CHAPTER - 3

Random Hyperbolic Curve
CHAPTER - 3
RANDOM HYPERBOLIC CURVE

3.1 AVERAGE NUMBER OF LEVEL CROSSINGS OF A RANDOM HYPERBOLIC CURVE

3.1.1 INTRODUCTION

Consider the random hyperbolic curve

\[ F_n(t, \omega) = \sum_{h=1}^{n} P_k(\omega) \cosh kt = c \]  \hspace{1cm} (3.1)

where \((P_k(\omega))_{k=0}^{n}\) form a sequence of mutually independent normally distributed random variables with mean zero and variance unity and characteristic function \(e^{\frac{-x^2}{2}}\) where \(c\) is any constant independent of \(t\) and not necessarily zero. Sambandham [54] studied the curve \(\sum P_k(\omega) \cosh kt = c\) where \(c\) is not necessarily zero and showed that if \(\frac{c}{n} \to 0\) then

\[ \text{EN}(t, 1) \sim \frac{1}{\pi} \log \left( \frac{n}{c^2} \right) \]

and \(\text{EN}(-\infty, -1) = \text{EN}(1, \infty) \sim \frac{1}{2\pi} \log n\).

In other words, all the earlier authors were interested in the average number of the crossings of the graph of the curve \(y = f_n(x, \omega)\) with \(x\)-axis while Farahamand for the first time studied the average number of
crossings of the graph of \( y = f_n(x, \omega) \) with that of the line \( y = c \). The study actually becomes interesting when 'c' becomes sufficiently large. It is to be noted that \( y = f_n(x, \omega) \) may have so many oscillations without crossing the x-axis. Therefore if we consider the intersection of the curve with the line \( y = c \), it opens a new avenue for research.

We have generalised the results of Sambandham [54] and proved the following theorem.

**THEOREM 3.1** The average number of level crossings of a random hyperbolic curve. \( \sum_{k=1}^{n} P_k(\omega) \cosh kt = c \) is less than or equal to

\[
\frac{1}{\pi} \log n + \Theta(\log \log n)^3 \text{ for all } n \text{ sufficiently large where } P_k's \text{ and } c \text{ are defined as above.}
\]

**3.1.2**

We need the following lemmas

**Lemma 3.1**

\[
EN_n(\alpha, \beta) \leq \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{(AC - B^2)^{1/2}}{A} dt
\]

where

\[
A = A_n(t) = \sum_{1}^{n} \cosh^2 kt, \\
B = B_n(t) = \sum_{1}^{n} k \cosh kt \sinh kt, \\
C = C_n(t) = \sum_{1}^{n} k^2 \sinh^2 kt
\]
Let \( N_n(\alpha, \beta) \) be the number of real roots of (3.1) in the interval \((\alpha, \beta)\). Following the procedure of Logan and Shepp [29], we get that the expected number of zeros of (3.1) in the interval \((\alpha, \beta)\) which is given by the Kac Rice formula as,

\[
E N_n(\alpha, \beta) = \left| \int_{\alpha}^{\beta} g_n(t) \, dt \right| \quad (3.2)
\]

where

\[
g_n(t) = \int_{-\infty}^{\infty} |y| p(0, y) \, dy \quad (3.3)
\]

and \( p(x, y) \) is the probability density for

\[
g(t) = \sum_{k=1}^{n} p_k \cosh kt - c = X \quad (3.4)
\]

\[
g'(t) = \sum_{k=1}^{n} kp_k \sinh kt = Y \quad (3.5)
\]

Let \( f(z, w) \) be the joint characteristic function of \( X \) and \( Y \). Then

\[
f(z, w) = E \{ \exp (i g(t) z + i g'(t) w) \}
\]

\[
= \exp \left\{ - \sum_{k=1}^{n} \frac{(z \cosh kt + kw \sinh kt)^2}{2} - izc \right\}
\]

\[
= \exp \left\{ - \sum_{k=1}^{n} \frac{(a_k z + b_k w)^2}{2} - izc \right\}
\]

where \( a_k = \cos h kt, \quad b_k = k \sin h kt. \)
By Fourier – Inversion formula we have

\[ P(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-ixz - iyw)f(z, w)dzdw \quad (3.6) \]

\[ \therefore p(0, y) \]

\[ = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-iyw)\exp\left(-\sum_{k=1}^{n} \frac{(a_k z + b_k w)^2}{2} - izc\right)dz\,dw. \]

For \( \varepsilon > 0 \) we have

\[ \int_{-\infty}^{\infty} |y|e^{-\varepsilon|y|}p(0, y)dy \]

\[ = \int_{-\infty}^{\infty} \left| y \right|e^{-\varepsilon|y|} \left[ \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} e^{-iyw} \exp\left(-\sum_{k=1}^{n} \frac{(a_k z + b_k w)^2}{2} - izc\right)dz \right] dy \quad (3.7) \]

but \[ \int_{-\infty}^{\infty} |y|e^{-\varepsilon|y|}e^{-iyw}dy = \frac{2(e^2 - w^2)}{(e^2 - w^2)^2}. \]

So for \( \varepsilon > 0 \) we get from (3.7), after some easy calculations,

\[ \int_{-\infty}^{\infty} |y|e^{-\varepsilon|y|}p(0, y)dy \]

\[ = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{(e^2 - w^2)}{(e^2 + w^2)^2} dw \int_{-\infty}^{\infty} \exp\left(-\sum_{k=1}^{n} \frac{(a_k z + b_k w)^2}{2}\right)\cos zc dz. \quad (3.8) \]

Next, we let \( \cosh kt \) and \( \sinh kt \) be arbitrary so that joint probability density \( p(x, y) \) of \( X \) and \( Y \) given by

\[ X = g(t) = M_x \]
and \( Y = g'(t) = N_y \) (say)
degenerates and we get from (3.8) the following identity valid for non-zero $M, N$ which can be chosen suitably so that we have

$$\int_{-\infty}^{\infty} |y|e^{-|y|}p(0, y)dy = 0.$$ 

Hence

$$0 = \frac{1}{\pi^2} \int_{0}^{\infty} \frac{(e^2 - w^2)}{(e^2 + w^2)^2} dw \int_{-\infty}^{\infty} \exp\left\{-\frac{|Mz + Nw|^2}{2}\right\} \cos zc \ dz. \quad (3.9)$$

Subtracting respective sides of (3.9) from that of (3.8) we get

$$\int_{-\infty}^{\infty} |y|e^{-|y|}p(0, y)dy = \frac{1}{\pi^2} \int_{0}^{\infty} \frac{(e^2 - w^2)}{(e^2 + w^2)^2} dw \left[ \int_{-\infty}^{\infty} \left( -e^{\frac{|aKz + bkw|^2}{2}} - e^{\frac{|Mz + Nw|^2}{2}} \right) \right] \cos zc \ dz. \quad (3.10)$$

When $\epsilon \to 0$ then (3.10) becomes

$$\int_{-\infty}^{\infty} |y|p(0, y)dy = \frac{1}{\pi^2} \int_{0}^{\infty} \frac{dw}{2} \left[ -e^{\frac{|Mz + Nw|^2}{2}} - e^{\frac{|aKz + bkw|^2}{2}} \right] \cos zc \ dz. \quad (3.11)$$

Putting $z = uw$ and hence $dz = wdu$

we have
Following the procedure of Das [7] we get

\[ E N_\alpha (\alpha, \beta) \leq \frac{1}{\pi} \int_\alpha^\beta \frac{(AC - B^2)^{1/2}}{A} \, dt. \quad (3.13) \]

Where 
\[ A = A_n(t) = \sum_{k=1}^n \cosh^2 kt, \]
\[ B = B_n(t) = \sum_{k=1}^n k \cosh kt \sinh kt, \]
\[ C = C_n(t) = \sum_{k=1}^n k^2 \sinh^2 kt. \]

To ensure the validity of B, it is necessary that \((AC - B^2) > 0\).

Hence we insist that the interval \((\alpha, \beta)\) does not contain the point \(t = 0\). Again since a simple consequence of the representation (3.13) is

\[ E N_\alpha (\alpha, \beta) \leq E N_\alpha (-\alpha, -\beta), \]

we need only consider the positive value of \(t\). Applying the method of finite differences, we obtain

\[ A = \frac{2n-1}{4} + \frac{1}{4} \frac{\sinh(2n+1)t}{\sinh t}, \]
\[ B = \frac{1}{8} \left[ (2n+1) \frac{\cosh(2n+1)t}{\sinh t} - \frac{\sinh(2n+1)t}{\sinh^2 t} \cosht \right], \]

\[ C = -\frac{n(n+1)(2n+1)}{12} + \frac{1}{16} \left[ (2n+1)^2 \frac{\sinh(2n+1)t}{\sinh t} \right. \]

\[ - 2(2n+1) \frac{\cosh(2n+1)t}{\sinh^2 t} \cosht \]

\[ + \frac{\sinh(2n+1)t}{\sinh t} \left( \frac{2\cosh^2 t}{\sinh^2 t} - 1 \right). \]  

(3.14)

Hence \((AC - B^2)\)

\[ = \frac{1}{64} \frac{\sinh^2 (2n+1)t}{\sinh^4 t} + \frac{2n+1}{4} \frac{\sinh(2n+1)t}{\sinh^3 t} (1 + \cosh^2 t) \]

\[ - \frac{1}{64} \left( \frac{(2n+1)^2}{\sinh^2 t} + \frac{(4n^2 - 1)}{\sinh^2 t} \frac{\cosh(2n+1)t \cosh t}{\sinh^2 t} \right) \]

\[ + \frac{(2n+1)(8n^2 - 4n - 3)}{192} + \frac{\sinh(2n+1)t}{\sinh t} - \frac{n(n+1)(4n^2 + 1)}{48}. \]  

(3.15)

Let \(t \geq 1\), then on taking \(n\) sufficiently large we get from (3.15) and (3.14) respectively.

\[ (AC - B^2) \leq \frac{1}{16} \sinh^2 (2n+1) t \cosech^4 t \]

and \(A \leq \frac{1}{4} \sinh (2n+1) t \cosech t \)

\[ \therefore \frac{(AC - B^2)^{1/2}}{A} \leq 2e^{-t}. \]
Hence from (3.13) we have
\[ E N_n (1, t') < 4 \text{ for any } t' > 1. \]
In particular, let \( t' = n^n > 1 \), so
\[ E N_n (1, n^n) < 4 . \tag{3.16} \]

3.1.3

Now we proceed to estimate \( E N_n \) in different ranges.

1. **The range** \( \left( \frac{\log \log n}{n}, 1 \right) \).

Now we will show that in the range
\[
\frac{\log \log n}{n} \leq t \leq 1,
\]
the function \( \sinh(nt) \cosech t^p \) is an increasing function for
\( p = 1, 2, 3 \) and 4. Because when \( t \) lies in the range \( \left( \frac{\log \log n}{n}, 1 \right) \)
then \( \coth (t) < \left( \frac{e}{t} \right) \)
\[ \leq e^{-1} (\log \log n)^{-1} \]
and derivative of \( \sinh(nt) \cosech t^p \) is
\[ \cosh nt (\cosech t^p) (n - p \tanh nt \coth t) \]
\[ \geq \cosh nt (\cosech t^p) \left( n - \frac{en}{\log \log n} \right) > 0 \text{ for all large } n. \]

Hence
\[ \frac{\sinh nt}{(\sinh t)^p} \]
\[ \geq \frac{\sinh(\log \log n)}{\sinh\left(\frac{\log \log n}{n}\right)^p} \]

\[ \geq \sinh(\log \log n)\left(\frac{n}{4 \log \log n}\right)^p \]

\[ > \frac{1}{300} \frac{n^p \log n}{(\log \log n)^p} \quad (3.17) \]

(\because \sinh t \leq 4t \text{ in } (0, 1) \text{ and } p = 1, 2, 3, 4)

Further we observe that

\[ 1 \leq \frac{\cosh nt}{\sinh nt} \leq \coth n \quad (3.18) \]

where \( \frac{1}{n} \leq t \leq 1 \)

using the results of (3.17) and (3.18) in the explicit value of A and

\[ AC - B^2 \text{ obtained in (3.14) and (3.15) respectively, we get} \]

\[ A = \frac{1}{4} \frac{\sinh n(2n + 1)t}{\sinh t} \left[ 1 + 0\left(\frac{\log \log n}{\log n}\right)\right] \]

and

\[ \left(AC - B^2\right) = \frac{1}{64} \frac{\sinh(2n + 1)t}{\sinh^2 t} \left[ 1 + 0\left(\frac{\log \log n}{\log n}\right)^3\right] \]

\( \because \) From (3.13) we get

\[ E N_n \left(\frac{\log \log n}{n}, 1\right) \]
\[
\log\log n = \frac{1}{\pi} \int_{\log\log n}^{n} \frac{1}{2\sinh t} \left[ 1 + O\left( \frac{(\log\log n)^3}{\log n} \right) \right] dt
\]

\[
= \frac{1}{2\pi} \log\cot\left( \frac{\log\log n}{2n} \right) \left[ 1 + O\left( \frac{(\log\log n)^3}{\log n} \right) \right]
\]

\[
= \frac{1}{2\pi} \log n + n + O((\log\log n)^3).
\]

(3.19)

2. The range \( \left( \frac{1}{2^n}, \frac{\log\log n}{n} \right) \):

Next, we will obtain the average number of zeros in the range

\[
\frac{1}{2n} \leq t \leq \frac{\log\log n}{n}.
\]

We know that

\[
\sqrt{\frac{AC - B^2}{A}} \leq \sqrt{\frac{C}{A}}
\]

\[
\leq \left( \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{1+1+1+\ldots+1} \right)^{1/2}
\]

\[
< \frac{1}{\sqrt{2}} n
\]

This is done by taking \( \sinh kt \leq 1 \leq \cosh kt \)

\[
\therefore E_n\left( \frac{1}{2^n}, \frac{\log\log n}{n} \right)
\]

\[
\leq n \left( \frac{\log\log n}{n} - \frac{1}{2^n} \right)
\]

\[
< \log\log n.
\]

(3.20)
Let the n-dimensional set of points $\tilde{p} = (p_1, p_2, \ldots, p_n)$ be denoted in $\mathbb{R}^n$. The probability mass attached to the "infinitesimal rectangle" $\pi(\tilde{p})$ is

$$dm(\tilde{p}) = \prod_{k=1}^{n} \left\{ \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} p_k^2 \right) dp_k \right\}$$

where $dp_1, \ldots, dp_n$ are the lengths of the sides of $\prod (\tilde{p})$.

Now, we can conveniently represent the random variables $p_k$ by $p_k(x)$, $0 \leq x \leq 1$ (from Kac [22] and Dunnage [11]).

Let $g(x; t) = g(p; t)$ and $N(x; \alpha, \beta) = N(p; \alpha, \beta)$

Clearly $p$'s are functions of $x$. It can be easily proven that

$$\int_{\mathbb{R}^n} N(p; \alpha, \beta) dp = \int_{0}^{1} N(x; \alpha, \beta) dx .$$

We shall use the second representation for the expectation of the number of zeros of $g(x; t)$ in $\alpha \leq t \leq \beta$.

3. **The range** $\left(0, \frac{1}{2^n}\right)$

Now, we will find the upper bound for the number of zeros of $g(x; t)$ in the interval $0 \leq t \leq \frac{1}{2^n}$, where $x$ does not belong to any exceptional set of small measure.

We apply Jensen's theorem (cf Titchmarsh [62]) to the function $g(x; z)$ of complex argument namely
\[ g(x; z) = \sum_{k=1}^{n} p_k(x) \cosh kz. \]

Let \( n(x; r) \) denote the number of real zeros of \( g(x; z) \) in \(|z| \leq r\) and let \( \varepsilon = \frac{1}{2^n} \). The number of zeros of \( g(x; z) \) in \( 0 \leq z \leq \varepsilon \) does not exceed \( n(x; \varepsilon) \). Let \( E_1 \) be the set of \( x \)'s for which \( g(x; 0) = 0 \), then \( P(E_1) = 0 \).

If \( x \not\in E_1 \) then we can apply Jensen's theorem to obtain

\[
\int_{\varepsilon}^{2\varepsilon} n(x; r) \frac{dr}{r} \leq \int_{0}^{\varepsilon} n(x; r) \frac{dr}{r}
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{g(x; 2e^{i\theta})}{g(x; 0)} \right| d\theta.
\]

Also since \( n(x; r) \) is an increasing function of \( r \), we have

\[
\log 2n(x; \varepsilon) \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{g(x; 2e^{i\theta})}{g(x; 0)} \right| d\theta
\]

(3.21)

Now

\[
g(x; 2e^{i\theta}) = \sum_{k=1}^{n} P_k(x) \cosh k(2e^{i\theta})
\]

\[
= \sum_{k=1}^{n} p_k(x) \lambda_k + i \sum_{k=1}^{n} p_k(x) \mu_k
\]

\[
= P + i Q
\]

where \( \lambda_k = \cosh(2k \cos \theta) \cos(2k \sin \theta) \)

\( \mu_k = \sinh(2k \cos \theta) \sin(2k \sin \theta) \)
\[ P = \sum_{k=1}^{n} p_k(x) \lambda_k, \quad Q = \sum_{k=1}^{n} p_k(x) \mu_k. \]

Further,

let \( R = g(x; 0) = \sum_{k=1}^{n} p_k(x) \)

For the particular value of \( e = \frac{1}{2^2} \), we have

\[ \lambda_k = 1 + C (k \epsilon)^2 \quad (|C| < 4) \]

\[ \mu_k \leq 8 (k \epsilon)^2 \quad k = 1, 2, 3, \ldots, n. \]

Hence we obtain

\[ a) \quad |P - R| = \left| \sum_{k=1}^{n} p_k(x) (\lambda_k - 1) \right| \]

\[ \leq 4n^3 \epsilon^2 \max_{1 \leq k \leq n} |p_k(x)|, \]

\[ b) \quad |Q| = \left| \sum_{k=1}^{n} p_k(x) \mu_k \right| \]

\[ \leq 8n^3 \epsilon^2 \max_{1 \leq k \leq n} |p_k(x)|. \]

The distribution function of \(|p_k(x)|\) is given by

\[ F(\xi) = \begin{cases} \sqrt{\frac{2}{\pi}} \int_{0}^{\xi} \exp \left( -\frac{1}{2} u^2 \right) \, du & (\xi \geq 0) \\ 0 & (\xi < 0) \end{cases} \]

Since the \( P_k \)'s are independent of each other, we have
\[ F(\xi) = \left( 1 - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left( -\frac{1}{2} u^2 \right) du \right)^n \]

\[ \geq 1 - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left( -\frac{1}{2} u^2 \right) du \]

\[ > 1 - \exp\left( -\frac{1}{4} \right) \quad \left( \text{since } \sqrt{\frac{2}{\pi}} < 1 \right) \]

Therefore, applying the fundamental inequalities to (a) and (b), we get

\[ |P - R| + |Q| \leq \frac{12n^4}{4^6} \quad (3.22) \]

outside a set \( E_2 \) of \( x \) of measure not exceeding \( \exp\left( -\frac{1}{2} n^2 \right) \) for all \( \theta \).

Again the distribution function of \( |g(x,0)| = \sum_{k=1}^{n} p_k(x) \) is

\[ G(x) = \begin{cases} \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \exp\left( -\frac{1}{2} u^2 / n^2 \right) du & x \geq 0 \\ 0 & x < 0 \end{cases} \]

\[ \therefore g(x,0) \geq 1 \quad (3.23) \]

except in a set \( E_3 \) of \( x \) of measure

\[ \sqrt{\frac{2}{\pi n^2}} \int_{-\infty}^{\infty} \exp\left( -\frac{1}{2} u^2 / n^2 \right) du \]

\[ \leq \sqrt{\frac{2}{\pi n^2}}. \]

Let \( E = E_1 \cup E_2 \cup E_3. \)
Then measure of the set $E$ does not exceed
\[
\exp\left(-\frac{1}{4}n^2\right) + \sqrt{\frac{2}{\pi}} \frac{1}{n} \leq \frac{1}{n}
\]
i.e. $P(E) < \frac{1}{n}$. \hfill (3.24)

Suppose $x$ does not belong to the set $E$ then using (3.22) and (3.23), we get
\[
\left| \frac{g(x, 2 \in e^\infty)}{g(x, 0)} \right| = \left| \frac{P + iQ}{R} \right| = \left| 1 + \frac{P - R + iQ}{R} \right|
\]
\[
\leq 1 + \frac{|P - R| + |Q|}{R}
\]
\[
\leq 1 + \frac{12n^4}{4^n}.
\]

Using this in (3.21) we find that when $x \notin E$,
\[
n(x; \in) \log 2 \leq \frac{1}{2\pi} \int_0^{2\pi} \log \left(1 + \frac{12n^4}{4^n}\right) d\theta
\]
\[
\leq \frac{12n^4}{4^n}.
\]

Since $n(x; \in)$ can take only non-negative integral values, we have $n(x; \in) = 0$ for all $x$ which do not belong to the set $E$. Therefore, $g(x; t)$ does not contain any zero in $0 \leq t \leq \frac{1}{2^n}$ when $x$ lies outside the set $E$. But when $x \in E$ then $g(x; t)$ can have at most $2n$ zeros in $0 \leq t \leq \frac{1}{2^n}$. Hence we get
\[
E N_n \left(0, \frac{1}{2^n}\right) \leq 2n \quad P(E) \leq 2
\]
4. The range \((n^n, \infty)\)

Lastly, we will show that the probability for \(g(x; t)\) to have at least one zero outside the range \(|t| \geq n^n\) is indeed very small. A little before we had seen that \(|p_n(x)| \geq \frac{1}{n}\) outside a set \(F_1\) of \(x\) of measure.

\[
\sqrt{\frac{2}{\pi}} \int_0^{4/n} e^{-\frac{1}{2} u^2} \, du < \sqrt{\frac{2}{\pi} \frac{1}{n}}.
\]

Further each of the independent functions \(|p_1(x)|, |p_2(x)| \ldots |p_{n-1}(x)|\) is less than \(\log n\) in a set of measure

\[
\{F(\log n)\}^{n-1}
\]

\[
= \left\{1 - \sqrt{\frac{2}{\pi}} \int_{\log n}^{\infty} e^{-\frac{1}{2} u^2} \, du \right\}^{n-1}
\]

\[
= \left\{1 - \sqrt{\frac{2}{\pi}} \int_0^{\infty} \exp \left[ -\frac{1}{2} u^2 - u \log n - \frac{1}{2} (\log n)^2 \right] du \right\}^{n-1}
\]

\[
\geq \left\{1 - e^{-\frac{1}{2} (\log n)^2} \right\}^{n-1}
\]

\[
> 1 - \frac{1}{n}.
\]

Therefore, each of \(p_k(x), k = 1, 2, \ldots, n-1\) is less than \(\log n\) if \(x\) lies outside a set \(F_2\) of measure not exceeding \(\frac{1}{n}\).

Let \(F = F_1 \cup F_2\). If \(x\) does not belong to \(F\) then

\[
|g(x; t)| \geq \frac{1}{2^n} e^{nt} - \log n \left[ e^t + e^{2t} + \ldots + e^{(n-1)t} \right]
\]
\[ N \left( \frac{1}{2^n} - \frac{2 \log n}{e^1} \right) e^{nt} \]

\[ > 0, \text{ when } t \geq n^n \]

\[ . \quad g(x; t) \neq 0 \text{ if } x \notin F \text{ and } t \geq n^n. \]

hence \( E N_n (n^n, \infty) \leq \int_{x \in F} n(x; n^n, \infty) \, dx \)

\[ \leq 2n \, P(F) \]

\[ < 4. \]

3.2 Proof of the theorem

Since the function \( g(t) \) is an even function of \( t \) and \( n(x; \alpha, \beta) \) is additive over \((\alpha, \beta)\).

\[ E N_n (-\infty, \infty) = 2 \, E N_n (0, \infty) \]

\[ = 2 \left[ E N_n \left( 0, \frac{1}{2^n} \right) + E N_n \left( \frac{1}{2^n}, \frac{\log \log n}{n} \right) \right] \]

\[ + E N_n \left( \frac{\log \log n}{n}, 1 \right) + E N_n \left( 1, n^n \right) + E N_n \left( n^n, \infty \right) \]

\[ \leq 2 \left[ \frac{1}{2\pi} \log n + O(\log \log n)^3 + O(\log \log n) + O(1) \right] \]

\[ = \frac{1}{\pi} \log n + O(\log \log n)^3. \]

Hence our required relation

\[ E N_n (-\infty, \infty) = \frac{1}{\pi} \log n + O(\log \log n)^3 \text{ is established.} \]

This proves the theorem.