CHAPTER - 5

On Random Polynomial with Stable Variate Coefficients
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ON RANDOM POLYNOMIAL WITH STABLE VARIATE COEFFICIENTS

5.1 Lower bound of level crossings of a random curve

5.1.1

Many authors have studied probable number of zeros of random polynomials with coefficients following different types of distributions. But a few of them have studied the case when random variables lie in domain of attraction of stable law. Domain of attraction is defined as follows.

Let $\xi_1(\omega), \xi_2(\omega), \ldots, \xi_n(\omega)$ be identically distributed independent random variables with common distribution function $F(x)$. If distribution function of the sums.

$$V_n = \frac{1}{B_n} \sum_{k=1}^{n} \xi_k(\omega) - A_n,$$

for suitably chosen constants $A_n$ and $B_n$ converges as $n \to \infty$ to the distribution function $V(x)$, then we say that $F(x)$ is attracted to $V(x)$. The totality of distribution functions attracted to $V(x)$ is called the Domain of attraction of the law $V(x)$. It has been shown (Gnedenko and Kolmogorov [17] p.172) that all stable laws and only these have nonempty domain of attraction. Also if a random variable lies in the domain of attraction of proper symmetric stable law its characteristic function admits the representation $\exp(-C|t|^\alpha h(t))$ where $C$ is a constant, $\alpha$ is index of stable
law and \( h(t) \) is a positive function slowly varying either in
neighbourhood of infinity or in the neighbourhood of origin as the case is
(cf Ibragimov and Linnik [19] p. 91). In connection with domain of
attraction of stable law we state and prove the following theorem:

**Theorem 5.1**

Let \( y = f(x, \omega) = \sum_{r=0}^{n} \xi_r(\omega)x^r \)

be an algebraic curve where \( (\xi_r(\omega))_{r=0}^{n} \) is a sequence of independent,
identically distributed random variables, lying in the domain of attraction
of symmetric proper stable law with index \( \alpha (0 < \alpha \leq 2) \) then the probable
number of level crossings of the random curve \( y = f(x, \omega) \) is at least

\[
\frac{C \log n}{\log \log n} \text{ except for a set of measure } \frac{C_1 \log \log n}{\log n}.
\]

**5.1.2 Introduction**

Ibragimov and Maslova [20] have studied first the average
number of real zeros of a random algebraic polynomial, when the
coefficients are random variables lying in the domain of attraction of
normal law. Then Mishra, Nayak and Pattanayak [32] studied the lower
bound of number of real zeros in the same situation and established that
the probable number of the real zeros is at least \( \frac{C_1 \log n}{\log \log n} \) outside a set of

measure utmost \( \frac{C_2}{(\log \log n)(\log n)^{1-\varepsilon}} \) for \( \varepsilon > 0 \)

Ali, Nayak and Rout [2] also studied lower bound for \( \alpha = 1 \).
Nayak and Mahanty [42] studied the upper bound of the same polynomial when the coefficients lie in the domain of attraction of stable law in the case $1 < \alpha \leq 2$, $\alpha$ being index of stable law and established strong result in the sense of Evans [14]. They also studied the lower bound when $1 < \alpha \leq 2$. But in this theorem, we extended their result to the case $0 < \alpha \leq 2$ which is most general, covering the case of all types of stable distributions. If $N_n$ denotes the probable number of real zeros we proved.

$$P\left(\frac{N_n}{\log \log n} < \frac{C \log n}{\log n}\right) < \frac{C_1 \log \log n}{\log n}$$

for $0 < \alpha \leq 2$


$$P\left(\frac{N_n}{384(\log n)^{5/4}} < \frac{768(\log \log n)^{3/2}}{\log n}\right)$$

for $\alpha = 1$, which shows that our exceptional set is far less than that of Ali, Nayak and Rout and also number of real zeros is more in our case. Also as written above, the result of Mishra, Nayak and Pattanayak shows that their number of real zeros is less and their exceptional set is greater than that of us. Thus we have obtained a better result.

### 5.1.3

For the proof of the theorem, we need the following definitions

We define a positive integer $M$ by

$$M = [(\log n)^2] + 1 \quad (5.1)$$
where \([x]\) means greatest integer \(\leq x\).

Let the integer \(k\) be defined by

\[M^{2k} \leq n < M^{2k+2}.\]  \hspace{1cm} (5.2)

From (5.2) we get

\[2k \log M \leq \log n \leq (2k + 2) \log M\]

i.e. \(\log n \geq 2k \log M\) and \(k \left(2 + \frac{2}{k}\right) \log M \geq \log n\),

or \(k \leq \frac{\log n}{2 \log M}\) and \(k \geq \frac{\log n}{\left(2 + \frac{2}{k}\right) \log M}\)

whence we get \(k \leq \frac{C_1 \log n}{\log M}\)

and

\(k \geq \frac{C_2 \log n}{\log M}\)

i.e., \(\frac{C_2 \log n}{\log M} \leq k \leq \frac{C_1 \log n}{\log M}\)  \hspace{1cm} (5.3)

for constants \(C_1\) and \(C_2\).

From (5.1) we get

\[C_3 (\log n)^2 \leq M \leq C_4 (\log n)^2\]

or

\[C_3' \log \log n \leq \log M \leq C_4' \log \log n.\]
Hence,

\[
\frac{1}{C_3 \log \log n} \geq \frac{1}{\log M} \quad \text{and} \quad \frac{1}{C_4 \log \log n} \leq \frac{1}{\log M}.
\]

Then from (5.3) we get

\[
\frac{C_1 \log n}{\log \log n} \leq k \leq \frac{C_2 \log n}{\log \log n}
\]

(5.4)

which implies that \( k \to \infty \) as \( n \to \infty \).

we define the set \( A \) by,

\[
A = \left\{ \left\lfloor \frac{k}{2} \right\rfloor + 1, \left\lfloor \frac{k}{2} \right\rfloor + 2, \ldots, k \right\}.
\]

There are \( \left\lfloor \frac{k}{2} \right\rfloor \) points in \( A \) if \( k \) is even

and \( \left\lfloor \frac{k+1}{2} \right\rfloor \) points if \( k \) is odd.

Define the set \( X \) by

\[
X = \{ x : x = x_m = (1 - M^{-2m})^{1/\alpha}, \ m \in A \}.
\]

Now, we consider the number of level crossings of the curve \( y = f(x, \omega) \) in the interval \([0, 1]\) for \( x \in X \) clearly the number of level crossings in \((-\infty, \infty)\) is equal to four times the number of level crossings in \([0, 1] \).

let \( f(x_m, \omega) = A_m(x_m, \omega) + R_m(x_m, \omega) \)
where,

\[ A_m(x_m, \omega) = \sum_{r=M^{2m-1}}^{M^{2m+1}} \xi_r(\omega)x_m^r = S_1(\omega) \quad \text{(say)} \quad (5.5) \]

and

\[ R_m(x_m, \omega) = \left( \sum_{r=0}^{n} + \sum_{r=M^{2m+1}+1}^{n} \right) \xi_r(\omega)x_m^r \\
= S_2(\omega) + S_3(\omega) \quad \text{(say)} \quad (5.6) \]

\( A_m, A_{m+1} \) are independent random variables.

since they have no common terms.

5.1.4

We need the following lemmas for the proof of the theorem.

**Lemma 5.1**

If \( L(u) \) is a positive slowly varying function in the neighbourhood of origin then for \( \rho > 0 \)

\[ \lim_{u \to 0} u^\rho L(u) = 0 \]

(i) \[ \lim_{u \to 0} u^{-\rho} L(u) = \infty \]

and

The lemma follows from Karamata's theorem (cf Ibragimov and Linnik [19] p 394) which is as follows:
A positive slowly varying function \( L(x) \) as \( x \to \infty \) which is integrable on any finite interval may be represented in the following form.

\[
L(x) = C(x) \exp \left\{ \int_a^x \frac{\epsilon(t)}{t} \, dt \right\}
\]

where

\[
\lim_{x \to \infty} C(x) = C \neq 0
\]

\[
\lim_{x \to \infty} \epsilon(x) = 0 \quad \text{and} \quad a > 0
\]

Corollary: Also similar results are obtained for any function \( L(t) \) slowly varying as \( t \to 0 \) putting \( t = \frac{1}{x} \) above whence we get

\[
L(t) = C(t) \exp \left\{ - \int_a^t \frac{\epsilon(u)}{u} \, du \right\}
\]

Lemma 5.2

If \( V_m \) denotes normalising constant defined by

\[
V_m^\alpha = \sum_{M^{2m-1} + 1}^{M^{2m+1}} x_m^{\alpha r} L \left( \frac{x_m^r \theta}{V_m} \right)
\]

for \( \theta > 0 \), then

\[
V_m > \left( \frac{b}{e M^{2m}} \right)^{\frac{1}{\alpha}} \quad \text{for} \quad b > 0
\]

where \( L(t) \) is positive function slowly varying in the neighbourhood of zero.
Proof: L(t) may be bounded or unbounded

Case I: If \( \lim_{t \to 0} L(t) = \infty \), then there exists constants \( t' \) and \( b' \) such that for \( t < t' \)

\[ L(t) > b' > 0. \]

Case II: If \( 0 < \lim L(t) < \infty \) (\( L(t) \) is positive)

then for \( t < t' \), \( L(t) > b'' > 0 \) for some constant \( b'' > 0 \).

Let \( b = \min (b', b'') \),

then in either case, \( L(t) > b > 0. \)

so

\[
V_m^\alpha = \sum_{M^{2m-1}+1}^{M^{2m+1}} x_m^{\alpha r} \left( \frac{x_m \theta}{V_m} \right) \\
> b \sum_{M^{2m-1}+1}^{M^{2m+1}} x_m^{\alpha r} \\
> b \sum_{M^{2m-1}+1}^{M^{2m}} x_m^{\alpha r}
\]

and \( 0 < x_m < 1 \) implies smallest term in the expansion of

\[
\sum_{M^{2m-1}+1}^{M^{2m}} (x_m^\alpha)^r \text{ is } (x_m^\alpha)^{M^{2m-1}+1} \text{ i.e. } \left(1 - \frac{1}{M^{2m}} \right)^{M^{2m}}.
\]
Number of terms in this expansion is $M^{2m} - M^{2m-1}$. So we have

$$V_m > b(M^{2m} - M^{2m-1})(1 - \frac{1}{M^{2m}})^{M^{2m}}$$

$$> bM^{2m}(1 - \frac{1}{M})(1 - \frac{1}{M^{2m}})^{M^{2m}}$$

$$> \frac{b}{e} M^{2m} \left( \text{as } 1 - \frac{1}{M} < 1 \text{ and } \left( 1 - \frac{1}{M^{2m}} \right)^{M^{2m}} \text{ increases to } e^{-1} \text{ with } M^{2m} \right),$$

so $V_m > \left( \frac{b}{e} M^{2m} \right)^{\frac{1}{\alpha}}$.

**Lemma 5.3**

Union of events $E_m$ and $F_m$ defined by

$$E_m = \left\{ \omega : A_{2m}(\omega) > V_{2m} \right\} \cap \left\{ \omega : A_{2m+1}(\omega) < -V_{2m+1} \right\}.$$  

$$F_m = \left\{ \omega : A_{2m}(\omega) < -V_{2m} \right\} \cap \left\{ \omega : A_{2m+1}(\omega) > V_{2m+1} \right\}.$$  

occur with positive probability.

**Proof:** Let $g_m(u)$ and $G_m(x)$ denote respectively characteristic function and distribution function of the random variables $\left( \frac{A_m(\omega)}{V_m} \right)$. Then

$$g_m(u) = \exp \left\{ -|u|^\alpha \frac{1}{V_m^\alpha} \sum_{m=1}^{M^{2m+1}} x_m^{ar} L \left( x_m^r \frac{u}{V_m} \right) \right\}.$$
Obviously $V_m \to \infty$ as $m \to \infty$ and as such

\[
x_m^\alpha \frac{u}{v_m} \to 0 \quad \text{and} \quad x_m^r \frac{\theta}{V_m} \to 0 \quad \text{as} \quad m \to \infty.
\]

We use lemma 5.1 here taking

\[
\lim_{u \to 0} C(x) = C \quad \text{and get}
\]

\[
\frac{L \left( \frac{x_m^r t}{V_m} \right)}{L \left( \frac{x_m^r \theta}{V_m} \right)} = \frac{C \left( \frac{x_m^r t}{V_m} \right) e^{-\int_{x_m^r t/V_m}^{x_m^r \theta/V_m} \frac{\epsilon(u)}{u} du}}{C \left( \frac{x_m^r \theta}{V_m} \right) e^{-\int_{x_m^r \theta/V_m}^{x_m^r t/V_m} \frac{\epsilon(u)}{u} du}}.
\]

Since

\[
\lim_{u \to 0} C(u) = C
\]

and

\[
\text{since} \quad \frac{x_m^r t}{V_m} \to 0 \quad \text{and} \quad \frac{x_m^r \theta}{V_m} \to 0 \quad \text{as} \quad m \to \infty, \text{we have}
\]

\[
C \left( \frac{x_m^r t}{V_m} \right) = (1 + o(1)) C \quad \text{as} \quad m \to \infty,
\]

and

\[
C \left( \frac{x_m^r \theta}{V_m} \right) = (1 + o(1)) C \quad \text{as} \quad m \to \infty.
\]
Also since \( \lim_{u \to 0} \epsilon(u) = 0 \),

\( \epsilon > 0 \) there exists \( t_0 > 0 \) such that for \( u < t_0 \),

\[ |\epsilon(u)| < \epsilon \]

so, \( \epsilon(u) = (1 + o(1)) \) as \( u \to 0 \).

Hence by corollary of lemma 5.1, we have

\[
\frac{L\left( \frac{x_{m,t}^{r}}{V_{m}} \right)}{C\left( \frac{x_{m,t}^{r}}{V_{m}} \right)} = \exp\left\{ -\frac{\int_{0}^{t} x_{m,t}^{r} V_{m}}{u} \right\} \epsilon(u) \]  
\[
\frac{L\left( \frac{x_{m}^{r} \theta}{V_{m}} \right)}{C\left( \frac{x_{m}^{r} \theta}{V_{m}} \right)} = \exp\left\{ -\frac{\int_{0}^{t} x_{m}^{r} \theta V_{m}}{u} \right\} \epsilon(u) \]  

\[
\frac{C\left( \frac{x_{m,t}^{r}}{V_{m}} \right)}{C\left( \frac{x_{m}^{r} \theta}{V_{m}} \right)} = \frac{(1 + o(1))}{(1 + o(1))} \exp\left\{ -\frac{\int_{0}^{t} x_{m,t}^{r} V_{m}}{u} \right\} (1 + o(1)) \]  

\[ = \epsilon \exp\left\{ -\frac{\int_{0}^{t} x_{m}^{r} \theta V_{m}}{u} \right\} (1 + o(1)) \]

\[ = (1 + o(1)) e^{\log |\theta|_{t}^{0(1)}} = |\theta|_{t}^{0(1)} \]  

as \( u \to 0 \) making \( \epsilon \to 0 \)

which implies

\[
L\left( \frac{x_{m,t}^{r}}{V_{m}} \right) = |\theta|_{t}^{0(1)} L\left( \frac{x_{m}^{r} \theta}{V_{m}} \right) \]  

as \( u \to 0 \).
Hence

\[ g_m(u) \to \exp \left\{ -|u|^{\alpha} \frac{1}{V_m^{\alpha}} \sum x_m^\alpha L \left( \frac{x_m^r \theta}{V_m} \right) \right\} \]

\[ = \exp \left\{ -|u|^{\alpha-0(1)} \theta^{0(1)} \frac{1}{V_m^{\alpha}} \sum x_m^\alpha L \left( \frac{x_m^r \theta}{V_m} \right) \right\} \]

\[ = \exp \left\{ -|u|^{\alpha-0(1)} \theta^{0(1)} \right\} \]

(by definition of \( V_m \)).

So as \( m \to \infty \), \( g_m(u) \to \exp (-|u|^\alpha) \) in any bounded interval of \( u \)-values. Since \( e^{|u|^\alpha} \) is continuous for all \( u \), \( G_m(x) \) will converge to a distribution \( F(x) \) (say) corresponding to the characteristic function \( g_m(u) \) (cf Gnedenko and Kolmogorov [17] p.83).

So,

\[
\text{sup } |G_m(x) - F(x)| = o(1). \tag{5.8}
\]

So, for \( \varepsilon > 0 \), \( |G_{2m}(-1) - F(-1)| < \varepsilon \)

\[ \Rightarrow F(-1) - \varepsilon < G_{2m}(-1) < F(-1) + \varepsilon \]

and \( |G_{2m+1}(-1) - F(-1)| < \varepsilon \)

implies \( F(-1) - \varepsilon < G_{2m+1}(-1) < F(-1) + \varepsilon \).

Then \( P(A_{2m} < -V_{2m}) = P \left( \frac{A_{2m}}{V_{2m}} < -1 \right) \)

\[ G_{2m}(-1) > F(-1) - \varepsilon \text{ as } m \to \infty \]
and similarly

\[ P(A_{2m+1} < -V_{2m+1}) > F(-1) - \varepsilon. \]

Also

\[ |G_{2m}(1) - F(1)| < \varepsilon \]

and so

\[ G_{2m}(1) < F(1) + \varepsilon \]

or,

\[ 1 - G_{2m}(1) > 1 - F(1) + \varepsilon. \]  

(5.9)

So,

\[ P(A_{2m} > V_{2m}) = 1 - P(A_{2m} < V_{2m}). \]

\[ = 1 - G_{2m}(1) \]

\[ > 1 - F(1) - \varepsilon \]  

(5.10)

similarly \( P(A_{2m+1} > V_{2m+1}) > 1 - F(1) - \varepsilon. \)  

(5.11)

Hence it follows from above that

\[ P(E_m U F_m) > 2 (F(-1) - \varepsilon)(1 - F(1) - \varepsilon) \text{ for } \varepsilon > 0. \]

Hence as \( m \to \infty, \ P(E_m U F_m) \geq 2 F(-1)(1 - F(1)) > 0 \)

Hence the proof of the lemma.
5.1.5

We define sets $H_1$, $H_2$, $H_3$, $H_4$ by

$$H_1 = \{ \omega : A_{2m}(\omega) > V_{2m} \} \cap \{ \omega : |R_{2m}(\omega)| < V_{2m} \},$$

$$H_2 = \{ \omega : A_{2m+1}(\omega) < -V_{2m+1} \} \cap \{ \omega : |R_{2m+1}(\omega)| < V_{2m+1} \},$$

$$H_3 = \{ \omega : A_{2m}(\omega) < -V_{2m} \} \cap \{ \omega : |R_{2m}(\omega)| < V_{2m} \},$$

$$H_4 = \{ \omega : A_{2m+1}(\omega) > V_{2m+1} \} \cap \{ \omega : |R_{2m+1}(\omega)| < V_{2m+1} \}.$$

Let $S'_1 = H_1 \cap H_2$ and $S'_2 = H_3 \cap H_4$. Clearly occurrence of $S'_1$ implies $f(x_{2m}) > 0$ and $f(x_{2m+1}) < 0$ and also occurrence of $S'_2$ implies $f(x_{2m}) < 0$ and $f(x_{2m+1}) > 0$ so that curve $y = f(x)$ crosses $x$-axis between $x_{2m}$ and $x_{2m+1}$.

If we show $|R_m| < V_m$ for all $m \in A$ except for a set of measure tending to zero as $m \to \infty$ then clearly $S'_1 \cap S'_2$ must occur with positive probability as we have already shown

$$P (E_m \cap F_m) > 0$$

Lemma 5.4

$$P (\omega : |R_m(\omega)| > V_m) < \frac{C}{\log \log n}$$

for all $m \in A$ except over a set $G_m$ such that

$$P (G_m) < \frac{C_1}{\log \log n},$$

$C$ and $C_1$ being constants.
Proof

If \( \phi(u) \) denote the characteristic function of \( S_3(\omega) \) then

\[
\phi(u) = \exp\left(-|u|^a \sum_{M^{2m+1}+1} x_{m}^{r} L(x_{m}^{r} u) \right)
\]

\[
= \exp (-|u|^a L_m(u)),
\]

where

\[
L_m(u) = \sum_{M^{2m+1}+1} x_{m}^{r} L(x_{m}^{r} u)
\]

Let \( T_1 = \left\{ \omega : S_2(\omega) > \frac{V_m}{2} \right\} \) and

\( T_2 = \{ \omega : S_3(\omega) > V_{m/2} \} \).

by lemma 5.1, \( \lim_{u \to 0} L(x_{m}^{r} u) = 0 \).

So we can find \( \delta > 0 \) such that

\[
\lim_{u \to 0} L(x_{m}^{r} u) < 1 \text{ for } |u| < \delta
\]

which implies

\[
L(x_{m}^{r} u) < \left( x_{m}^{r} u \right)^a \text{ when } u \to 0.
\]

Now \( |1 - \phi_m(u)| = |1 - e^{-|u|^a L_m(u)}| \)
\[
1 - \left(1 - |u|^\alpha L_m(u) + \frac{1}{2!} |u|^\alpha L_m^2(u) + \ldots\right)
\]

\[
= |u|^\alpha \left(1 - \frac{1}{2!} L_m(u) + \frac{1}{3!} L_m^2(u) + \ldots\right)
\]

\[
= |u|^\alpha L_m(u) \left(1 - \frac{1}{2!} L_m(u) + \frac{1}{3!} L_m^2(u) + \ldots\right)
\]

\[
= |u_m|^\alpha |L_m(u)(1 + o(1))| \text{ as } u \to 0
\]

\[
= |u|^\alpha \sum_{M^{2m+1}+1}^n x_m^r L(x_m u)^-\epsilon (1 + o(1)) \text{ as } u \to 0
\]

(Putting the value of \(L_m(u)\))

\[
<|u|^\alpha \sum_{M^{2m+1}+1}^n x_m^r(x_m u)^-\epsilon (1 + o(1)) \quad \text{(by 5.12)},
\]

\[
=|u|^{\alpha-\epsilon} \sum_{M^{2m+1}+1}^n x_m^{r(\alpha-\epsilon)} (1 + o(1)) \text{ as } x_m^{\alpha-\epsilon} > 0 \quad \text{(by definition of } x_m)\]

\[
\leq 2 |u|^{\alpha-\epsilon} \sum_{M^{2m+1}+1}^n x_m^{r(\alpha-\epsilon)}. \quad (5.13a)
\]

It follows from Loeve [28] that

\[
P(T_2) = P \left( \left\{ \omega : S_3(\omega) > \frac{V_m}{2} \right\} \right) < \frac{V_m}{2} \int_0^{2/V_m} (1 - \phi_m(u)) du
\]
\[ < C \frac{V_m}{2} \int_0^{2/V_m} u^{\alpha-e} \sum_{M^{2m+1}+1} x_m^{r(\alpha-e)} du \quad \text{(by 5.13a)} \]

\[ = C \frac{V_m}{2} \sum_{M^{2m+1}+1} x_m^{r(\alpha-e)} \int_0^{2/V_m} u^{\alpha-e} du \]

\[ = C \frac{V_m}{2} \left( \frac{2}{V_m} \right)^{\alpha-e+1} \sum_{M^{2m+1}+1} x_m^{r(\alpha-e)} \]

\[ = C \frac{V_m}{2(\alpha-e+1)} 2^{\alpha-e+1} \frac{1}{V_m^{\alpha-e+1}} \sum_{M^{2m+1}+1} x_m^{(\alpha-e)r}. \]

Substituting value of \( V_m \) we get

\[ P(T_2) < \frac{C'}{2^m \alpha(e)} \sum_{M^{2m+1}+1} x_m^{(\alpha-e)r}. \]

Now

\[ \sum_{M^{2m+1}+1} x_m^{(\alpha-e)r} < \sum_{M^{2m+1}} x_m^{(\alpha-e)r} \]

\[ = x_m^{(\alpha-e)M^{2m+1}} \frac{1 - x_m^{\alpha-e}}{1 - x_m} \quad (5.13b) \]

(since \( \sum x_m^{(\alpha-e)r} \) is an infinite G.P. series with common ratio < 1).

Taking \( 0 \leq \varepsilon < \frac{\alpha}{2} \) we have \( \varepsilon > -\frac{\alpha}{2} \).

so \( \alpha - \varepsilon > \frac{\alpha}{2} \)

\[ \frac{\alpha}{2} \]
\[ x_m^{\alpha-e} < x_m^{\alpha/2} \quad \text{(since } x_m < 1) \]

\[ -x_m^{\alpha-e} > -x_m^{\alpha/2} \]

\[ 1 - x_m^{\alpha-e} > 1 - x_m^{\alpha/2} \]

\[ \frac{1}{1 - x_m^{\alpha-e}} < \frac{1}{1 - x_m^{\alpha/2}} \]

\[
\frac{1 + x_m^{\alpha/2}}{(1 - x_m^{\alpha/2}) (1 + x_m^{\alpha/2})} < \frac{1 + x_m^{\alpha/2}}{1 - x_m^{\alpha/2}}
\]

\[ = M^{2m} (1 + x_m^{\alpha/2}) \quad \text{(since } x_m^{\alpha} = 1 - \frac{1}{M^{2m}} \text{ (by definition of } x_m) \]

\[ \leq 2M^{2m} \quad \text{(since } x_m < 1, \ 1 + x_m^{\alpha/2} < 2). \quad \text{(5.14)} \]

By (5.14) we get from (5.13b) that

\[ \sum_{M^{2m+1}+1}^{n} x_m^{(\alpha-e)r} < 2M^{2m} x_m^{(\alpha-e)M^{2m+1}} \]

\[ = 2M^{2m} \left(1 - \frac{1}{M^{2m}}\right)^{\frac{\alpha-e}{\alpha}} M^{2m+1} \]

\[ = 2M^{2m} \left(1 - \frac{1}{M^{2m}}\right) M^{2m} \left(\frac{\alpha-e}{\alpha}\right) \]
\[ \leq 2M^{2m} e^{-1}\left(\frac{\alpha - \varepsilon}{\alpha}\right) \left(1 - \frac{1}{M^{2m}}\right)^{M^{2m}} \text{ increase to } e^{-1} \]

\[ = 2M^{2m} e^{-N} \quad \text{(where } N = M\left(\frac{\alpha - \varepsilon}{\alpha}\right)\text{)} \]

\[ \leq 2M^{2m} N^{-1} \quad \text{(since } e^{-N} < N^{-1}\text{)} \]

\[ = 2M^{2m} M^{-1} \left(\frac{\alpha - \varepsilon}{\alpha}\right)^{-1} \]

\[ = 2M^{2m-1} \frac{\alpha}{\alpha - \varepsilon}. \quad (5.15) \]

Hence from above we have by (5.15)

\[ P(T_2) < \frac{C'}{\frac{2m}{(\alpha - \varepsilon)\alpha - \varepsilon}} \frac{2\alpha}{M^{2m-1}} \]

\[ = \left(\frac{2C'\alpha}{\alpha - \varepsilon}\right) \frac{1}{\frac{2m\varepsilon}{\alpha}} M^{2m-1} \]

\[ = \left(\frac{2C'\alpha}{\alpha - \varepsilon}\right) \frac{1}{\frac{2m\varepsilon}{\alpha}}. \]

(Take for fixed \( m, \varepsilon = \frac{\alpha}{4m} \))

\[ = \left(\frac{2C'\alpha}{\alpha - \varepsilon}\right) \frac{1}{\frac{1}{M^2}}. \]
But by (5.1), $M > C_1 (\log n)^2$

so $P(T_2) < \frac{C_2}{\log n}$

again

$$\sum_{0}^{M^{2m-1}} x_m^{(\alpha-\epsilon)r} \leq 1 + x_m^{\alpha-\epsilon} + x_m^{2(\alpha-\epsilon)} + \ldots$$

There are $(M^{2m-1} + 1)$ terms in the expansion and highest term is $\leq 1$ as $x_m \leq 1$.

So $\sum_{0}^{M^{2m-1}} x_m^{(\alpha-\epsilon)r} < (M^{2m-1} + 1) < 2M^{2m-1}$.

$\therefore$ As before

$$P(T_1) = P(\{\omega: S_2(\omega) > \frac{V_m}{2}\}) < \frac{C'}{M^{2m-2m\epsilon}} \sum_{0}^{M^{2m-1}} x_m^{(\alpha-\epsilon)r}$$

$$< \frac{C'}{2^{m-2m\epsilon} M^{2m-1}} = \frac{2C'}{M^{1-2m\epsilon}}$$

$$< \frac{C''}{(\log n)^2 \left(\frac{1-2m\epsilon}{\alpha}\right)} = \frac{C_3}{\log n} \quad \left(\text{since } \epsilon = \frac{\alpha}{4m}\right) \quad (5.16)$$

(as $M > (\log n)^2$ by 5.1).

Then we have

$$P(T_m) = P(\{\omega: R_m(\omega) > V_m\}) = P(T_1) + P(T_2)$$
\[
= P\left(\left\{ \omega : S_2(\omega) > \frac{V_m}{2} \right\}\right) + P\left(\left\{ \omega : S_3(\omega) > \frac{V_m}{2} \right\}\right)
\]

\[
\leq \frac{C_3}{\log n} + \frac{C_2}{\log n} < \frac{C_4}{\log n}
\]

(5.17)

If \(G_m\) denotes the exceptional set then using (5.4), we have

\[
P(G_m) \leq \sum_{m=\left[\frac{k}{2}\right]+1}^{k} P(T_m)
\]

\[
\leq k P(T_m) < C_4 \frac{k}{\log n} \quad \text{by (5.17)}
\]

\[
< C_1 \frac{\log n}{\log \log n} \cdot \frac{1}{\log n} \quad \text{(using 5.4)}
\]

\[
= \frac{C_1}{\log \log n}.
\]

Therefore,

\[
R_m(\omega) < V_m \quad \text{for all } m \in A \text{ except for a set of measure}
\]

\[
\leq \frac{C_1}{\log \log n}
\]

(5.18)

5.1.6 Proof of the theorem

We have from lemma 5.3

\[
P(E_m U F_m) > 0
\]

Let \(P(E_m U F_m) = \delta_m > \delta\), \(\delta\) being absolute constant.
Let \( i_m \) denote indicator function of \( E_m \cup F_m \)

Then

\[
P(\{\omega: i_m = 1\}) = \delta_m \quad \text{and} \quad P(\{\omega: i_m = 0\}) = 1 - \delta_m.
\]

Therefore, \( i_m \)'s are independent random variables with \( E(i_m) = \delta_m \) and \( V(i_m) = \delta_m - \delta_m^2 \)

It follows from Samal [48], p 439) that there are at least

\[
\frac{1}{2} \left\{ k - \left[ \frac{k}{2} - 2 \right] \right\} \text{pairs } (A_{2p}, A_{2p+1}) \quad \text{such that}
\]

\[
\left\lfloor \frac{k}{2} \right\rfloor + 1 \leq 2p \leq 2p + 1 \leq k.
\]

If \( q \) be denotes the number of such pairs then

\[
q \geq \frac{1}{2} \left( k - \left[ \frac{k}{2} \right] - 2 \right)
\]

\[
\leq \frac{1}{2} \left( \frac{k}{2} - 2 \right) = \frac{k}{4} - 1
\]

\[
= \frac{k}{4} \left( 1 - \frac{4}{k} \right) \geq \frac{k}{8}
\]

\[
\geq \frac{C \log n}{\log \log n} \quad \text{for large } k \text{ by (5.4).} \quad (5.19)
\]

Let for \( \varepsilon > 0 \),

\[
H = \{ \omega: |i_m(\omega) - E(i_m(\omega))| > q \varepsilon \}
\]
Then by Chebyshev’s inequality we have for $0 < \epsilon < \delta_m$

$$P(H) \leq \frac{\sum \delta_m}{\sqrt{\epsilon}} < \frac{q}{q^2} \frac{\epsilon}{\epsilon^2}$$

$$< \frac{q}{q^2} \frac{\epsilon}{\epsilon^2} \quad (as \ \delta_m \leq 1)$$

$$= \frac{1}{q \epsilon^2}$$

$$\leq \frac{C_1 \log \log n}{\log n} \quad (by \ 5.19).$$

If $\eta = \sum_{q} i_m$

then

$$\eta \geq E(i_m) - q \epsilon \geq q(\delta - \epsilon) \geq \frac{C \log n}{\log \log n} \quad by \ (5.19).$$

Therefore, the number of level crossings of the curve $y = f(x, \omega)$ is

at least $\frac{C \log n}{\log \log n}$ outside a set of measure less than $C_1 \frac{\log \log n}{\log n}$ for

constants $C$ and $C_1$. 

---
5.2 STRONG RESULT FOR LOWER BOUND OF LEVEL CROSSING OF A RANDOM ALGEBRAIC CURVE.

5.2.1 Theorem 5.2

Let \( f(x, \omega) = \sum_{r=0}^{n} \xi_r(\omega)x^r \)

be a random algebraic curve where coefficients \( \xi_r(\omega) \)'s are identically distributed random variables belonging to the domain of attraction of symmetric proper stable law with common characteristic function \( \exp(-c|t|^\alpha h(t)) \), \( c > 0 \), \( h(t) \) being positive function slowly varying in the neighbourhood of origin where \( 0 < \alpha \leq 2 \). If \( N_n \) be the number of level crossings of the curve \( f(x, \omega) \) then for \( n_0 \in N \),

\[
N_n > \mu \left( \varepsilon_n \log n \right) \text{ outside an exceptional set } G \text{ with }
\]

\[
P(G) \leq \frac{\mu_1}{\left( \varepsilon_{n_0} \log n_0 \right)^{\frac{\delta}{\alpha}}} \]

where \( (\varepsilon_n) \) is a sequence such that as \( n \to \infty, \varepsilon_n \log n \to \infty \) but \( \varepsilon_n^2 \log n \to 0 \) and \( 0 < \delta < \frac{\alpha}{2} \).

5.2.2 Introduction

In the previous chapter, we have studied the lower bound of the number of level crossings of the random algebraic curve \( f(x, \omega) = \sum_{r=0}^{n} \xi_r(\omega)x^r \) and gave a weak result. In this section, we establish a strong result in the sense
of Evans when the random variables lie in the domain of attraction of proper stable law with index $\alpha (0 < \alpha \leq 2)$. The result is strong because we showed that

$$P\left\{ \sup_{n} \frac{N_n}{\log n} \rightarrow \mu \right\} \rightarrow 1 \text{ as } n \rightarrow \infty$$

and here the exceptional set is independent of degree of the polynomial.

Evans [14] first established strong result for the bounds of the numbers of roots when the coefficients of the polynomial are normal random variables. Samal and Mishra [50] studied the case when the random coefficients follow stable distribution. But Nayak and Das [41] first established strong result when the coefficients are random variables lying in the domain of attraction of normal law. Therefore, our work generalizes the works of our predecessors.

### 5.2.3 Proof of the theorem

The random variables $\xi_k$ as defined above have characteristics function $\phi (x)$ which is real and admits the representation.

$$\phi(t) = \exp (-c |t|^\alpha h(t))$$

(5.20)

where $c > 0$, $h(t)$ is slowly varying in the neighbourhood of origin.

Let $(e_n)$ be a sequence such that $e_n$ and $\frac{\log n}{e_n}$ tend to $\infty$ as $n \rightarrow \infty$. 

We define

\[ \Lambda_n = m^{2/\alpha} e_n \]

and \((M_n)\) is a sequence defined by

\[ M_n = \left[ b e_{n}^{\alpha} \right] + 1 \quad (5.21) \]

where \([x]\) means greatest integer \(\leq x\) and choice of \(b > 0\) will be made later.

Let \(p(x) = x^x\)

Let \(k \in \mathbb{N}\) be defined by

\[ p(4k + 5) M_n^{4k+5} \leq n \leq p(4k + 7) M_n^{4k+7} \quad (5.22) \]

\[ \Rightarrow (4k + 5)^{4k+5} M_n^{4k+5} \leq n \leq (4k + 7)^{4k+7} M_n^{4k+7} \]

Taking logarithm we get

\[ (4k + 5) \log (4k + 5) + (4k + 5) \log M_n \leq \log n \]

\[ < (4k + 7) \log (4k + 7) + (4k + 7) \log M_n \]

\[ \Rightarrow (4k + 5) \log M_n \leq \log n \]

\[ \Rightarrow k \left( 4 + \frac{5}{k} \right) \log M_n \leq \log n \]

\[ \Rightarrow k \leq \frac{\mu_1 \log n}{\log M_n} \]
Again since \( \log (4k + 7) \leq 4k + 7 \) we have

\[
(4k+7)^2 + (4k + 7) \log M_n \geq \log n
\]

\[
\Rightarrow k^2 \left( \frac{4 + 7}{\mu} \right)^2 + \left( \frac{4 + 7}{k^2} \right) \log M_n \geq \log n
\]

or \( k \geq \mu_2 \sqrt{\frac{\log n}{\log M_n}} \)

hence \( \mu_2 \sqrt{\frac{\log n}{\log M_n}} \leq k \leq \mu_1 \frac{\log n}{\log M_n} \) (5.23)

But by (5.21) we get

\[
\mu_3 \epsilon_n^\alpha \leq M_n \leq \mu_4 \epsilon_n^\alpha
\]

\[
\Rightarrow \mu_5 \log \epsilon_n \leq \log M_n \leq \mu_6 \log \epsilon_n
\]

So from (5.23) we get

\[
\mu_7 \left( \frac{\log n}{\log \epsilon_n} \right)^{1/2} \leq k \leq \mu_8 \frac{\log n}{\log \epsilon_n}
\]

setting \( \epsilon_n = \exp \left( \frac{\mu}{\epsilon_n^2 \log n} \right) \) we get

\[
\mu' (\epsilon_n \log n) \leq k \leq \mu'' (\epsilon_n \log n)^2
\] (5.24)

The number of roots in \( (-\infty, \infty) \) is 4 times the number of roots in \([0, 1]\) so considered points \( x'_m \)'s in \([0,1]\) defined by
\[ x_m = \left(1 - \frac{1}{p(2m + 1)M_n^{2n}}\right)^{1/\alpha} \quad (5.25) \]

for \( m = \lceil k/2 \rceil + 1, \lceil k/2 \rceil + 2, \ldots, k \)

Let \( f(x, \omega) = A_m(\omega) + R_m(\omega) \), where

\[
A_m(\omega) = \sum_{r} \xi_r(\omega)x_m^r
\]

\[
R_m(\omega) = \left(\sum_{2} + \sum_{3}\right) \xi_r(\omega)x_m^r \quad (5.26)
\]

where index \( r \) ranges from \( p(2m - 1) M_n^{2m-1} + 1 \) to

\[
p(2m + 3) M_n^{2m+1} \text{ in } \sum_1 \text{, from } 0 \text{ to } p(2m - 1) M_n^{2m-1} \text{ in } \sum_3 \text{ and from }
\]

\[
p(2m + 3) M_n^{2m+1} + 1 \text{ to } n \text{ in } \sum_3
\]

So we have

\[
f(x_{2n}, \omega) = A_{2m}(\omega) + R_{2m}(\omega)
\]

\[
F(x_{2n}, \omega) = A_{2m+1}(\omega) + R_{m+1}(\omega) \quad (5.27)
\]

And \((2k + 1) \leq 2\mu'' (\varepsilon_n \log n)^2 + 1 \) by (5.24)

\[
\leq (\varepsilon_n \log n)^2 \left(2\mu'' + \frac{1}{(\varepsilon_n \log n)^2}\right)
\]
\[ \leq \mu \left( \epsilon_n \log n \right)^2 \text{ as } \epsilon_n \log n \to \infty, \quad \frac{1}{(\epsilon_n \log n)^2} \to 0. \]

\[ A_m(\omega) = \sum_{p(2m+3)M_n^{2m+3}} \xi_{m+1}(\omega)x_m^r \]

Therefore, \[ A_{2m+1}(\omega) = \sum_{p(4m+5)M_n^{4m+5}} \xi_{m+1}(\omega)x_m^r \]

Hence maximum index in \[ A_{2m+1}(\omega) \text{ for } m = k \text{ is } p(4m+5)M_n^{4m+5} \] which by (5.23) is consistent with (5.27).

5.2.4

The following lemmas are necessary for the proof of the theorem:

Lemma 5.5

If we define normalizing constant \( V_m \) by

\[ V_m^\alpha = \sum_1 x_m^{\alpha r} \left( \frac{x_m^r \theta}{V_m} \right) \quad (5.28) \]

then \( V_m^\alpha > \mu_1 p(m+1)M_n^{2m} \quad (5.29) \)

where \( \theta \) is a small positive number

\textbf{Proof} : The slowly varying function \( h(t) \) may be bounded or unbounded.
Case I: If \( \lim_{t \to 0} h(t) = \infty \), then there exists constants \( c, d \) such that for \( t < c \)

\[ h(t) > d > 0 \]

Case II: If \( 0 < \lim_{t \to 0} h(t) < \infty \) (since \( h(t) \) positive function),

then

for \( t < c \)

\[ h(t) > d' > 0 \] for some constant \( d' > 0 \).

If \( d_1 = \min (d, d') \) then in either case

\[ h(t) > d_1 > 0. \]

Therefore, \( V_m^\alpha = \sum_i x_m^{\alpha r_i} \left( x_m^r \frac{\theta}{V_m} \right) \)

\[ > d_1 \sum_i x_m^{\alpha r_i} \]

\[ > d_1 \left\{ p(2m+1)M_n^{2m} - p(2m-1)M_n^{2m-1} \right\} x_m^{p(2m+1)M_n^{2m}} \]

\[ = d_1 p (2m + 1) M_n^{2m} \left( 1 - \frac{p(2m-1)}{p(2m+1)M_n} \right) \left( 1 - \frac{1}{p(2m+1)M_n^{2m}} \right) \]

\[ > d_1 p (2m + 1) M_n^{2m} \left( \frac{d''e}{a} \right)^{-1} \]

(where \( 0 < d'' < 1, \ a > 1 \))
= p(2m + 1) M_n^{2m} \left( \frac{ad_1}{d^e} \right)

Hence the proof

Lemma 2

\[
P\left( \left| \sum_{r=2}^{R} r \frac{V_{m}}{\Lambda_{m} V_{m}} \right| > \Lambda_{m} \bar{V}_{m} \right) < \frac{\mu}{\Lambda_{m}^{\alpha-\delta}} \tag{5.30}
\]

For \( \delta > 0 \) where

\[
\bar{V}_{m} = \left\{ \sum_{r=2}^{R} x_{m}^{\alpha r} h\left( \frac{x_{m}^{r} \Theta}{V_{m}} \right) \right\}^{1/\alpha} \tag{5.31}
\]

Where \( \Theta \) has some meaning as \( \theta \).

Proof: Let \( \phi_{m}(t) \) be the characteristic function of

\[
\sum_{r=2}^{R} \xi_{r} (\omega) \left( \frac{x_{m}^{r}}{V_{m}} \right)
\]

Then \( \phi_{m}(t) = \exp (-c |t|^{\alpha} h_{m}(t)) \),

where \( h_{m}(t) = \frac{1}{V_{m}^{\alpha}} \sum_{r=2}^{R} x_{m}^{\alpha r} h\left( \frac{x_{m}^{r} t}{V_{m}} \right) \).

Now \( (1 - \phi_{m}(t)) = 1 - \exp (-c |t|^{\alpha} h_{m}(t)) \)

\[
= 1 - \left\{ 1 - c |t|^{\alpha} h_{m}(t) + \frac{1}{2} (c |t|^{\alpha} h_{m}(t))^{2} + \ldots \ldots \right\}
\]

\[
= c |t|^{\alpha} h_{m}(t) (1 + o(1)) \text{ as } t \to 0
\]
\[ \leq c|t|^{\alpha} \frac{t^{-\delta}}{|\theta|} \]

(as derived in lemma 5.3)

Hence \(1 - \phi_m(t) \leq \mu_1|t|^{\alpha-\delta}\)

It follows from Loeve ([28] P 196) that

\[
P \left( \left| \sum_{2} \xi_r(\omega)x_r^m \right| > \Lambda_m \overline{V}_m \right)
\]

\[< \mu' \Lambda_m \int_0^{1/\Lambda_m} \left(1 - \phi_{mt}\right) dt \quad \text{ (} \phi_m(t) \text{ being real)}
\]

\[< \mu' \Lambda_m \int_0^{1/\Lambda_m} |t|^{\alpha-\delta} dt
\]

\[\leq \frac{\mu}{\Lambda_m^{\alpha-\delta}}
\]

**Lemma 5.7**

\[
P \left( \left| \sum_{2} \xi_r(\omega)x_r^m \right| > \Lambda_m \overline{V}_m \right) < \frac{\mu}{\Lambda_m^{\alpha-\delta}}.
\]

For \(\delta > 0\), where,

\[
\overline{V}_m = \left\{ \sum_{3} x_m^{\alpha_r} \left( x_m^{r} \frac{\theta}{\overline{V}_m} \right) \right\}^{1/2}
\]

where \(\overline{\theta}\) has some meaning as \(\theta\)

**Prof:** Proceeding exactly as in lemma 5.6, we can prove lemma 5.7.
Lemma 5.8

\[ P(|R_m| > V_m) < \frac{\mu}{\Lambda_{m}^{\alpha-\delta}} \]

for \( \delta > 0 \) and \( m \in \left[ \left\lfloor \frac{k}{2} \right\rfloor + 1, \left\lfloor \frac{k}{2} \right\rfloor + 2, \ldots, k \right] \).

**Proof:** Let \( \max_{0 \leq r \leq n} h \left( \frac{x_m^r \bar{\theta}}{V_m} \right), h \left( \frac{x_m^r \bar{\theta}}{\overline{V}_m} \right) \leq d_2 \)

for \( d_2 > 0 \).

By lemma 5.6 and lemma 5.7,

\[ |R_m| < \Lambda_m (\overline{V}_m + \overline{V}_m) \]

\[ |R_m| \Rightarrow |R_m| < \Lambda_m \left\{ \sum_2 x_m^r h \left( \frac{x_m^r \bar{\theta}}{V_m} \right) \right\} + \sum_3 x_m^r h \left( \frac{x_m^r \bar{\theta}}{\overline{V}_m} \right) \]

\[ < \Lambda_m d_2^{1/\alpha} \left\{ \left( \sum_2 x_m^r \right)^{1/\alpha} + \left( \sum_3 x_m^r \right)^{1/\alpha} \right\} \]

(5.32)

Again \( \frac{p(2m + 1)}{p(2m + 1)} = \frac{p(2m + 1)^{2m+1}}{p(2m + 1)^{2m-1}} = \frac{(2m + 1)^{2m-1}}{2m - 1} \cdot \frac{(2m + 1)^2}{(2m + 1)^2} \)

\[ > m^2 \]

\[ \Rightarrow p(2m + 1) > m^2 p(2m - 1) \]

(5.33)

Similarly it can be proved that

\[ p (2m + 3) > m^2 p (2m + 1) \]
Now \( \sum_{2} x_{m}^{\alpha r} = \sum_{0}^{p(2m-1)} M_{n}^{2m-1} x_{m}^{\alpha r} \)

\( \leq 1 + x_{n}^{\alpha} + x_{m}^{2\alpha} + x_{m}^{3\alpha} \ldots + x_{m}^{\alpha p(2m-1)} M_{n}^{2m-1} \)

\( \leq 1 + 1.p(2m - 1)M_{n}^{(2m-1)} \) (since \( x_{m} < 1 \) and number of terms in the series \( x_{m}^{\alpha} + x_{m}^{2\alpha} + \ldots + x_{m}^{\alpha p(2m-1)} \) is \( p(2m - 1) M_{n}^{2m-1} \))

\( \leq 1 + p(2m - 1)M_{n}^{2m-1} \) (since \( x_{m} \leq 1 \))

\( \leq 2p(2m - 1)M_{n}^{2m-1} \)

\( \leq \frac{2}{m^{2}} p(2m + 1)M_{n}^{2m-1} \) by (5.33). \( (5.34) \)

Again \( \sum_{3} x_{m}^{\alpha r} \leq \sum_{m^{2}p(2m+1)M_{n}^{2m-1}}^{\infty} x_{m}^{\alpha r} \)

\( \leq \frac{x_{m}^{m^{2}p(2m+1)M_{n}^{2m+1}}}{1 - x_{m}^{\alpha}} = \frac{1}{1 - \frac{1}{p(2m + 1)M_{n}^{2m}}^{\frac{1}{p(2m + 1)M_{n}^{2m+1}}}} \)

\( \left( \text{since} \left( x_{m} = 1 - \frac{1}{p(2m + 1)M_{n}^{2\alpha}} \right)^{1/\alpha} \right) \)

\( = p(2m + 1) M_{n}^{2m} \left( 1 - \frac{1}{p(2m + 1)M_{n}^{2m}} \right)^{m^{2}p(2m+1)M_{n}^{2m}M_{n}} \)
<p>(2m + 1) M_n^{2m} e^{-m^2 M_n} \quad < p (2m + 1) M_n^{2m} (m^2 M_n)^{-1} \quad \text{as } e^x < x^{-1} \quad \Rightarrow \quad e_n < b^{1/\alpha} M_n^{1/\alpha} \quad \text{i.e.} \quad M_n^{1/\alpha} > \frac{e_n}{b^{1/\alpha}} \quad \Rightarrow \quad \frac{e_n}{M_n^{1/\alpha}} < b^{1/\alpha} \Rightarrow \frac{e_n}{M_n^{1/\alpha}} < b^{1/\alpha} . \quad (5.36)

Hence by (5.36) we get

\[ |R_m| < V_m, \]
where we make choice of \( b = \left( \frac{d_2 \text{ad}_1}{d^e} \right)^{1/\alpha} \).

Hence \( |R_m| < V_m \) for \( m = \left[ \frac{k}{2} \right] + 1, \left[ \frac{k}{2} \right] + 2, \ldots, k \).

outside a set of measure at most \( \frac{\mu}{\Lambda_n^{\alpha-\delta}} \).

**Lemma 5.9**

Define events \( E_m \) and \( F_m \) by

\[
E_m = \{ \omega : A_{2m}(\omega) > V_{2m}, A_{2m+1}(\omega) < -V_{2m+1} \} \\
F_m = \{ \omega : A_m(\omega) < -V_{2m}, A_{2m+1}(\omega) > V_{2m+1} \}
\]

Then \( P(E_m \cup F_m) > 0 \)

Let \( G_m(x) \) and \( g_m(t) \) denote respectively the distribution function and characteristic function of \( \frac{A_m(\omega)}{V_m} \).

When \( m \to \infty \), \( V_m \to \infty \) and so \( \frac{x_m^r}{V_m} \to 0 \)

\[
g_m(t) = \exp \left\{ -c \left| t \right|^\alpha \frac{1}{V_m^\alpha} \sum_x x_m^\alpha h \left( \frac{x_m^r t}{V_m} \right) \right\} \\
= \exp \left\{ -C \left| t \right|^\alpha \frac{1}{V_m^\alpha} \sum_x x_m^\alpha h \left( \frac{x_m^r \theta}{V_m} \right)^{1+O(1)} \right\}
\]
(by consequence of Karamata's theorem)

\[ = \exp \{-c |t|^{\alpha} \theta^{0(1)} (1 + o(1)) \} \text{ (by definition of } V_m) \]

Therefore as \( m \to \infty \), \( g_m(t) \to \exp (-c |t|^\alpha) \) in any bounded interval of \( t \) values. Let \( F(x) \) be distribution function corresponding to the characteristic function \( \exp (-c |t|^\alpha) \). Since \( \exp (-c |t|^\alpha) \) is continuous for all \( t \) and is characteristics function of a proper stable distribution \( G_m(x) \) converges to \( F(x) \) weakly. Again since all proper stable laws are continuous at every point

\[ G_m(x) \to F(x) \text{ for all } x \]

Hence \( \sup \{ G_m(x) - F(x) \} = o(1) \) \hspace{1cm} (5.37)

Since \( E_m \cap F_m \) is empty, \( E_m \) and \( F_m \) are mutually exclusive

\[ P(E_m \cup F_m) \]

\[ = P\left( \{ \omega : A_{2m}(\omega) > V_{2m} , A_{2m+1}(\omega) < -V_{2m+1} \} \cup \{ A_{2m}(\omega) < -V_{2m} , A_{2m+1}(\omega) > V_{2m+1} \} \right) \]

\[ = P\left( \{ \omega : A_{2m}(\omega) > V_{2m} , A_{2m+1}(\omega) < -V_{2m+1} \} \right) \]

\[ + P\left( \{ \omega : A_{2m}(\omega) < -V_{2m} , A_{2m+1}(\omega) > V_{2m+1} \} \right) \]

But \( P\left( \{ \omega : A_{2m}(\omega) > V_{2m} \} \right) = P\left( \frac{A_{2m}}{V_{2m}} > 1 \right) \)

\[ = 1 - P\left( \frac{A_{2m}}{V_{2m}} \leq 1 \right) \]
\[ P = P\left( \omega : A_{2m} (\omega) < -V_{2m} \right) = P\left( \frac{A_{2m}}{V_{2m}} < -1 \right) = G_{2m}(-1) \]

Also \( |G_{2m}(l) - F(l)| < \varepsilon_1 \) for \( \varepsilon_1 > 0 \) by (5.38)

\[ \Rightarrow F(l) - \varepsilon_1 < G_{2m}(l) < F(l) + \varepsilon_1 \]

\[ \Rightarrow -G_{2m}(l) < F(l) - \varepsilon_1 \]

\[ \Rightarrow 1 - G_{2m}(l) > 1 - F(l) - \varepsilon_1 \]

Similarly \( P\left( A_{2m+1} > V_{2m+1} \right) > 1 - F(1) - \varepsilon_1 \)

\[ P\left( \omega : A_{2m} (\omega) < -V_{2m} \right) = G_{2m} \left( l \right) > F(-1) - \varepsilon_1 \]

\[ P\left( \omega : A_{2m+1} (\omega) < -V_{2m+1} \right) > F(-1) - \varepsilon_1 \]

Thus \( P(E_m UF_m) > 2 \left( F(-1) - \varepsilon_1 \right) \left( 1 - F(1) - \varepsilon_1 \right) \)

Therefore, as \( m \to \infty \)

\[ P\left( F_m UF_m \right) \to 2 \left( F(-1)(1-F(1)) > 0 \right) \]

let \( P\left( E_m U F_m \right) = \tau_m \).

Then we can find absolute constant \( \tau > 0 \) such that

\[ P\left( E_m U F_m \right) = \tau_m > \tau > 0 \]
Lemma 5.10.

Let \( \eta_m(\omega) \), \( \rho_m(\omega) \), \( \xi_m(\omega) \) be random variable defined by

\[
\eta_m(\omega) = \begin{cases} 
1 & \text{if } \omega \in E_m \cup F_m \\
0 & \text{if } \omega \in (E_m \cup F_m)^c
\end{cases}
\]

\[
\rho_m(\omega) = \begin{cases} 
0 & \text{if } |R_{2m}(\omega)| < V_{2m} \text{ and } |R_{2m+1}(\omega)| < V_{2m+1} \\
1 & \text{otherwise}
\end{cases}
\]

and let \( \xi_m(\omega) = \eta_m(\omega) - \rho_m(\omega) \eta_m(\omega) \)

Then number of level crossings in the interval

\((x_{2m_0}, x_{2F+1})\) must exceed \( \sum_{m=m_0}^{k} \xi_m \), where

\[ m_0 = \left\lfloor \frac{k}{2} \right\rfloor + 1 \]

Proof

Since \( P(E_m \cup F_m) = \tau_m \) and \( \eta_m \) s are independent random variables, we have

\[
E(\eta_m) = \tau_m \\
V(\eta_m) = E(\eta_m^2) - (E(\eta_m))^2 = (\tau_m - \tau_m^2)
\]

Again \( \xi_m(\omega) = 1 \) if \( \eta_m(\omega) = 1 \), \( \rho_m(\omega) = 0 \) which implies that there is a root of the polynomial \( \sum \xi_r(\omega)x^r \) i.e., the random curve has a level crossing in the interval \((x_{2m}, x_{2m+1})\), since \( \eta_r(\omega) = 1 \) and \( \rho_m(\omega) = 0 \) implies the occurrence of one of the events,
Clearly (i) implies $f(x_{2m}) > 0$, $f(x_{2m + 1}) < 0$

and (ii) implies $f(x_{2n}) < 0$, $f(x_{2m + 1}) > 0$

Then in both the cases, there is a root of the polynomial in the interval $(x_{2m}, x_{2m + 1})$. Hence the number of the roots in the interval $(x_{2m_0}, x_{2m+1})$

where $m_0 = \left[ \frac{k}{2} \right] + 1$ must exceed $\sum_{m=m_0}^{k} \xi_n$

Hence the proof.

5.2.5

The strong law of large numbers is necessary in the following form

**Lemma 5.11**

Let $\eta_1, \eta_2, \ldots$ be a sequence of random variables identically distributed with $V(\eta_i) < 1$ for all $i$, Then for $\epsilon > 0$

$$P\left\{ \sup_{k > k_0} \left| \frac{1}{k} \sum_{i=1}^{k} (\eta_i - E(\eta_1)) \right| \geq \epsilon \right\}$$

where $B$ is a positive constant

The lemma is a consequence of Hajek – Renyi [47] inequality.
Lemma 5.12

\[ \sup_{(k-m_0+1) \geq k} \left| \frac{1}{(k-m_0+1)} \sum_{m=m_0}^{k} (\xi_m - E(\eta_m)) \right| < \varepsilon \]

**Proof:** Since \( \xi_r(\omega) = \eta_m(\omega) - \rho_m(\omega)\eta_m(\omega) \) (defined in lemma 5.10)

We have

\[ \sum_{m=0}^{k} \left( \xi_m - E(\eta_m) \right) \]

\[ = \sum_{m=0}^{k} \left( \eta_m(\omega) - \rho_m(\omega)\eta_m(\omega) \right) - E(\eta_m) \]

\[ = \sum_{m=m_0}^{k} \left( \eta_m(\omega) - E(\eta_m(\omega)) \right) - \sum_{m=m_0}^{k} \eta_m(\omega)\rho_m(\omega) \]

Hence

\[ \sum_{m=m_0}^{k} \left( \xi_m - E(\eta_m) \right) \]

\[ \leq \left| \sum_{m=m_0}^{k} \left( \eta_m - E(\eta_m) \right) + \sum_{m=m_0}^{k} \eta_m\rho_m \right| \]

\[ \leq \left| \sum_{m=m_0}^{k} \left( \eta_m - E(\eta_m) \right) \right| + \left| \sum_{m=m_0}^{k} \rho_m \right| . \quad \text{(since } \eta_m \leq 1) \]

Again \( E(\rho_m) = 1 \times P(\rho_m = 1) \)

\[ \leq P\left( \left| R_m(\omega) \right| \geq V_m \right) \]

\[ \leq \frac{\mu}{\Lambda_m^{\alpha-\delta}} \quad \text{(by lemma 4).} \]
Therefore,

\[
P \left( \sum_{m=m_0}^k \rho_m \geq (k - m_0 + 1) \varepsilon_1 \right)
\]

\[
\sum_{m=m_0}^k E(\rho_m) < \frac{1}{\varepsilon_1 (k - m_0 + 1)} \sum_{m=m_0}^k \frac{\mu}{\Lambda^\alpha - \sigma}
\]

(by Bienagme - Chebyshev inequality)

\[
< \frac{1}{\varepsilon_1 (k - m_0 + 1)} \sum_{m=m_0}^k \frac{\mu}{\Lambda^\alpha - \sigma}
\]

\[
< \frac{\mu}{\varepsilon_1 \Lambda^\alpha - \sigma}
\]

\[
\sum_{m=m_0}^k \rho_m < (k - m_0 + \varepsilon)
\]

outside a set of measure at most \(\frac{\mu}{\Lambda^\alpha - \sigma} \).

So outside, an exceptional set of measure at most

\[
\sum_{(k=m_0+1)\in k_0} \frac{\mu}{\Lambda^\alpha - \delta}
\]

we have 

\[
\text{Sup} \frac{1}{(k-m_0+1) k - m_0 + 1} \left\{ \left| \sum_{m=m_0}^k (\xi_m - E(\eta_m)) \right| \right\}
\]
Now by use of strong law of large numbers, we have

\[
P\left( \sup_{k \geq k_0} \frac{1}{k-m_0+1} \left| \sum_{m=m_0}^{k} (\eta_m - E(\eta_m)) \right| < \varepsilon \right) 
\]

\[
\leq P\left( \sup_{k \geq k_0} \frac{1}{k-m_0+1} \left| \sum_{m=m_0}^{k} (\eta_m - E(\eta_m)) \right| > \varepsilon \right) 
\]

\[
\leq P\left( \sup_{k \geq k_0} \frac{1}{k-m_0+1} \left| \sum_{m=m_0}^{k} (\eta_m - E(\eta_m)) \right| > (\varepsilon - \varepsilon_1) \right) 
\]

\[
\leq \frac{B}{(\varepsilon - \varepsilon_1)^\alpha} = \frac{\mu}{k_0}. 
\]

Therefore, outside a set \( G \), where

\[
P(G) \leq \frac{\mu}{k_0} + \sum_{k \geq k_0} \frac{\mu}{\Lambda_{m_0}^{\alpha-\sigma}},
\]

we have

\[
\sup_{k \geq k_0} \frac{1}{k-m_0+1} \left| \sum_{m=m_0}^{k} (\xi_m - E(\eta_k)) \right| > \varepsilon
\]

Hence the proof.

5.2.6 Now we proceed to estimate \( N_n \) outside the set \( G \),

Now we have

\[
\frac{1}{(k-m_0+1)} \sum_{m=m_0}^{k} \xi_m > \frac{1}{k-m_0+1} \sum_{m=m_0}^{k} (E(\eta_m) - \varepsilon)
\]

for all \( k \) such that \( k-m_0+1 > k_0 \)

We know that \( E(\eta_m) > \tau_m > \tau \) for some absolute constant \( \tau \).
Therefore, \( N_n > \sum_{m=m_0}^{k} \xi_m \)

\[
= \sum_{m=m_0}^{k} \eta_m - \sum_{m=m_0}^{k} \eta_m \rho_m
\]

\[
> \sum_{m=m_0}^{k} \tau_m -(k-m_0+1) \in
\]

\[
>(k-m_0+1) \tau - (k-m_0+1) \in
\]

\[
=(k-m_0+1)(\tau - \epsilon)
\]

\[
= \mu'(k-m_0+1)
\]

\[
= \mu'(k-[\frac{k}{2}])
\]

\[
> \mu'k
\]

\[
> \mu (\epsilon_n \log n)
\]

for all \( k \) such that \((k - m_0 + 1) \geq k_0\) considering \( m_0 = \left[\frac{k}{2}\right] + 1\). We state that when \( k \) is even the statements \((k - m_0 + 1) \geq k_0\) and \( k \geq 2k_0\) are equivalent. And if \( k \) is odd the statements \( k - m_0 + 1 \geq k_0\) and \( k \geq 2k_0 - 1\) are equivalent.

Therefore, \( N_n > \mu (\epsilon_n \log n) \) for \( k \geq 2k_0\). If \( n = n_0\) corresponding to \( k = 2k_0\), all \( \eta > \eta_0\) will correspond to \( k > 2k_0\). Thus, \( N_n > \mu (\epsilon_n \log n) \) for all \( \eta > \eta_0\) except for a set \( G \), where

\[
P_\epsilon (G) < \frac{\mu}{k_0} - \mu \sum_{k \geq (2k_0 - 1)} \frac{1}{\Lambda_{m}^{\alpha - \delta}}.
\]
Now consider $m_0 = \left\lfloor \frac{k}{2} \right\rfloor + 1$, where $k$ takes values $2k_0 - 1$, $2k_0$, $2k_0 + 1$.

(i) when $k = 2k_0 - 1$, we have

\[ \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k_0 - 1}{2} \right\rfloor = (k_0 - 1) \]

\[ \Rightarrow \left\lfloor \frac{k}{2} \right\rfloor + 1 = k_0 \]

\[ \Rightarrow m_0 = k_0. \]

(ii) When $k = 2k_0$

\[ \left\lfloor \frac{k}{2} \right\rfloor = k_0 \]

\[ \left\lfloor \frac{k}{2} \right\rfloor + 1 = k_0 + 1 \]

or $m_0 = k_0 + 1$.

(iii) When $k = 2k_0 + 1$

\[ \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k_0 + 1}{2} \right\rfloor = k_0 \]

\[ \Rightarrow \left\lfloor \frac{k}{2} \right\rfloor + 1 = k_0 + 1 \]

\[ \Rightarrow m_0 = k_0 + 1. \]

Thus we find that

\[ m_0 = k + 1 \text{ when } k = 2k_0 \text{ or } 2k_0 + 1. \]

(iv) Similarly when $k = 2k_0 + 2$, we get

\[ \left\lfloor \frac{k}{2} \right\rfloor = \lfloor k_0 + 1 \rfloor = k_0 + 1 \]
or \( \left\lceil \frac{k}{2} \right\rceil + 1 = k_0 + 2 \)

or \( m_0 = k_0 + 2. \)

(v) When \( k = 2k_0 + 3 \)

\[
\left\lceil \frac{k}{2} \right\rceil = \left\lceil k_0 + \frac{3}{2} \right\rceil = k_0 + 1
\]

or \( \left\lceil \frac{k}{2} \right\rceil + 1 = k_0 + 2 \)

\( \Rightarrow m_0 = k_0 + 2. \)

Thus, we find that \( m_0 = k_0 + 2 \) when \( k = 2k_0 + 2 \) or \( 2k_0 + 3 \).

We have

\[
P(G) < \frac{\mu}{k_0} + \mu \sum_{k=2k_0-1} \frac{1}{\Lambda_m^{a-\delta}}.
\]

Again \( \Lambda_m = m^{2/\alpha}e_n \)

\[
> m^{2/\alpha} \text{ for large } n
\]

\[
\frac{1}{\Lambda_m^{a-\delta}} > \frac{1}{2(a-\delta)} = \frac{1}{m^{\frac{2-2\delta}{\alpha}}}
\]

Thus,

\[
\sum_{k \geq (2k_0-1)} \frac{1}{m_0^{\frac{2\delta}{\alpha}}}
\]

\[
= \sum_{m_0 \geq k_0} \frac{1}{m_0^{\frac{2\delta}{\alpha}}}
\]
\[
\begin{align*}
&= \frac{1}{2^\frac{28}{\alpha}} + \frac{1}{k_0^\alpha + 1} + \frac{1}{k_0^\alpha + 2} + \ldots \nonumber \\
&< \frac{1}{2} \sum_{k \geq k_0} \frac{1}{k^{\frac{28}{\alpha}}} \\
\text{But } \sum_{k \geq k_0} \left[ \frac{1}{k} \right]^{\frac{28}{\alpha}} < \int_{k_0}^{\infty} \frac{dx}{x^{\frac{28}{\alpha}}} = \int_{k_0}^{\infty} x^{-\frac{28}{\alpha}} dx \\
&= \left[ \frac{x^{-\frac{28}{\alpha}}}{1 + 2\delta} \right]_{k_0}^{\infty} = \frac{1}{\left(1 - \frac{2\delta}{\alpha}\right)^{\frac{28}{\alpha}}} k_0^{-\frac{28}{\alpha}} \\
\text{let } 0 < \delta < \frac{\alpha}{2} \\
\text{So } P(G) < \frac{\mu}{k_0} + \frac{\mu'}{k_0^{\frac{1}{2} - \frac{28}{\alpha}}} \\
&< \frac{1}{k_0^{\frac{1}{2} - \frac{28}{\alpha}}} \left( \frac{\mu}{\frac{28}{\alpha}} + \mu' \right) < \frac{\mu''}{k_0^{\frac{1}{2} - \frac{28}{\alpha}}} \\
&< \frac{\mu'}{(\epsilon_{\eta_0} \log n_0)^{\frac{28}{\alpha}}} \text{ by (5.24)} \\
\text{where } 0 < \delta < \frac{\alpha}{2} \text{ and } 0 < \alpha \leq 2. 
\end{align*}
\]