( C \in \mathcal{F} \) and \( \lim X_n \) is a random variable.

Thus it is clear that \( C \) is the set of all \( \omega \in \Omega \), at which, wherever be \( \varepsilon > 0 \), \( |X_n(\omega) - X(\omega)| < \varepsilon \) for all \( n \) greater than some \( N_\varepsilon(\omega) \) sufficiently large. Symbolically, for \( r = n + m \) and \( m \geq 1 \),

\[
C = \{\omega : X_n(\omega) \to X(\omega)\}
= \bigcap_{\varepsilon > 0} \bigcup_n \bigcap_m \{\omega : |X_{n+m}(\omega) - X(\omega)| < \varepsilon\} \tag{1.1.1}
\]

Equivalently, replacing ‘for every \( \varepsilon > 0 \)’ by every \( \frac{1}{k} \), \( k = 1, 2, \ldots \) we have

\[
C = \bigcap_k \bigcup_n \bigcap_m \left\{\omega : |X_{n+m}(\omega) - X(\omega)| < \frac{1}{k}\right\} \tag{1.1.2}
\]

Since \( C \) is obtained from countable operations on measurable sets, \( C \) is measurable. This provides a direct proof that \( C \in \mathcal{F} \).
This is a known fact that if \( X_n(\omega) \to X(\omega) \) where \( X(\omega) \) is finite, then \( X_m(\omega) - X_n(\omega) \to 0 \) as \( m,n \to \infty \) and conversely if \( X_m(\omega) - X_n(\omega) \to 0 \) as \( m,n \to \infty \) then \( X_n(\omega) \to X(\omega) \), where \( X(\omega) \) is finite. This is called the Cauchy criterion for convergence. Thus \( X_n \to X < \infty \) if and only if \( X_m - X_n \to 0 \) or equivalently \( X_{n+m} - X_n \to 0 \) as \( n \to \infty \) uniformly in \( m \). If a sequence of random variables converges in Cauchy sense, it is said to converge mutually. If \( C' \) is the set of mutual convergence, then

\[
C' = \{ \omega : |X_{n+m}(\omega) - X_n(\omega)| \to 0 \text{ as } n \to \infty \}
= \bigcap_{k} \bigcup_{n} \bigcap_{m} \left\{ \omega : |X_{n+m}(\omega) - X(\omega)| < \frac{1}{k} \right\}
\tag{1.1.3}
\]

So \( C' \subset C \) as mutual convergence implies convergence. Again since \( X \) is finite on \( C \), we have \( C' = C \).

**Definition 1.1.1:** A sequence of random variables \( \{X_n\} \) is said to converge to \( X \) in probability, denoted by \( X_n \overset{p}{\to} X \), if for every \( \varepsilon > 0 \),

\[
P\{|X_n - X| \geq \varepsilon\} \to 0 \text{ as } n \to \infty
\tag{1.1.4}
\]

That is for every \( \varepsilon > 0 \),

\[
P\{|X_n - X| < \varepsilon\} \to 1 \text{ as } n \to \infty.
\]

This concept plays an important role in statistics. Weak consistency of estimators and weak law of large numbers are instances of this concept.
Definition 1.1.2: A sequence of random variables \( \{X_n\} \) is said to converge to \( X \) almost surely(a.s.) denoted by \( X_n(\omega) \to X(\omega) \), if \( X_n(\omega) \to X(\omega) \) for all \( \omega \) except those belonging to a null set \( N \). Thus symbolically, \( X_n \to X \) if and only if \( X_n(\omega) \to X(\omega) < \infty \) for all \( \omega \in N^c \), where \( P(N) = 0 \). Here the set of convergence of \( \{X_n\} \) has probability unity.

Definition 1.1.3: A sequence of random variables \( \{X_n\} \) is said to converge to \( X \) in \( r^{th} \) mean, denoted by \( X_n \to^r X \) if \( E|X_n - X|^r \to 0 \) as \( n \to \infty \). For \( r = 2 \), it is called convergence in quadratic mean(q.m.) or mean square convergence and for \( r = 1 \), it is convergence in the mean.

Definition 1.1.4: Two sequences of random variables \( \{X_n\} \) and \( \{Y_n\} \) are called equivalent if \( \sum_{n=1}^{\infty} P[X_n \neq Y_n] < \infty \). If \( \{X_n\} \) and \( \{Y_n\} \) are equivalent sequences of random variables, the Borel-Cantelli Lemma ensures that \( P[X_n \neq Y_n \text{ i.o.}] = 0 \).

Definition 1.1.5: A sequence of random variables \( \{X_n\} \) is said to be uniformly integrable if
\[
\limsup_{a \to \infty} \sup_{n \geq 0} E\left[|X_n| I_{\{|X_n| \geq a\}}\right] = 0.
\]
We have the following convergence results for \( \{X_n\} \), a sequence of random variables.

**Results 1.1.1:**

(a) \( X_n - X_m \xrightarrow{P} 0 \) if and only if \( X_n \xrightarrow{P} X \), where \( X \) is some random variable.

(b) \( X_n \xrightarrow{a.s.} X \), then \( X_n \xrightarrow{P} X \).

(c) \( X_n \xrightarrow{P} X \), then there exists a sub-sequence \( \{X_{n_k}\} \) of \( \{X_n\} \) such that \( X_{n_k} \xrightarrow{a.s.} X \).

(d) \( X_n \xrightarrow{r} X \), then \( X_n \xrightarrow{P} X \). Again if \( X_n \)'s are almost surely bounded and \( X_n \xrightarrow{P} X \), then \( X_n \xrightarrow{r} X \) for all \( r \).

(e) If \( \{X_n\} \) is a sequence of random variables with finite a.s. expectations, if \( X_n \xrightarrow{P} X \) for some random variable \( X \) and if \( \{X_n\} \) is uniformly integrable, then \( X_n \xrightarrow{r} X \), where \( r = 1 \).

(f) \( X_n \xrightarrow{r} X \), then \( E|X_n| \xrightarrow{r} E|X| \).

(g) Let the random variables \( X_n \) be orthogonal. Then

(i) The series \( \sum X_n \) converges in quadratic mean if and only if \( \sum E|X_n|^2 < \infty \) and then \( E|\sum X_n|^2 = \sum E|X_n|^2 \).
(ii) If \( \sum \frac{E|X_n|^2}{b_n^2} < \infty, b_n \uparrow \infty, \) then \( \frac{1}{b_n} \sum_{k=1}^{n} X_k \to 0. \)

The above convergence results can be found in any elementary book on probability theory.

1.2. MARTINGALE AND CONVERGENCE:

The concept of martingale was introduced into the modern probability literature by Ville in 1939. Most of the major works in the field are due to Doob in the 1940’s and early 1950’s. The theory, like probability theory, has its origin partly in gambling theory and the idea of martingale expresses the concept of a fair game.

The basic tool in defining a martingale is the concept of conditional expectation, which is one of the most fundamental notions in probability theory. Conditional expectation could not be properly defined until the Radon-Nikodym theorem was stated and was proved in its abstract theoretic setting. We state the theorem below.

**Theorem 1.2.1. (Radon-Nikodym):** Let \( \phi \) be a finite signed measure (i.e. a difference of two finite measures) over a finite measure space \( (X, \mathcal{M}, \mu) \) and let us assume that \( \phi \) is absolutely continuous with respect to \( \mu \), that is, \( M \in \mathcal{M} \) and \( \mu(M) = 0 \) implies \( \phi(M) = 0. \) Then there exists an \( \mathcal{M} \)-measurable function \( f \), uniquely determined except over an \( \mathcal{M} \)-measurable set of \( \mu \)-measure zero such that

\[
\phi(M) = \int_M f \, d\mu \quad \text{for all } M \in \mathcal{M}.
\]
Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(\mathcal{B}\) be a sigma field of events such that \(\mathcal{B} \subset \mathcal{F}\). Let \(X\) be a random variable with finite expectation. We define a set function \(\varphi\) over \(\mathcal{B}\) by

\[
\varphi(B) = \int_B X \, dp = E(X \mathbf{1}_B) \quad \text{for all } B \in \mathcal{B},
\]

where \(\mathbf{1}_B\) is the indicator function of the event \(B\). Here \(\varphi\) is a signed measure over \((\Omega, \mathcal{B})\) and is absolutely continuous with respect to \(P\). By using the Radon-Nykodym theorem, we finally define conditional expectation as follows.

**Definition 1.2.1:** Let \(X\) be a random variable with finite expectation and let \(\mathcal{B}\) be a sub-sigma field of \(\mathcal{F}\). The conditional expectation of \(X\) given \(\mathcal{B}\) i.e. \(E[X|\mathcal{B}]\) is a \(\mathcal{B}\) -measurable random variable uniquely defined except over a \(\mathcal{B}\) -measurable event of probability zero, such that

\[
\int_B X \, dp = \int_B E[X|\mathcal{B}] \, dP \quad \text{for all } B \in \mathcal{B}.
\]

**Remark 1.2.1:** If \(X \in L_1(\Omega, \mathcal{F}, P)\) and if \(\mathcal{B}\) is a sub-sigma field of \(\mathcal{F}\), then \(E[X] = E[E(X|\mathcal{B})]\). This follows immediately from the above identity in the case \(B = \Omega\).

**Remark 1.2.2:** If \(X \in L_1(\Omega, \mathcal{F}, P)\) and if \(\mathcal{B}\) is a sub-sigma field of \(\mathcal{F}\) consisting only of events of probability zero or one, then \(E[X|\mathcal{B}] = E[X]\) except over a \(\mathcal{B}\) -measurable event of probability zero.
We state below some of the important properties of conditional expectation those are used for derivation of important results in the thesis.

**Property 1.2.1:** If $X$ and $Y$ are in $L_1(\Omega, \mathcal{F}, P)$, $a$ and $b$ are constants and $\mathcal{B}$ is a sub-sigma field of $\mathcal{F}$, then

$$E[aX + bY|\mathcal{B}] = aE[X|\mathcal{B}] + bE[Y|\mathcal{B}] \text{ a.s.}$$

and if $X \leq Y$ a.s., then

$$E[X|\mathcal{B}] \leq E[Y|\mathcal{B}] \text{ a.s.}$$

**Property 1.2.2:** If $X \in L^1(\Omega, \mathcal{F}, P)$ and if $\mathcal{B}$ and $\mathcal{C}$ are sub-sigma fields of $\mathcal{F}$ such that $\mathcal{B} \subseteq C$, then

$$E[X|\mathcal{B}] = E[E(X|\mathcal{C})|\mathcal{B}] \text{ a.s.}$$

**Property 1.2.3:** If $X$ and $Y$ are random variables, $Y$ and $XY$ have finite expectations, $\mathcal{B}$ is a sub-sigma field of $\mathcal{F}$ and if $X$ is $\mathcal{B}$-measurable, then

$$E[XY|\mathcal{B}] = XE[Y|\mathcal{B}] \text{ a.s.}$$

**Property 1.2.4:** If $X$ and $Y$ are independent random variables and $X \in L_1(\Omega, \mathcal{F}, P)$, then $E[X|Y] = E[X]$.

**Property 1.2.5:** If $\mathcal{B}$ and $\mathcal{C}$ are two sub-sigma fields of $\mathcal{F}$ such that $\mathcal{B} \subseteq C$ and if $X \in L_1(\Omega, \mathcal{F}, P)$, then

$$E[X|\mathcal{B}] = E[E(X|\mathcal{B})|\mathcal{C}] \text{ a.s.}$$
Property 1.2.6: Let \( \{X_n\} \) be a non-decreasing sequence of non-negative random variables such that \( X_n \to X \) a.s. as \( n \to \infty \) and \( X \in L_1(\Omega, \mathcal{F}, P) \). Then for any sub-sigma field \( \mathcal{B} \subset \mathcal{F} \),
\[
E[X_n|\mathcal{B}] \to E[X|\mathcal{B}] \quad \text{a.s. as } n \to \infty.
\]

Property 1.2.7 (Conditional form of Jensen’s inequality): Let \( g \) be a convex function defined over \(( -\infty, +\infty)\) and \( X \) be a random variable such that \( X \) and \( g(X) \) have finite expectations. If \( \mathcal{B} \) is a sub-sigma field of \( \mathcal{F} \), then
\[
g(E[X|\mathcal{B}]) \leq E[g(X)|\mathcal{B}] \quad \text{a.s.}
\]

Definition 1.2.2: Let \( \{X_n\} \) be a sequence of random variables defined on a probability space \((\Omega, \mathcal{F}, P)\). Let \( \{\mathcal{F}_n\} \) be a sequence of sub-sigma fields of \( \mathcal{F} \) with \( \mathcal{F}_n \subset \mathcal{F}_{n+1} \) for all \( n \). Then \( \{X_n\} \) is called a martingale with respect to \( \{\mathcal{F}_n\} \) if
(i) \( E[|X_n|] < \infty \) for all \( n \) and
(ii) \( E[X_{n+1}|\mathcal{F}_n] = X_n \) for all \( n \).

Definition 1.2.3: Under the set up of Definition 1.2.2 \( \{X_n\} \) is called a sub-martingale with respect to \( \{\mathcal{F}_n\} \) if
(i) \( \{X_n\} \) is adapted to \( \{\mathcal{F}_n\} \) (i.e. \( X_n \) is \( \mathcal{F}_n \)-measurable.)

(ii) \( E[X_n^+] < \infty \) for all \( n \) and

(iii) \( E[X_{n+1} | \mathcal{F}_n] \geq X_n \) for all \( n \).

\( \{X_n\} \) is called a super-martingale with respect to \( \{\mathcal{F}_n\} \) if

(i) \( \{X_n\} \) is adapted to \( \{\mathcal{F}_n\} \)

(ii) \( E[X_n^-] < \infty \) for all \( n \) and

(iii) \( E[X_{n+1} | \mathcal{F}_n] \leq X_n \) for all \( n \), where

\[ x^+ = \max(x, 0) \quad \text{and} \quad x^- = \min(x, 0). \]

It is clear that \( \{X_n\} \) is a sub-martingale if and only if \( \{-X_n\} \) is a super-martingale. Similarly \( \{X_n\} \) is a martingale if and only if \( \{X_n\} \) is both a sub-martingale and a super-martingale.

Using Remark 1.2.1 in the definition of martingale, we get

\[ E[X_{n+1}] = E[X_n] \]

and so by induction \( E[X_n] = E[X_0] \) for all \( n \).

**Example 1.2.1:** Let \( Y_0 = 0 \), and \( Y_1, Y_2, \ldots \) be independent random variables with \( E|Y_n| < \infty \) and \( E[Y_n] = 0 \) for all \( n \). If \( X_0 = 0 \) and \( X_n = Y_1 + Y_2 + \cdots + Y_n \) for \( n \geq 1 \), then \( \{X_n\} \) is a martingale with respect to \( \{Y_n\} \).

Again using Jensen's inequality for conditional expectation, it can be shown that if \( \{X_n\} \) is a martingale and \( g \) is a convex function for which \( E[g^+(X_n)] < \infty \) for all \( n \) then \( \{g(X_n)\} \) is a sub-martingale.
Under general conditions, a martingale \( \{X_n\} \) will converge to a limit random variable \( X \) as \( n \) increases. We state below some of the basic martingale convergence theorems.

**Theorem 1.2.1:** (a) Let \( \{X_n\} \) be a sub-martingale satisfying \( \sup_{n \geq 0} E[|X_n|] < \infty \). Then there exists a random variable \( X \) to which \( \{X_n\} \) converges with probability one i.e.
\[
P\left( \lim_{n \to \infty} X_n = X \right) = 1.
\]
(b) If \( \{X_n\} \) is a martingale and is uniformly integrable then
\[
P\left( \lim_{n \to \infty} X_n = X \right) = 1,
\]
\[
\lim_{n \to \infty} E[|X_n - X|] = 0 \quad \text{and}
\]
\[
E[X_n] = E[X] \quad \text{for all } n
\]
i.e. \( \{X_n\} \) also converges in the mean.

**Theorem 1.2.2:** Let \( \{X_n\} \) be a martingale satisfying \( E[X_n^2] \leq K < \infty \) for all \( n \) and some constant \( K \). Then \( \{X_n\} \) converges as \( n \to \infty \) to a limit random variable \( X \) both with probability one and in mean square. That is
\[
P\left( \lim_{n \to \infty} X_n = X \right) = 1 \quad \text{and}
\]
\[
\lim_{n \to \infty} E[|X_n - X|^2] = 0.
\]
Finally \( E[X_0] = E[X_n] = E[X] \) for all \( n \).
For proof of the theorems, we refer to Karlin and Taylor (1975).

1.3. CONVERGENCE OF OTHER DEPENDENT SEQUENCES:

In this section we introduce some other dependent sequences of random variables such as martingale difference sequence, martingale transforms, mixing sequences, mixing transforms, mixing difference sequences etc. and discuss their convergence properties.

A. Martingale Difference Sequences:

A martingale difference sequence can be viewed as a mathematical model for a game of chance that is a fair game in a certain precise way. By game we mean a sequence of trials for which money is won or lost in each trial.

**Definition 1.3.1:** Let \( \{X_n, n \geq 1\} \) be a sequence of random variables and \( \{\mathcal{N}_n, n \geq 1\} \) an increasing sequence of sub-sigma fields with \( \mathcal{N}_n \subset \mathcal{N} \). If \( X_n \) is \( \mathcal{N}_n \)-measurable for each \( n \geq 1 \), then the sequence \( \{X_n, n \geq 1\} \) said to be adapted to \( \mathcal{N}_n \) and \( \{X_n, \mathcal{N}_n, n \geq 1\} \) is said to be an adapted sequence of random variables.

**Definition 1.3.2:** Let \( \{X_n, \mathcal{N}_n, n \geq 1\} \) be an adapted sequence of random variables. Then \( \{X_n, \mathcal{N}_n, n \geq 1\} \) is called a martingale difference sequence if \( E[|X_n|; \mathcal{N}_{n-1}] = 0 \) a.s.
Let \( \{X_n, \mathcal{F}_n, n \geq 1\} \) be a martingale difference sequence and
\[
S_n = \sum_{k=1}^{n} X_k \quad \text{for} \quad n \geq 1.
\]
Then
\[
E[S_n | \mathcal{F}_{n-1}] = S_{n-1} + E[X_n | \mathcal{F}_{n-1}]
\]
\[= S_{n-1} \quad \text{a.s.}
\]
That is \( \{S_n\} \) for \( n \geq 2 \) is a martingale.

Again let \( \{S_n, n \geq 1\} \) be a martingale. Let \( S_0 = 0 \) and
\[
X_n = S_n - S_{n-1} \quad \text{for} \quad n \geq 1.
\]
Then
\[
E[X_n | \mathcal{F}_{n-1}] = E[S_n - S_{n-1} | \mathcal{F}_{n-1}]
\]
\[= E[S_n | \mathcal{F}_{n-1}] - E[S_{n-1} | \mathcal{F}_{n-1}]
\]
\[= S_{n-1} - S_{n-1}
\]
\[= 0.
\]
That is \( \{X_n\} \) is a martingale difference sequence.

We state below some of the important convergence results for martingale difference sequences those are used in subsequent chapters.

**Theorem 1.3.1 (Doob, 1953):** Let \( \{X_k, \mathcal{F}_k, k \geq 1\} \) be a martingale difference sequence. Then \( \sum_{k=1}^{\infty} E[X_k^2 | \mathcal{F}_{k-1}] < \infty \) implies \( S_n \) converges, where
\[
S_n = \sum_{k=1}^{n} X_k.
\]
Theorem 1.3.2 (Chow, 1965): Let \( \{X_k, \mathcal{F}_k, k \geq 1\} \) be a martingale difference sequence. Then
\[
\sum_{k=1}^{\infty} E \left[ |X_k|^p \left| \mathcal{F}_{k-1} \right| \right] < \infty \quad \text{for some } 1 \leq p < 2
\]
implies \( S_n \) converges where \( S_n = \sum_{k=1}^{n} X_k \).

A basic tool in establishing convergence results for martingale difference sequences and other such sequences is the martingale decomposition theorem due to Gundy (1968) which is stated below.

Theorem 1.3.3: Let \( \{Q_n, \mathcal{F}_n, n \geq 1\} \) be a martingale satisfying \( \sup |Q_n| < \infty \). Then for every \( K > 0 \), \( Q_n \) may be decomposed: \( Q_n = T_n + U_n + V_n \) for \( n \geq 1 \), where \( \{T_n, n \geq 1\} \), \( \{U_n, n \geq 1\} \) and \( \{V_n, n \geq 1\} \) are martingales such that \( \sup E |T_n| < \infty \), \( P[\sup |T_n| > 0] \leq \sup \frac{E |Q_n|}{K} \), \( E \sum_{n=1}^{\infty} |U_n - U_{n-1}| \leq 4 \sup E |Q_n| \) (where \( U_0 = 0 \)) and \( E[V_n]^2 \leq 2K \sup E |Q_n| < \infty \).

Theorem 1.3.4 (Austin, 1966): Let \( \{X_k, \mathcal{F}_k, k \geq 1\} \) be a martingale difference sequence and \( S_n = \sum_{k=1}^{n} X_k \) with \( \sup |S_n| < \infty \) a.s. Then
\[
\sum_{k=1}^{\infty} X_k^2 < \infty \quad \text{a.s.}
\]

The proof of the above theorems are found in Stout (1974).
B. Martingale Transforms:

Definition 1.3.3: Let \( \{X_k, \mathcal{F}_k, k \geq 1\} \) be a martingale difference sequence and \( v_k \) be \( \mathcal{F}_{k-1} \)-measurable for each \( k \geq 1 \). Then \( \{T_n, n \geq 1\} \) defined by \( T_n = \sum_{k=1}^{n} v_k X_k \) is called a martingale transform and \( \{v_k, k \geq 1\} \) is called the transforming sequence.

The martingale transform \( \{T_n, n \geq 1\} \) is a martingale if \( E[v_k X_k] < \infty \) for each \( k \geq 1 \) as

\[
E \left[ T_n \mid \mathcal{F}_{n-1} \right] = E \left[ \sum_{k=1}^{n} v_k X_k \mid \mathcal{F}_{n-1} \right] = \sum_{k=1}^{n-1} v_k X_k + E[v_n X_n \mid \mathcal{F}_{n-1}] = T_{n-1} + v_n E[X_n \mid \mathcal{F}_{n-1}] = T_{n-1}.
\]

Let \( X_k \) be gambler’s winning at trial \( k \) given that he bets one rupee at each trial. Let \( \mathcal{G}_k = \mathcal{B}(X_1, X_2, \ldots, X_k) \) for each \( k \geq 1 \) and \( \mathcal{G}_0 = \{\emptyset, \Omega\} \). Suppose that the game is fair in the sense that \( E[X_k \mid \mathcal{G}_{k-1}] = 0 \) a.s. for each \( k \geq 1 \). In order to make the gambler’s role more interesting we can allow him to vary his bet from trial to trial. Let \( v_k X_k \) be gambler’s winning at trial \( k \) given that he bets \( v_k \) rupees at trial \( k \). Suppose the gambler chooses the amount he bets at each trial on
the basis of his past luck with the game. That is the gambler’s betting system \( \{v_k, k \geq 1\} \) is a stochastic sequence or a sequence of random variables with each \( v_k \) being \( \mathcal{G}_{k-1} \)-measurable. The gambler’s accumulated winning at trial \( n \geq 1 \) is given by \( \sum_{k=1}^{n} v_k X_k \). All the above assumptions imply that \( \{T_n, n \geq 1\} \) is a martingale transform.

**Definition 1.3.4:** A martingale difference sequence \( \{X_k, \mathcal{F}_k, k \geq 1\} \) is said to be regular in the sense of Marcinkiewicz and Zygmund (regular MZ) if

\[
\infty > E \left[ \|X_k\|_{\mathcal{F}_{k-1}} \right] \geq \delta E^{1/2} \left[ X_k^2 | \mathcal{F}_{k-1} \right] \quad \text{a.s.}
\]

for all \( k \geq 1 \) and some \( \delta > 0 \).

Further, a regular MZ martingale difference sequence \( \{X_k, \mathcal{F}_k, k \geq 1\} \) is called normed regular MZ if \( E \left[ X_k^2 | \mathcal{F}_{k-1} \right] = 1 \) a.s. for each \( k \geq 1 \).

A martingale transform \( \{T_n, n \geq 1\} \) with \( X_k \) regular MZ may always be represented as a transform of a normed regular MZ martingale difference sequence without loss of generality.[cf. Stout(1974)]

We have the following convergence results relating to martingale transform (Stout(1974)).

**Theorem 1.3.5:** Let the martingale difference sequence \( \{X_k, \mathcal{F}_k, k \geq 1\} \) have a transforming sequence \( \{v_k, k \geq 1\} \). Let us assume that
sup \(E|S_n| < \infty\) and sup \(V_k < \infty\) a.s., where \(S_n = \sum_{k=1}^{n} X_k\). Then the martingale transform \(T_n = \sum_{k=1}^{n} V_k X_k\) converges almost surely.

**Theorem 1.3.6:** Let \(\{X_k, \mathcal{F}_k, k \geq 1\}\) be a regular MZ. Then there exists a martingale difference sequence \(\{Y_k, G_k, k \geq 1\}\) with transforming sequence \(\left\{E^{1/2} \left[ X_k^2 | \mathcal{F}_{k-1} \right], k \geq 1 \right\}\) such that \(\{Y_k, G_k, k \geq 1\}\) is normed regular MZ and \(\left\{E^{1/2} \left[ X_k^2 | \mathcal{F}_{k-1} \right] Y_k, k \geq 1 \right\}\) is a representation of \(\{X_k, k \geq 1\}\).

**Theorem 1.3.7:** (i) Let \(\{X_k, \mathcal{F}_k, k \geq 1\}\) be a martingale difference sequence satisfying \(|X_k| < \infty\) a.s. and \(E \left[ X_k^2 | \mathcal{F}_{k-1} \right] = 1\) a.s. for all \(k \geq 1\). Then \(\{X_k, \mathcal{F}_k, k \geq 1\}\) is a normed regular MZ.

(ii) Let \(X_k\) be independent with \(E[X_k] = 0\) and \(E[X_k^2] = 1\) for all \(k \geq 1\) and \(X_k^2\) is uniformly integrable (i.e. \(E[X_k^2 I_{\{X_k \geq N\}}] \to 0\) as \(N \to \infty\) uniformly in \(k\)). Then \(\{X_k, G_k, k \geq 1\}\) is normed regular MZ.

(iii) Let \(X_k\) be independent and symmetric with \(E[X_k^2] < \infty\) for all \(k \geq 1\). Then \(\{X_k, k \geq 1\}\) can be represented by a martingale difference sequence \(\{\omega_k Y_k, k \geq 1\}\) with \(\omega_k, \mathcal{H}_{k-1}\)-measurable for each \(k \geq 1\) and \(\{Y_k, \mathcal{H}_k, k \geq 1\}\) normed regular MZ.
**Theorem 1.3.8:** Let \( \{X_k, \mathcal{S}_k, k \geq 1\} \) be a normed regular MZ martingale difference sequence. Then

(i) \( \sum_{k=1}^{\infty} v_k X_k \) converges if and only if

(ii) \( \sup_{k=1}^{n} \sum_{k=1}^{n} v_k X_k < \infty \) if and only if

(iii) \( \sup_{k=1}^{n} \left| \sum_{k=1}^{n} v_k X_k \right| < \infty \) if and only if

(iv) \( \sum_{k=1}^{\infty} v_k^2 < \infty \) if and only if

(v) \( \sum_{k=1}^{\infty} v_k X_k^2 < \infty \).

**Remark 1.3.1:** The equivalence of conditions \( \sum_{k=1}^{\infty} v_k X_k \) converges,

\( \sum_{k=1}^{\infty} v_k^2 X_k^2 < \infty \) and \( \sum_{k=1}^{\infty} v_k^2 < \infty \) is due to Gundy(1967). Chow(1969) established the equivalence of \( \sum_{k=1}^{\infty} v_k X_k \) converging and

\( \sup_{k=1}^{\infty} \left| \sum_{k=1}^{\infty} v_k X_k \right| < \infty \). Davis(1969) improved Chow’s result by establishing the equivalence of \( \sum_{k=1}^{\infty} v_k X_k \) converging and \( \sup_{k=1}^{\infty} \sum_{k=1}^{\infty} v_k X_k < \infty \).
Definition 1.3.5: Let $\mathcal{B}_{ij} = \mathcal{B} \{X_k, i \leq k \leq j\}$ for all $1 \leq i \leq j < \infty$. 
$\{X_k, k \geq 1\}$ is said to be $\varphi$-mixing if there exists an integer $M$ and a function $\varphi$ for which $\varphi(m) \to 0$ as $m \to \infty$ and $A \in \mathcal{B}_{1n}, B \in \mathcal{B}_{m+n,\infty}$ implies

$$|P(A \cap B) - P(A)P(B)| \leq \varphi(m)P(A)$$

(1.3.1)

for all $m \geq M$ and all $n \geq 1$.

If instead of equation 1.3.1,

$$|P(A \cap B) - P(A)P(B)| \leq \varphi(m)P(A)P(B)$$

for all $m \geq M$ and all $n \geq 1$, then $\{X_k, k \geq 1\}$ is said to be strong $\varphi$-mixing and if

$$|P(A \cap B) - P(A)P(B)| \leq \varphi(m)$$

for all $m \geq M$ and all $n \geq 1$, then $\{X_k, k \geq 1\}$ is said to be weak $\varphi$-mixing.

Definition 1.3.6: Given a sequence of random variables $\{X_k, k \geq 1\}$, let

$\mathcal{G}_{ij} = \mathcal{B} \{X_k, i \leq k \leq j\}$ for all $1 \leq i \leq j < \infty$. $\{X_k, k \geq 1\}$ is said to be $^*$-mixing if there exists an integer $M$ and a function $\varphi$ for which $\varphi(m) \to 0$ as $m \to \infty$ and $A \in \mathcal{G}_n, B \in \mathcal{G}_{m+n,m+n}$ implies

$$|P(A \cap B) - P(A)P(B)| \leq \varphi(m)P(A)P(B)$$

for all $m \geq M$ and all $n \geq 1$. 
Intuitively \{X_k, k \geq 1\} is a mixing sequence if the \(X_k\)'s with indices far apart are almost independent.

**Lemma 1.3.1:** If \{X_k, k \geq 1\} is strong \(\varphi\)-mixing, then \{X_k, k \geq 1\} is weak \(\varphi\)-mixing and also \(*\)-mixing.

**Lemma 1.3.2:** Let \{X_k, k \geq 1\} be \(*\)-mixing with respect to an integer \(M\) and a function \(\varphi\) with \(E[|X_k|] < \infty\) for each \(k \geq 1\). Then

\[
|E[X_{n+m}|G] - E[X_{n+m}]| \leq \varphi(m)E|X_{n+m}| \text{ a.s.}
\]

for each sigma field \(G \subseteq \mathcal{G}_n\), each \(n \geq 1\) and \(m \geq M\).

**Remark 1.3.2:** The concept of \(*\)-mixing has been fitted conveniently to martingale framework because of the concrete bound that is possible to obtain for \(E[X_{n+m}|G]\).

All the convergence results on mixing sequences discussed in the subsequent chapters relate to \(*\)-mixing sequences.

**Theorem 1.3.9:** Let \{X_k, k \geq 1\} be a \(*\)-mixing sequence with \(E[X_k] = 0, \ E[|X_k|] < \infty\) for each \(k \geq 1\) and \(\sum_{k=1}^{\infty} \frac{E[X_k^2]}{k^2} < \infty\).

Then \(\sum_{k=1}^{n} \frac{X_k}{n} \to 0\) as \(n \to \infty\).

Proof of the above theorems and lemmas are found in Stout(1974).
D. Mixingale Difference Sequences:

**Definition 1.3.7:** The pair \( \{ (X_n, n \geq 1), \mathcal{F}_n : n = 0, \pm 1, \pm 2, \ldots \} \) is called an \( L_p \)-mixingale difference sequence if there exists a sequence of constants \( \{c_n, n \geq 1\} \) and \( \{\psi_m, m \geq 0\} \) such that \( \psi_m \to 0 \) as \( m \to \infty \) and

\[
\begin{align*}
& (a) \quad \|E[X_n|\mathcal{F}_{n-m}]\|_p \leq c_n\psi_m \\
& (b) \quad \|X_n - E[X_n|\mathcal{F}_{n+m}]\|_p \leq c_n\psi_{m+1}
\end{align*}
\]

where \( \|X\|_p = \left(\mathbb{E}|X|^p\right)^{1/p} \) for \( p > 0 \).

We have the following convergence theorem for mixingales.

**Theorem 1.3.10:** If \( \{X_n, \mathcal{F}_n\} \) is a mixingale such that \( \sum_{k=1}^{\infty} c_k^2 < \infty \),

\[
\psi_n = O\left(n^{1/2} [\log n]^{-2}\right)
\]
as \( n \to \infty \), then \( S_n = \sum_{k=1}^{n} X_k \) converges a.s. to a finite limit as \( n \to \infty \).

1.4. CONVERGENCE OF WEIGHTED SUMS OF RANDOM VARIABLES:

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, \( \{\mathcal{F}_n, n \geq 1\} \) be an increasing sequence of sub-sigma fields of \( \mathcal{F} \) and \( \{X_n, n \geq 1\} \) be a sequence of real valued random variables adapted to \( \{\mathcal{F}_n, n \geq 1\} \) i.e. each \( X_n \) is \( \mathcal{F}_n \)-measurable.
An array \( (a_{nk}) \) of random numbers is said to be a Toeplitz matrix, if for some \( M < \infty \), the following conditions are satisfied.

\[
\begin{align*}
(i) \quad & \lim_{n \to \infty} a_{nk} = 0, \quad k \geq 1 \\
(ii) \quad & \sum_{k=1}^{\infty} a_{nk} \leq M, \quad n \geq 2
\end{align*}
\]  

The following deterministic lemma on Toeplitz sequence is from Stout(1974).

**Lemma 1.4.1:** Let \( \{x_n\} \) be a sequence of real numbers and let \( \{a_{nk}\} \) be a Toeplitz sequence.

(a) (The Toeplitz Lemma): If \( x_n \to 0 \) as \( n \to \infty \), then \( \sum_{k=1}^{n} a_{nk} x_k \to 0 \) as \( n \to \infty \).

(b) If \( x_n \to x \in \mathbb{R} \) and \( \sum_{k=1}^{n} a_{nk} = 1 \), then \( \sum_{k=1}^{n} a_{nk} x_k \to x \) as \( n \to \infty \).

(c) (The Kronecker's Lemma): Let \( \sum_{k=1}^{\infty} \left( \frac{x_k}{a_k} \right) \) converges for given \( 0 < a_k \uparrow \infty \). Then \( \sum_{k=1}^{n} \left( \frac{x_k}{a_n} \right) \to 0 \) as \( n \to \infty \).

Finally if \( \{a_k\} \subset \mathbb{R}^+ \) such that \( b_n = \sum_{k=1}^{n} a_k \to \infty \) then,

\[
\frac{1}{b_n} \sum_{k=1}^{n} a_k x_k \to x \quad \text{as} \quad n \to \infty.
\]
The Toeplitz sequence

\[ a_{nk} = \begin{cases} 
1 & \text{if } k = 1, 2, \ldots, n \\
0 & \text{if } k > n
\end{cases} \]

is often used.

Jamison, Orey and Pruitt (1965) have considered independent and identically distributed random variables and weights \( \{a_k\} \subset \mathbb{R}^+ \). Since

\[ \frac{1}{b_n} \sum_{k=1}^{n} a_k x_k = \frac{b_{n-1}}{b_n} \frac{1}{b_{n-1}} \sum_{k=1}^{n-1} a_k x_k + \frac{a_n x_n}{b_n}, \]

converges in probability to a constant for non-degenerate random variables it necessitates that \( \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = 0 \). Thus assuming that

\[ \max_{1 \leq k \leq n} \left( \frac{a_k}{b_n} \right) \to 0 \text{ as } b_n \to \infty, \]

the following three results were obtained by Jamison, Orey and Pruitt (1965).

**Theorem 1.4.1:** For independent and identically distributed random variables \( \{X_k\} \), \( \lim_{t \to \infty} t P[|X_1| \geq t] = 0 \) and \( \lim_{t \to \infty} \int_{|X_1| < t} X_1 \, dP \) exists if and only if \( \frac{1}{b_n} \sum_{k=1}^{n} a_k x_k \to c \) in probability for some constant \( c \).

**Theorem 1.4.2:** Let \( \{X_k\} \) be independent and identically distributed random variables and let \( N(y) \) be the number of subscripts \( n \) such that

\[ \frac{b_n}{a_n} \leq y. \]
\[ |X_1| < \infty \text{ and } \int_{(y \geq |x|)} N(y) \frac{dy}{y^3} dF_{X_1}(x) < \infty \quad (1.4.2) \]

Then \( \frac{1}{b_n} \sum_{k=1}^{n} a_k x_k \to c \) as \( n \to \infty \) with probability one for some constant \( c \).

Condition 1.4.2 implies that \( E[N(|X_1|)] < \infty \).

**Theorem 1.4.3:** Let \( \{X_k\} \) be independent and identically distributed random variables such that \( E[|X_1| \log^+ |X_1|] < \infty \). Then

\[
\frac{1}{b_n} \sum_{k=1}^{n} a_k x_k \to c \text{ as } n \to \infty
\]

with probability one for some constant \( c \).

Pruitt(1966) has proved the following two theorems for independent and identically distributed random variables and Toeplitz sequences.

**Theorem 1.4.4:** Let \( \{X_k\} \) be independent and identically distributed random variables such that \( E[|X_1|] < \infty \) and \( (a_{nk}) \) be a Toeplitz sequence such that \( \lim_{n \to \infty} \sum_{k=1}^{n} a_{nk} = 1 \). A necessary and sufficient condition that \( Y_n = \sum_{k=1}^{n} a_{nk} X_k \to E[X_1] \) in probability is that \( \max_k |a_{nk}| \to 0 \).
A sequence \( \{t_n\} \) is said to be \( O(f(n)) \) if \( \sup_n \frac{|t_n|}{f(n)} < \infty \) and said to be \( o(f(n)) \) if \( \lim_{n \to \infty} \frac{|t_n|}{f(n)} = 0 \).

**Theorem 1.4.5:** Let \( \{X_k\} \) be independent and identically distributed random variables and \( \{a_{nk}\} \) be a Toeplitz sequence. If
\[
\max_k |a_{nk}| = O(n^{-\gamma}) \quad \text{for some} \quad \gamma > 0,
\]
then \( E\left[|X_1|\right]^{1+\frac{1}{\gamma}} < \infty \) implies that \( Y_n = \sum_{k=1}^{n} a_{nk} X_k \to E[X_1] \) as \( n \to \infty \) with probability one.

Hung and Tien (1988) have studied the convergence problem for weighted sums of martingale difference sequences. They have studied \( L_1 \)-convergence of \( S_n = \sum_{k=1}^{n} a_{nk} X_k \) to zero for \( \{a_{nk}\} \), a Toeplitz matrix. Also they have studied the almost sure convergence of \( A_n^{-1} \sum_{k=1}^{n} a_k X_k, \ n \geq 1, \) where \( \{X_n, \ n \geq 1\} \) is a martingale difference sequence, \( \{a_k\} \) is a sequence of positive real numbers and \( A_n > 0, \ A_n \uparrow \infty. \)

Again let \( \{X_n, \ n \geq 1\} \) be a sequence of random variables and \( S_n = X_1 + X_2 + \cdots + X_n. \) Pyke and Root (1968) have shown that if \( \{X_n, \ n \geq 1\} \) is an independent and identically distributed sequence of random variables and \( E\left[|X_1|^p\right] < \infty \) for some \( 0 < p < 2, \) then
\[ n^{-1}E\left[|S_n - a_n|^p\right] \to 0 \text{ as } n \to \infty, \text{ when } a_n = 0 \text{ for } 0 < p < 1 \text{ and } a_n = nE[X_1] \text{ for } 1 \leq p < 2. \]

Considering \( \{X_n, n \geq 1\} \) to be dominated in distribution by a random variable \( X \) such that \( E[|X|^p] < \infty \) and taking \( a_n = \sum_{k=1}^{n} E[X_k | X_1, X_2, \ldots, X_k] \), Chatterjee(1969) proved the above results for \( 1 \leq p < 2 \). Chow(1971) strengthened the result by replacing the domination condition by the condition of Uniform Integrability(UI) of \( \{|X_n|^p, n \geq 1\} \) for some \( 0 < p < 2 \).

Bose and Chandra(1993) proved \( L_p \) (\( 0 < p < 2 \))-convergence for some pair-wise independent and dependent sequences under the condition of Cesaro Uniform Integrability(CUI) as introduced by Chandra(1989) which is defined as follows.

**Definition 1.4.1:** A sequence of real valued random variables \( \{X_n, n \geq 1\} \) on a probability space \((\Omega, \mathcal{F}, P)\) is said to be Cesaro uniform integrable (CUI) if

\[
\lim_{a \to \infty} \limsup_{n \to \infty} \left( n^{-1} \sum_{k=1}^{n} E\left[|X_k| I_{\{|X_k| > a\}}\right]\right) = 0.
\]

**Remark 1.4.1:** In order that WLLN or SLLN holds for \( \{X_n, n \geq 1\} \) it should be possible to allow a few of the \( X_n \)'s to take large values. The CUI condition is capable (at least to a certain extent) of allowing such sequences.

Bose and Chandra(1993) have proved the following convergence results.
Theorem 1.4.6: Let \( \{X_n, n \geq 1\} \) be a sequence of independent and identically distributed random variables such that \( \{ |X_n|^p, n \geq 1\} \) is CUI for some \( 0 < p < 1 \). Then \( n^{-1}E\left[|S_n|^p\right] \to 0 \) as \( n \to \infty \), where \( S_n = \sum_{k=1}^{n} X_k \).

Theorem 1.4.7: Let \( \{X_n, n \geq 1\} \) be a martingale difference sequence such that \( \{ |X_n|^p, n \geq 1\} \) is CUI for some \( 1 \leq p < 2 \). Then \( n^{-1}E\left[|S_n|^p\right] \to 0 \) as \( n \to \infty \), where \( S_n = \sum_{k=1}^{n} X_k \).

Theorem 1.4.8: Let \( \{(X_n, n \geq 1), (\mathbb{S}_n : n = 0, \pm 1, \pm 2, \cdots)\} \) be a \( L_p \)-mixingale difference sequence and \( \{ |X_n|^p, n \geq 1\} \) be CUI for some \( 1 \leq p < 2 \). Further assume that \( \limsup_n n^{-1} \left( \sum_{k=1}^{n} c_k \right)^p < \infty \). Then \( E\left[n^{-1}|S_n|^p\right] \to 0 \) as \( n \to \infty \) where \( S_n = \sum_{k=1}^{n} X_k \).
1.5. LAW OF THE ITERATED LOGARITHM:

The law of large numbers and the central limit theorem study the fluctuation behavior of suitable normalised sums

\[ S_n = X_1 + X_2 + \cdots + X_n \]

derived from an independent sequence \( \{X_n\} \) of identically distributed random variables. This fluctuation behavior reaches its climax in various manifestations of the law of the iterated logarithm. The modern period of the law of the iterated logarithm was started by Strassen[(1964), (1965), (1966)] with his discovery of almost sure invariance principle, his deep functional law of the iterated logarithm and his converse to the law of Hartman–Wintner law of the iterated logarithm. One of the characteristics of modern period has been an emphasis on the law of the iterated logarithm for dependent random variables.

Let \( \{X_k, k \geq 1\} \) be a sequence of independent and identically distributed random variables assumed to be symmetric to avoid the question of centering constants. Then \( \{X_k, k \geq 1\} \) is said to exhibit behavior of the iterated logarithm type if

\[-\infty < \lim \inf \left( \frac{S_n}{a_n} \right) < \lim \sup \left( \frac{S_n}{a_n} \right) < \infty \text{ a.s.}\]

for some \( 0 < a_n \uparrow \infty \).

**Law of the iterated logarithm:**

Let \( X_1, X_2, \cdots \) be a sequence of random variables with mean zero. Set

\[ S_n = X_1 + X_2 + \cdots + X_n, \quad E\left[ S_n^2 \right] = B_n. \]
\[ P \left( \limsup_{n} \frac{|S_n|}{(2B_n \log \log B_n)^{1/2}} = 1 \right) = 1, \]

then we say that the sequence \( X_1, X_2, \ldots \) satisfies the law of iterated logarithm.

A.N. Kolmogorov in 1929 showed that if \( B_n \to \infty \) and

\[ |X_n| \leq m_n = o \left( \frac{B_n}{\log \log B_n} \right)^{1/2}, \]

then the law of iterated logarithm holds.

Law of the iterated logarithm also exists for the weighted sums of random variables. The first such result was due to Gal(1951) and Stackelberg(1965). Extensions of the results to independent and uniformly bounded random variables with mean zero and equal variance was due to Gaposhkin(1965). Chow and Teicher(1973), and Klesov(1986) have extended the results for general classes of weights for independent and identically distributed sequences of random variables. Their results are stated below.

**Theorem 1.5.1 (Chow and Teicher, 1973):** If \( \{X_k, k \geq 1\} \) are independent and identically distributed random variables with

\( E[X_k] = 0, E[X_k^2] = 1 \) and \( \{a_k, k \geq 1\} \) are real constants satisfying

(i) \( \frac{a_k^2}{n} \leq \frac{c}{n}, \quad n \geq 1, \)
Let \( \{X_k, k \geq 1\} \) be a sequence of independent and identically distributed random variables with a distribution function \( F(x) \) and let \( \{b_k, k \geq 1\} \) be a sequence of real numbers. Define

\[
S_n = b_1 X_1 + b_2 X_2 + \cdots + b_n X_n \quad \text{and} \quad B_n = b_1^2 + b_2^2 + \cdots + b_n^2.
\]

Suppose that \( E[X_1] = 0, \ E[X_1^2] = 1 \) and \( B_n \to \infty \) as \( n \to \infty \). Then the following theorem holds.

**Theorem 1.5.2 (Klesov, 1986)**: If there exists a \( \delta \in (0,1) \) and a numerical sequence \( \{\phi(n), n \geq 1\} \) for which \( \phi(n+1) = O(\phi(n)) \), \( \phi(n) \uparrow \infty \) as \( n \to \infty \) and

\[
\begin{align*}
(i) \quad & \sum P[|X_1| \geq \phi(n)] < \infty \quad \text{and} \\
(ii) \quad & \sum \left[ \frac{b_n^2}{X^2(B_n)} \right]^{1+\frac{3+\delta}{2}} \int_{|x|<\phi(n)} |x|^{2+\delta} dF(x) < \infty.
\end{align*}
\]

Then

\[
\sum a_k^2 \to \infty \quad \text{as} \quad n \to \infty, \quad \text{for some} \quad c \in (0, \infty).
\]
\[
\limsup_{n \to \infty} \frac{S_n}{\chi(B_n)} = 1 \text{ a.s., where } \chi(B_n) = (2B_n \log \log B_n)^{1/2}.
\]

In the present thesis we have tried to prove some convergence results for weighted sums of dependent sequences of random variables under different conditions. The types of random variables are martingale difference sequences, martingale transforms, mixing sequences, mixing transforms and mixingale difference sequences. The main technique adopted in proving the convergence results is that different dependent sequences have been fitted conveniently into the martingale framework. Chapters II, III, and IV deal with the convergence results under different conditions for different sequences. In Chapter V we have proved the law of iterated logarithm for weighted sums of a martingale difference sequences and *-mixing sequences.