Chapter 5

Triangular Numbers

5.1 Introduction

From ancient times, triangular numbers attracts the attention of people all over the world. The interest is not limited to mathematicians and researchers.

Each of the numbers $1 = 1$, $3 = 1 + 2$, $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$, $15 = 1 + 2 + 3 + 4 + 5, \ldots$ represents the number of dots that can be arranged in an equilateral triangle.

![Figure 5.1: Triangular Numbers in terms of Dots](image)

These numbers are called triangular numbers. Thus there are infinitely many triangular numbers $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, \ldots$.

First prime triangular number is 3. Triangular numbers $1, 36, 1225, \ldots$ are perfect square integers.

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5.2 Elementary Properties of Triangular Numbers

Let \( t_n \) denote the \( n^{th} \) triangular number i.e \( t_1 = 1, t_2 = 3, \ldots \).

(5.2.1) Then \( t_n = t_{n-1} + n \), \( n \geq 2 \). Moreover \( t_n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \left( \frac{n+1}{2} \right) \); for \( n \in \mathbb{N} \). Also note that \( t_n \) is even if \( n \equiv 0 \text{(mod 4)} \) or \( n \equiv -1 \text{(mod 4)} \) and \( t_n \) is odd if \( n \equiv 1 \text{(mod 4)} \) or \( n \equiv 2 \text{(mod 4)} \).

(5.2.2) Thus \( n \) is triangular number iff \( n = \frac{k(k+1)}{2} \) (i.e \( n = t_k \)) for some \( k \in \mathbb{N} \) iff \( 8n + 1 = (2k + 1)^2 \) is a perfect square.

(5.2.3) \( t_n + t_{n-1} = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2 \) \( (n \geq 2) \).

i.e the sum of any two consecutive triangular numbers is a perfect square.

For example, the following figure shows the the square number \( S_9 \) can be written as a sum of two triangular numbers \( t_9 \) and \( t_8 \).

![Figure 5.2: \( S_9 \) in terms of \( t_8 \) and \( t_9 \)](image)

15, 21, 28 are three consecutive triangular numbers whose sum is a perfect square.

(5.2.4) **Polygonal Numbers** [21]:

A polygonal number denoted by \( P_d(n) \) and is of the form
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\[ P_d(n) = 1 + [1 + (d - 2)] + [1 + 2(d - 2)] + \cdots + [1 + (n - 1)(d - 2)] \]

\[ P_d(n) \] is a \( d \)-gonal number of order \( n \) (nth \( d \)-gonal number).

Figure 5.3: \( P_5(5) \) and \( P_6(6) \) in terms of Triangular Numbers

From the above examples one may observe that \( P_d(n) = t_n + (d - 3)t_{n-1} \) which is proved in the following theorem.

**Theorem 5.2.1.** A polygonal number with \( d \) sides and of order \( n \) is generated by the formula

\[ P_d(n) = \frac{(d - 2)n^2 + (4 - d)n}{2} \]

\[ t_n^2 - t_{n-1}^2 = \frac{n^2(n + 1)^2}{4} - \frac{n^2(n - 1)^2}{4} = n^3 \quad (n \geq 2). \]

i.e the difference between the squares of two consecutive triangular numbers is a perfect cube [58].

**Note:** A standard problem in elementary number theory is to determine all the numbers that are both square and triangular.

\[ t_n = \frac{n(n + 1)}{2} = m^2 \text{ (say), is a perfect square, where as by property (5.2.2),} \]
8m^2 + 1 is also a perfect square.  
⇒ t_{4m(n+1)} = t_{8m^2} = \frac{8m^2(8m^2 + 1)}{2} = (2m)^2(8m^2 + 1) is a perfect square.

(5.2.7) 15 and 21 are two consecutive triangular numbers such that their difference 6 and their sum 36 are also triangular numbers. 6, 10 and 15 are three consecutive triangular numbers whose product is 900 = 30^2, a perfect square. Integers 1, 3 and 6 are three consecutive triangular numbers whose sum is 10, the next triangular number. The sum of first eight triangular numbers is 120, the 15th triangular number.

(5.2.8) [Aryabhata 500 A.D.] 
\[ t_1 + t_2 + \ldots + t_n = \frac{n(n+1)(n+2)}{6} \]

\textbf{Proof.} Consider L.H.S= \[ \sum_{r=1}^{n} t_r = \sum_{r=1}^{n} \frac{r(r+1)}{2} = \frac{1}{2} \left[ \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r \right] \]
\[ = \frac{1}{2} \left[ \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right] = \frac{n(n+1)(n+2)}{6} = \text{R.H.S} \]

This also gives product of any three consecutive positive integers is divisible by 3!.

(5.2.9) If \( n \) is any triangular number and \( m \in \mathbb{N} \) (any), then \((2m + 1)^2n + t_m\) is a triangular number.

\textbf{Hint:} For \( n = t_k \), \((2m + 1)^2n + t_m\)
\[ = \frac{1}{2} \left[ (2m + 1)k + m \right] [((2m + 1)k + m + 1] = t_{(2m+1)k+m}. \]

Taking \( m = 1, 2, 3 \) and noting \( t_1 = 1, t_2 = 3, t_3 = 6 \) we have; [Euler, 1775] if \( n \) is a triangular number, then \( 9n + 1, 25n + 3, 49n + 6 \) are triangular numbers.

(5.2.10) There are infinitely many triangular numbers that are the difference of other two triangular numbers.

For any integer \( x \geq 2 \), by property (5.2.1) \( x = t_x - t_{x-1} \). Taking \( x = \frac{n(n+1)}{2} = t_n \) we get the result. This can also be stated as \([t_x = t_{x-1} + x]\) there are infinitely many triangular numbers that are the sum of other two triangular numbers.
(5.2.11) The square of any odd multiple of 3 is the difference of two triangular numbers; i.e. \(9(2n + 1)^2 = t_{9n+4} - t_{3n+1}\).

Also \(t_{132}, t_{143}, t_{164}\) is a pythagorean triple.

Every odd square is a sum of eight times a triangular number and 1 i.e \((2n + 1)^2 = 8 \times \frac{n(n+1)}{2} + 1\).

(5.2.12) Pascal triangle and other triangle of natural numbers shown below with the sequence of triangular numbers:

![Triangular Numbers in terms of Pascal triangle and other triangle](image)

Figure 5.4: Triangular Numbers in terms of Pascal triangle and other triangle

(5.2.13) Gauss has proved [27]: Any natural number can be expressed as the sum of less than or equal to three triangular numbers.

(5.2.14) \( t_{n+2k} \equiv t_n \pmod{k} \)

Proof. We have \( t_n = \frac{n(n+1)}{2} \), \( t_{n+2k} = \frac{1}{2}(n + 2k)(n + 2k + 1) \) and \( t_{n+2k} - t_n = k(2n + 2k + 1) \) divisible by \( k \), showing \( t_{n+2k} \equiv t_n \pmod{k} \).

Moreover for \( k = 10 \), \( t_{n+20} \equiv t_n \pmod{10} \), i.e \( t_{n+20} \) and \( t_n \) ends with same digit.

(5.2.15) Theorem: There do not exist four distinct triangular numbers in geometric progression [13].

(5.2.16) Every even perfect number is triangular [68] given by

\[ M_p 2^{p-1} = \frac{M_p(M_p + 1)}{2} = T_{M_p} \]
where $M_p$ is a Mersenne prime number.
All the known perfect numbers are even. So all the known perfect numbers are triangular.

(5.2.17) The square of nth triangular number is same as the sum of cubes of first $n$ natural numbers [33].

\[ i.e. \sum_{j=1}^{n} j^3 = \left( \sum_{j=1}^{n} j \right)^2 \quad (5.2.1) \]

Such tuples of numbers are called square triangular numbers.
Equation (5.2.1) is proved arithmetically by Nicomachus of Gerasa, the Greek mathematician of the Hellenistic age. Hence equation (5.2.1) is called Nicomachauss theorem.

**Remark 5.2.1.** To count the number of rectangles with horizontal and vertical sides formed in an $n \times n$ array, squared triangular numbers are used.

(5.2.18) [Triangular Numbers and Pythagorean Triples] [5]

**Theorem 5.2.2.** Three triangular numbers form a Pythagorean triples $x = t_a$, $y = t_b$, $z = t_c$ iff there exist integers $n$ and $k$ such that

\[ t_b^2 = n^3 + (n+1)^3 + \cdots + (n+k)^3 \]

i.e. $t_b^2$ is a sum of $k + 1$ consecutive cubes.

### 5.3 Further Properties of Triangular Numbers $t_n$.

(5.3.1) Sum of squares of two consecutive triangular numbers is a triangular number.

For $n \geq 2$, \[ t_n^2 + t_{n-1}^2 = \frac{n^2(n+1)^2}{4} + \frac{n^2(n-1)^2}{4} = \frac{n^2}{4}[2n^2 + 2] = t_{n^2}. \]

(5.3.2) Any three consecutive triangular numbers do not form a Pythagorean triple.

**Proof.** Suppose there exists a Pythagorean triple $(t_n, t_{n+1}, t_{n+2})$ of three con-
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consecutive triangular numbers, for some \( n \in \mathbb{N} \).

So \( t_n^2 + t_{n+1}^2 = t_{n+2}^2 \) \( (5.3.1) \)

\[ \Rightarrow \frac{n^2(n+1)^2}{4} + \frac{(n+1)^2(n+2)^2}{4} = \frac{(n+2)^2(n+3)^2}{4}, \text{ since } t_n = \frac{n(n+1)}{2} \]

\[ \Rightarrow (n+1)^2[n^2 + (n+2)^2] = (n+2)^2(n+3)^2 \]

\[ \Rightarrow n^2 + (n+2)^2 = 2n^2 + 4n + 4 \text{ is a perfect square and it is even.} \]

\[ \Rightarrow \frac{n^2}{2} + n + 1 \text{ is a perfect square and } n = 2m \text{ for some } m \in \mathbb{N} \]

\[ \Rightarrow m^2 + (m+1)^2 = 2m^2 + 2m + 1 \text{ is a perfect square.} \]

\[ \Rightarrow m = 3 \text{ only.} \]

Then \( n = 2m = 6 \) and equation (5.3.1) gives \( t_6^2 + t_7^2 = t_8^2 \) i.e. \( 21^2 + 28^2 = 36^2 \)

which is impossible, since LHS is an odd number, while RHS is an even number.

So the supposition is false and result follows. \( \square \)

(5.3.3) \( n^p \) in terms of triangular numbers

For any integer \( n \geq 2 \), by (5.2.1), \( n = t_n - t_{n-1} \) i.e \( n^p = (t_n - t_{n-1})^p \).

By (5.2.3), \( n^2 = t_n + t_{n-1} \) \( (n \geq 2) \) and by (5.2.4), \( n^3 = t_n^2 - t_{n-1}^2 \) \( (n \geq 2) \)

\[ \Rightarrow n^4 = (t_n + t_{n-1})^2, \quad n^5 = (t_n + t_{n-1})^2(t_n - t_{n-1}), \quad n^6 = (t_n^2 - t_{n-1}^2)^2, \quad (n \geq 2) \]

(5.3.4) The series \( \sum_{n=1}^{\infty} \frac{1}{t_n^p} \)

\[ \frac{t_n}{n^2} = \frac{n(n+1)}{2n^2} = \frac{1}{2} + \frac{1}{2n} \to \frac{1}{2} \text{ as } n \to \infty. \]

\[ \sum \frac{1}{t_n^p} \text{ converges iff } \sum \frac{1}{n^{2p}} \text{ converges. (Comparison Test)} \]

iff \( 2p > 1 \) i.e \( p > \frac{1}{2} \) [p- series test].

Thus the series \( \sum \frac{1}{t_n^p} \) converges if \( p > \frac{1}{2} \) and diverges if \( p \leq \frac{1}{2} \).

In particular , \( i \) \( \sum \frac{1}{t_n} \) converges and it is a telescopic series as

\[ \frac{1}{t_n} = \frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1} = a_n - a_{n+1}, \text{ where } a_n = \frac{2}{n}. \]

\[ \sum_{n=1}^{\infty} \frac{1}{t_n} = a_1 - \lim_{n \to \infty} a_n = 2 - 0 = 2 \quad (5.3.2) \]
(ii) \[ \sum_{n=1}^{\infty} \frac{1}{t^2_n} = 4 \sum_{n=1}^{\infty} \left[ \frac{1}{n} - \frac{1}{n+1} \right]^2 = 4 \left[ \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=2}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^2} \right] = 4 \left[ \frac{2\pi^2}{6} - 1 - 2 \right] \]

\[ \text{(since } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \text{ and (5.3.2)) = } 4 \left( \frac{\pi^2}{3} - 3 \right) \]  

(5.3.3)

(iii) \[ \sum_{n=1}^{\infty} \frac{1}{t^3_n} = 8 \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)^3 \]

\[ = 8 \sum_{n=1}^{\infty} \left[ \frac{1}{n^3} - \frac{3}{n^2(n+1)} + \frac{3}{n(n+1)^2} - \frac{1}{(n+1)^3} \right] \]

\[ = 8 \left[ 1 - \sum_{n=1}^{\infty} \frac{3}{n(n+1)} \left( \frac{1}{n} - \frac{1}{n+1} \right) \right] \]

\[ = 8 \left[ 1 - \sum_{n=1}^{\infty} \frac{3}{n^2(n+1)^2} \right] \]

\[ = 8 \left[ 1 - \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{t^2_n} \right] \]

\[ = 8 \left[ 1 - \frac{3}{4} \times 4 \left( \frac{\pi^2}{3} - 3 \right) \right] \quad \text{by equation (5.3.3)} \]

\[ = 8 \left[ 10 - \pi^2 \right] \]

(iv) \[ \sum_{n=1}^{\infty} \frac{1}{t^4_n} = 16 \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)^4 \]

\[ = 16 \sum_{n=1}^{\infty} \left[ \frac{1}{n^4} - \frac{4}{n^3(n+1)} + \frac{6}{n^2(n+1)^2} - \frac{4}{n(n+1)^3} + \frac{1}{(n+1)^4} \right] \]

\[ = 16 \left[ 2 \sum_{n=1}^{\infty} \frac{1}{n^4} - 1 + \sum_{n=1}^{\infty} \left( \frac{6}{n^2(n+1)^2} - \frac{4}{n(n+1)} \left[ \frac{1}{n^2} + \frac{1}{(n+1)^2} \right] \right) \right] \]

\[ = 16 \left[ \frac{2\pi^4}{90} - 1 + \sum_{n=1}^{\infty} \left[ \frac{6}{n^2(n+1)^2} - 4 \left( \frac{1}{n^3} - \frac{1}{(n+1)^3} + \frac{1}{n(n+1)} - \left( \frac{1}{n+1} - \frac{1}{n} \right) \right) \right] \right] \]

\[ \left( \because \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad \text{and} \quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \right) \]
\[
= 16 \left[ \sum_{n=1}^{\infty} \frac{\pi^4}{45} - 4 + \frac{5}{4} \sum_{n=1}^{\infty} \frac{4}{n^2(n+1)^2} \right]
\]
\[
= 16 \left[ \frac{\pi^4}{45} - 5 \cdot \frac{\pi^2}{3} - 3 \right] \quad \text{by equation (5.3.3)}
\]
\[
= 16 \left[ \frac{\pi^4}{45} + \frac{5\pi^2}{3} - 20 \right]
\]

Similarly, we can determine \(\sum_{n=1}^{\infty} \frac{1}{t_n^6}\), \(\sum_{n=1}^{\infty} \frac{1}{t_n^6}\) etc.

(5.3.5) Finite Series \(\sum_{r=1}^{n} t_r^p\), \((n, p \in \mathbb{N})\).

By the property (5.2.8), \(\sum_{r=1}^{n} t_r = \frac{n(n+1)(n+2)}{6}\)

\[
\sum_{r=1}^{n} t_r^2 = \frac{1}{4} \sum_{r=1}^{n} r^2(r+1)^2 = \frac{1}{4} \left[ \sum_{r=1}^{n} r^4 + 2 \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2 \right]
\]
\[
= \frac{1}{4} \left[ \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} + \frac{1}{2} (n^4 + 2n^3 + n^2) + \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right]
\]
\[
= \frac{1}{4} \left[ \frac{n^5}{5} + n^4 + \frac{5}{3} n^3 + n^2 + \frac{2}{15} n \right]
\]

\[
\sum_{r=1}^{n} t_r^3 = \frac{1}{8} \sum_{r=1}^{n} r^3(r+1)^3 = 8 \left[ \sum_{r=1}^{n} r^6 + 3 \sum_{r=1}^{n} r^5 + 3 \sum_{r=1}^{n} r^4 + \sum_{r=1}^{n} r^3 \right]
\]
\[
= 8 \left( \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} \right) + 24 \left( \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12} \right) + \frac{24}{7} \left( \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \right) + 8 \left( \frac{n^4}{4} + \frac{n^3}{3} + \frac{n^2}{4} \right)
\]
\[
= 8 \left[ \frac{n^7}{7} + n^6 + \frac{13}{5} n^5 + 3n^4 + \frac{4}{3} n^3 - \frac{1}{105} n \right]
\]

In this way we can calculate \(\sum_{r=1}^{n} t_r^4, \sum_{r=1}^{n} t_r^5\) etc.
(5.3.6) Ratio \( \frac{t_{an+r}}{t_{bn+s}} \) where \( a, b, m, n, r \in \mathbb{N} \).

\[
\frac{t_{an+r}}{t_{bn+s}} = \frac{(an + r)(an + r + 1)}{(bm + s)(bm + s + 1)}
\]

In particular \( \frac{t_{an+r}}{t_{bn+r}} = \frac{(a + \frac{r}{n})(a + \frac{r+1}{n})}{(b + \frac{s}{n})(b + \frac{s+1}{n})} \rightarrow \frac{a^2}{b^2} \) as \( n \rightarrow \infty \).

And \( \frac{t_{n+r}}{t_{n+s}} \rightarrow 1 \) as \( n \rightarrow \infty \). \( \frac{t_{a1n^2+2a2n+a3}}{t_{b1n^2+b2n+b3}} \rightarrow \frac{a^2}{b^2} \) as \( n \rightarrow \infty \).

(5.3.7) Differences of \( t_n \), \( \sum_{r=1}^{n} t_r \).

Using notation \( n^{(0)} = 1, n^{(1)} = n, n^{(r)} = n(n - 1) \ldots (n - r + 1) \) where integer \( r \geq 2 \). We get, \( \Delta n^{(r)} = r n^{(r-1)} \) etc.

Now \( t_n = \frac{(n+1)^{2}}{2!}, \sum_{r=1}^{n} t_r = \frac{(n+2)^{(3)}}{3!} \), we have

\[
\Delta t_n = n + 1, \Delta^2 t_n = 1, \Delta^r t_n = 0 \text{ for all } r \geq 3 \text{ and }
\]

\[
\Delta \sum_{r=1}^{n} t_r = \frac{(n+2)^{(2)}}{2!}, \Delta^2 \sum_{r=1}^{n} t_r = n + 2, \Delta^3 \sum_{r=1}^{n} t_r = 1 \text{ and } \Delta^r \sum_{r=1}^{n} t_r = 0 \text{ for all } r \geq 4.
\]

(5.3.8) The product of any two consecutive triangular numbers is not a perfect square.

For any integer \( n \geq 2, \frac{(n^2 - 1)n^2}{4} \) is not a perfect square.(since \( n^2 - 1 \) is not a perfect square as \( n \geq 2 \)).

(5.3.9) \( t_n \) divides \( t_1 + t_2 + \ldots + t_n \) iff \( n = 1, 4, 7, 10, \ldots \) and

\( t_{n+1} \) divides \( t_1 + t_2 + \ldots + t_n \) iff \( 3 \mid n \) i.e \( n = 3, 6, 9, 12, \ldots \)

Proof. As \( t_n = \frac{n(n+1)}{2} \) and \( t_1 + t_2 + \ldots + t_n = \frac{n(n+1)(n+2)}{6} \)

\[
\frac{t_1 + t_2 + \ldots + t_n}{t_n} = \frac{n+2}{3} \text{ is an integer iff } n = 1, 4, 7, 10, \ldots
\]

\[
\frac{t_1 + t_2 + \ldots + t_n}{t_{n+1}} = \frac{n}{3} \text{ is an integer iff } n = 3, 6, 9, 12, \ldots \]

(5.3.10) The unit digits of a triangular number is 0, 1, 3, 5, 6 or 8.
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Proof. For \( n \in \mathbb{N} \), \( n^{th} \) triangular number is \( t_n = \frac{n(n+1)}{2} \). Now \( n \) has one of the following form: \( 10k, 10k + 1, 10k + 2, \ldots, 10k + 9 \) where \( k \in \mathbb{N} \cup \{0\} \).

For \( n = 10k; t_n = \frac{10k(10k+1)}{2} = (5k)10k + 5k \) has unit digit 0 or 5.

For \( n = 10k + 1; t_n = \frac{(10k+1)(10k+2)}{2} = (5k + 1)10k + (5k + 1) \)

has unit digit 1 or 6.

For \( n = 10k + 2; t_n = \frac{(10k+2)(10k+3)}{2} = (5k + 1)10k + 3(5k + 2) \)

has unit digit 3 or 8.

For \( n = 10k + 3; t_n = \frac{(10k+3)(10k+4)}{2} = (5k + 2)10k + 3(5k + 2) \)

has unit digit 1 or 6.

For \( n = 10k + 4; t_n = \frac{(10k+4)(10k+5)}{2} = (5k + 2)10k + 5(5k + 2) \)

has unit digit 0 or 5.

For \( n = 10k + 5; t_n = \frac{(10k+5)(10k+6)}{2} = (5k + 3)10k + 5(5k + 3) \)

has unit digit 0 or 5.

For \( n = 10k + 6; t_n = \frac{(10k+6)(10k+7)}{2} = (5k + 3)10k + 7(5k + 3) \)

has unit digit 1 or 6.

For \( n = 10k + 7; t_n = \frac{(10k+7)(10k+8)}{2} = (5k + 4)10k + 7(5k + 4) \)

has unit digit 3 or 8.

For \( n = 10k + 8; t_n = \frac{(10k+8)(10k+9)}{2} = (5k + 4)10k + 9(5k + 4) \)

has unit digit 1 or 6.

For \( n = 10k + 9; t_n = \frac{(10k+9)(10k+10)}{2} = (5k + 5)10k + 9(5k + 5) \)

has unit digit 0 or 5.

Hence unit digit of a triangular number is 0, 1, 3, 5, 6, or 8.

\((5.3.11)\) \( t_n \) divides \( t_1^2 + t_2^2 + \cdots + t_n^2 \) iff \( n \equiv 1, 7, \text{or } 13(\text{mod}15) \).

Proof. We have \( t_n = \frac{n(n+1)}{2} \) and

\[
t_1^2 + t_2^2 + \cdots + t_n^2 = \frac{1}{4} \left[ n^5 + n^4 + \frac{5}{3} n^3 + n^2 + \frac{2}{15} n \right]
\]

\[
t_n \mid \sum_{r=1}^{n} t_r^2 \iff \frac{\sum_{r=1}^{n} t_r^2}{t_n} = \frac{3n^3 + 12n^2 + 13n + 2}{30} \text{ is an integer}
\]

iff \( 30 \mid (3n^3 + 12n^2 + 13n + 2) \) iff \( 15 \mid 3n^3 + 12n^2 + 13n + 2 \).
i.e. $15|3n^3 - 3n^2 - 2n + 2$ since $f(n) = 3n^3 - 3n^2 - 2n + 2$ is an even integer.

Any $n \in \mathbb{N}$ has the form $n = 15q + r$ where $r \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7\}$
i.e. $n \equiv r(\text{mod}15)$ and so $f(n) \equiv f(r)(\text{mod}15)$.

Hence $15|f(n)$ iff $15|f(r)$.

For $n = 15q; f(n) \equiv f(0)(\text{mod}15)$ and $f(0) = 2$, so $15 \nmid f(n)$.

For $n = 15q + 1; f(n) \equiv f(1)(\text{mod}15)$ and $f(1) = 0$, so $15|f(n)$.

For $n = 15q - 1; f(n) \equiv f(-1)(\text{mod}15)$ and $f(-1) = -6$, so $15 \nmid f(n)$.

For $n = 15q + 2; f(n) \equiv f(2)(\text{mod}15)$ and $f(2) = 10$, so $15 \nmid f(n)$.

For $n = 15q - 2; f(n) \equiv f(-2)(\text{mod}15)$ and $f(-2) = -30$, so $15|f(n)$.

For $n = 15q + 3; f(n) \equiv f(3)(\text{mod}15)$ and $f(3) = 50$, so $15 \nmid f(n)$.

For $n = 15q - 3; f(n) \equiv f(-3)(\text{mod}15)$ and $f(-3) = -100$, so $15 \nmid f(n)$.

For $n = 15q + 4; f(n) \equiv f(4)(\text{mod}15)$ and $f(4) = 138$, so $15 \nmid f(n)$.

For $n = 15q - 4; f(n) \equiv f(-4)(\text{mod}15)$ and $f(-4) = -230$, so $15 \nmid f(n)$.

For $n = 15q + 5; f(n) \equiv f(5)(\text{mod}15)$ and $f(5) = 292$, so $15 \nmid f(n)$.

For $n = 15q - 5; f(n) \equiv f(-5)(\text{mod}15)$ and $f(-5) = -438$, so $15 \nmid f(n)$.

For $n = 15q + 6; f(n) \equiv f(6)(\text{mod}15)$ and $f(6) = 530$, so $15 \nmid f(n)$.

For $n = 15q - 6; f(n) \equiv f(-6)(\text{mod}15)$ and $f(-6) = -742$, so $15 \nmid f(n)$.

For $n = 15q + 7; f(n) \equiv f(7)(\text{mod}15)$ and $f(7) = 870$, so $15|f(n)$.

For $n = 15q - 7; f(n) \equiv f(-7)(\text{mod}15)$ and $f(-7) = -1160$, so $15 \nmid f(n)$. Thus

$t_n|\left(t^2_n + t^2_2 + \cdots + t^2_n\right)$ iff $n$ has the form $15q + 1, 15q - 2, 15q + 7$
i.e. $n \equiv 1, 7, -2(\text{mod}15)$ i.e $n \equiv 1, 7, 13(\text{mod}15)$.

\hfill \square

(5.3.12) From the list of triangular numbers we observe that (conjecture) in between any two consecutive triangular numbers there is a prime number.