Chapter 6

Multifractal Analysis in Image Denoising

6.1 Introduction

One of the most eye-catching problems in image processing is the recognition of noise free and noisy images. GFD is the measure of complexity, irregularity and chaotic nature in the fractal or corrupted images. This chapter invokes the multifractal analysis by using the GFD in the recognition of noise less & noisy gray scale images and color images with RGB components. In this technique, median filter is used to denoise the images corrupted by the salt and pepper noise. It is shown that GFD measure discriminate the noisy and noise free images very accurately in both gray scale and RGB color images; and also demonstrated the classification by the graphical methods for the standard images. Also the fuzzy multifractal theory for gray scale and RGB color images is developed in order to define the Fuzzy Generalized Fractal Dimensions by introducing fuzzy membership function in the classical GFD method.
6.2 Filtering Techniques

Image noise is random variation of brightness or color information in images, and is usually an aspect of electronic noise. It can be produced by the sensor of a scanner or digital camera. Image noise can also originate in film grain and in the unavoidable shot noise of an ideal photon detector. Image noise is an undesirable by-product of image capture that adds spurious and extraneous information [42].

In this chapter, we have used the Salt and Pepper noise to make the standard gray scale and RGB color images as noisy images and denoised the corrupted experimental images by the Median Filter [42, 22].

Salt and Pepper noise is a form of noise typically seen on images. It represents itself as randomly occurring white and black pixels. An effective noise reduction method for this type of noise involves the usage of a median filter, morphological filter or contra harmonic mean filter. Salt and pepper noise creeps into images in situations where quick transients, such as faulty switching, take place [42].

Median filtering is a nonlinear operation often used in image processing to reduce “salt and pepper” noise. Such noise reduction is a typical pre-processing step to improve the results of later processing. Median filtering is very widely used in digital image processing because, under certain conditions, it preserves edges while removing noise [42, 22].

6.3 Multifractal Analysis for Gray Scale Images

In this section, we invoke the multifractal analysis in the recognition of noise free images by using Generalized Fractal Dimensions (GFD).
6.3.1 Generalized Fractal Dimensions for Gray Scale Images

Now we define a probability distribution of a experimental gray scale image by the following construction.

Let $N$ be the number of boxes to cover the tested gray scale image with box size $r$. The probability $p_i$ for $i^{th}$ box of size $r$ in the gray scale image is defined as,

$$p_i = \frac{M_i}{M}$$

where $M_i$ is the mass of the tested gray scale image included in the corresponding $i^{th}$ box of size $r$ and $M$ is the total mass of the tested gray scale image.

Then, the Renyi Fractal Dimensions or Generalized Fractal Dimensions (GFD) of order $q \in (-\infty, \infty)$ such that $q \neq 1$ of the tested gray scale image for the known probability distribution can be defined as

$$D_q = \lim_{r \to 0} \frac{1}{q-1} \frac{\log_2 \left( \sum_{i=1}^{N} p_i^q \right)}{\log_2 r}$$

(6.1)

Here $D_q$ is defined in terms of generalized Renyi Entropy.

Some Special Cases of Generalized Fractal Dimensions for Gray Scale Images

- If $q = 0$, then

$$D_0 = -\frac{\log_2 N}{\log_2 r}$$

which is nothing but the Fractal Dimension of the gray scale image.

- As $q \to 1$, $D_q$ converges to $D_1$, which is given by

$$D_1 = \lim_{r \to 0} \frac{\sum_{i=1}^{N} p_i \log_2 p_i}{\log_2 r}.$$

This is called as Information Dimension of the gray scale image.

- If $q = 2$, then $D_q$ is called the Correlation Dimension of the gray scale image.
• There are two limit cases of the gray scale image when \( q = -\infty \) and \( q = \infty \), which is given as

\[
D_{-\infty} = \lim_{r \to 0} \frac{\log_2 (p_{\min})}{\log_2 r}
\]

\[
D_{\infty} = \lim_{r \to 0} \frac{\log_2 (p_{\max})}{\log_2 r}
\]

where

\[
p_{\min} = \min\{p_1, p_2, \ldots, p_N\}
\]

\[
p_{\max} = \max\{p_1, p_2, \ldots, p_N\}.
\]

6.3.2 Qualitative Measures in Gray Scale Image Denoising

The Mean Absolute Error (MAE) and Peak Signal to Noise Ratio (PSNR) have been used to assess the performance of the denoising process in gray scale images by using the Median Filter. These quality measures [42] are defined as

\[
\text{MAE} = \frac{\sum_{j=1}^{N} \sum_{i=1}^{M} |I_O(i,j) - I_F(i,j)|}{M \cdot N}
\]

and

\[
\text{PSNR} = 20 \cdot \log \left( \frac{255}{\sqrt{\frac{1}{M \cdot N} \sum_{j=1}^{N} \sum_{i=1}^{M} (I_O(i,j) - I_F(i,j))^2}} \right)
\]

where \( M \) and \( N \) are the image dimensions, and \( I_O(i,j) \) and \( I_F(i,j) \) denote the original and filtered or restored image, at pixel position \((i,j)\), respectively.

6.3.3 Results and Discussions

In this section, simulations are carried out in MATLAB software using four natural gray scale images. We have tested the four standard gray level images such as Lena image, Camera Man Image, Mandril Image and Pirate Image in order to analyze and detect the level of noise in gray scale images (Figure 6.1).
Figure 6.1: Standard Gray Scale Images
Table 6.1: Comparison of the Denoising Performance of Tested Gray Scale Images in Terms of MAE and PSNR Measures

<table>
<thead>
<tr>
<th>Image</th>
<th>Noisy Image Corrupted by Salt and Pepper Noise with Density 0.05</th>
<th>Denoised Image using Median Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MAE</td>
<td>PSNR</td>
</tr>
<tr>
<td>Lena</td>
<td>6.3159</td>
<td>18.4841</td>
</tr>
<tr>
<td>Camera Man</td>
<td>6.4095</td>
<td>18.0435</td>
</tr>
<tr>
<td>Mandril</td>
<td>6.3844</td>
<td>18.6178</td>
</tr>
<tr>
<td>Pirate</td>
<td>6.3922</td>
<td>18.3885</td>
</tr>
</tbody>
</table>

Firstly, the experimental natural gray scale images are contaminated by the Salt and Pepper noise with density 0.05. Then the corrupted gray scale images are denoised by the filtering technique using the Median Filter. Noisy gray scale images with Salt and Pepper noise of density 0.05 and Denoised images by Median Filter of four standard gray scale images such as Lena, Cameraman, Mandril and Pirate are depicted in Figures 6.2, 6.4, 6.6 and 6.8 respectively. Figures 6.2, 6.4, 6.6 and 6.8 showed that the Median filter performs the denoising process significantly on tested gray scale images. The comparison of the performance of denoising process for all tested gray scale images are analyzed and depicted in Table 6.1 by using the quality measures such as MAE and PSNR. The quality measures in Table 6.1 also showed that the denoising process is good for all categories of tested gray scale images.

In Figures 6.3, 6.5, 6.7 and 6.9 the GFD values of Original and Denoised Images are significantly differed from the GFD Values of Noisy images in all categories of standard gray scale images. Also the GFD values of Original and Denoised gray scale images are coincided in some cases. It is observed by graphically in Figures 6.3, 6.5, 6.7 and 6.9.
Figure 6.2: (a). Noisy Gray Scale Lena Image Corrupted by Salt and Pepper Noise; (b). Denoised Gray Scale Lena Image using Median Filter

Figure 6.3: Generalized Fractal Spectra for Original, Noisy and Denoised Gray Scale Lena Image
Figure 6.4: (a). Noisy Gray Scale Camera Man Image Corrupted by Salt and Pepper Noise; (b). Denoised Gray Scale Camera Man Image using Median Filter

Figure 6.5: Generalized Fractal Spectra for Original, Noisy and Denoised Gray Scale Camera Man Image
Figure 6.6: (a). Noisy Gray Scale Mandril Image Corrupted by Salt and Pepper Noise; (b). Denoised Gray Scale Mandril Image using Median Filter

Figure 6.7: Generalized Fractal Spectra for Original, Noisy and Denoised Gray Scale Mandril Image
Figure 6.8: (a). Noisy Gray Scale Pirate Image Corrupted by Salt and Pepper Noise; (b). Denoised Gray Scale Pirate Image using Median Filter

Figure 6.9: Generalized Fractal Spectra for Original, Noisy and Denoised Gray Scale Pirate Image
Hence, Figures 6.3, 6.5, 6.7 and 6.9 are the evidence that the multifractal measure plays an efficient role in the discrimination of Noisy and Noise free gray scale images by using Generalized Fractal Dimensions.

6.4 Multifractal Analysis for RGB Color Images

The focal theme of this section is to present the multifractal analysis using Generalized Fractal Dimensions (GFD) for color images with RGB components and invoke the GFD in the detection of noise free color images.

6.4.1 Generalized Fractal Dimensions for RGB Color Images

Now we define a probability distribution of a given color image with (R,G,B) triplet components by the following construction. The three-dimensional space of color coordinates (R,G,B) possesses $256^3$ distinct colors.

Let $N$ be the number of boxes to cover the tested RGB color image with box size $r$. The probability $p_{ik}$ for $i^{th}$ box of size $r$ for the corresponding $k^{th}$ component in the tested RGB color image is defined as,

$$p_{ik} = \frac{M_{ik}}{M_k}, \quad k \in \{R, G, B\}$$

where $M_{ik}$ is the mass of the tested color image included in the $i^{th}$ box of size $r$ for the corresponding $k^{th}$ component and $M_k$ is the total mass for the corresponding $k^{th}$ component in the tested RGB color image.
Then, the Renyi Fractal Dimensions or Generalized Fractal Dimensions (GFD) of order \( q \in (-\infty, \infty) \) such that \( q \neq 1 \) of the RGB color image for the known probability distribution can be defined as

\[
D_q = \lim_{r \to 0} \frac{1}{q - 1} \log_2 \left( \frac{\sum_{k \in \{R,G,B\}} \sum_{i=1}^{N} p^q_{ik}}{\log_2 r} \right).
\]  

(6.2)

Here \( D_q \) is defined in terms of generalized Renyi Entropy.

Some Special Cases of Generalized Fractal Dimensions for RGB Color Images

Here we have discussed the special cases of GFD for RGB color image as in the classical method.

- If \( q = 0 \), then
  \[
  D_0 = -\frac{\log_2 \left( \sum_{k \in \{R,G,B\}} N \right)}{\log_2 r}
  \]
  which is nothing but the Fractal Dimension of the RGB color image.

- As \( q \to 1 \), \( D_q \) converges to \( D_1 \), which is given by
  \[
  D_1 = \lim_{r \to 0} \frac{\sum_{k \in \{R,G,B\}} \sum_{i=1}^{N} p_{ik} \cdot \log_2 p_{ik}}{\log_2 r}
  \]
  This is called as Information Dimension of the RGB color image.

- If \( q = 2 \), then \( D_q \) is called the Correlation Dimension of the RGB color image.

- There are two limit cases of the RGB color image when \( q = -\infty \) and \( q = \infty \), which is given as
  \[
  D_{-\infty} = \lim_{r \to 0} \frac{\log_2 \left( \sum_{k \in \{R,G,B\}} p_{\min} \right)}{\log_2 r}
  \]
  \[
  D_{\infty} = \lim_{r \to 0} \frac{\log_2 \left( \sum_{k \in \{R,G,B\}} p_{\max} \right)}{\log_2 r}
  \]
  where
  \[
  p_{\min} = \min\{p_{1k}, p_{2k}, \ldots, p_{Nk}\}, \quad k \in \{R, G, B\}
  \]
  \[
  p_{\max} = \max\{p_{1k}, p_{2k}, \ldots, p_{Nk}\}, \quad k \in \{R, G, B\}.
  \]
6.4.2 Qualitative Measures in RGB Color Image Denoising

The Mean Absolute Error (MAE) and Peak Signal to Noise Ratio (PSNR) have been used to assess the performance of the denoising process in RGB color images by using the Median Filter. These quality measures [42] are defined as

\[
MAE = \frac{\sum_{k=1}^{Q} \sum_{j=1}^{N} \sum_{i=1}^{M} |I_O(i, j, k) - I_F(i, j, k)|}{M \cdot N \cdot Q}
\]

and

\[
PSNR = 20 \cdot \log \left( \frac{255}{\sqrt{\frac{1}{M \cdot N \cdot Q} \sum_{k=1}^{Q} \sum_{j=1}^{N} \sum_{i=1}^{M} (I_O(i, j, k) - I_F(i, j, k))^2}} \right)
\]

where \(M\) and \(N\) are the image dimensions, \(Q\) is the number of channels of the image (\(Q = 3\) for RGB color image), and \(I_O(i, j, k)\) and \(I_F(i, j, k)\) denote the \(k^{th}\) component of the original and filtered or restored image, at pixel position \((i, j, k)\), respectively.

6.4.3 Results and Discussions

In this section, simulations are carried out in MATLAB software using five natural RGB color images. We have tested the five standard RGB Color images such as Lena Image, Barbara Image, Mandril Image, Peppers Image and Pills Image in order to analyze and detect the level of noise in RGB color images (Figure 6.10).

Firstly, the experimental natural RGB color images are contaminated by the Salt and Pepper noise with density 0.09. Then the corrupted RGB color images are denoised by the filtering technique using the Median Filter. Noisy RGB color images with Salt and Pepper noise with density 0.09 and Denoised images by the Median Filter of five natural RGB color images such as Lena, Barbara, Mandril, Peppers and Pills are depicted in Figures 6.11, 6.13, 6.15, 6.17 and 6.19 respectively. Figures 6.11, 6.13, 6.15, 6.17 and 6.19 showed that the Median filter performs the denoising process in significant manner on tested RGB color images.
Figure 6.10: Standard RGB Color Images
Figure 6.11: (a). Noisy Color Lena Image Corrupted by Salt and Pepper Noise; (b). Denoised Color Lena Image using Median Filter

Figure 6.12: Generalized Fractal Spectra for Original, Noisy and Denoised Color Lena Image
Figure 6.13: (a). Noisy Color Barbara Image Corrupted by Salt and Pepper; (b). Denoised Color Barbara Image using Median Filter

Figure 6.14: Generalized Fractal Spectra for Original, Noisy and Denoised Color Barbara Image
Figure 6.15: (a). Noisy Color Mandril Image Corrupted by Salt and Pepper Noise; (b). Denoised Color Mandril Image using Median Filter

Figure 6.16: Generalized Fractal Spectra for Original, Noisy and Denoised Color Mandril Image
Figure 6.17: (a). Noisy Color Peppers Image Corrupted by Salt and Pepper Noise; (b). Denoised Color Peppers Image using Median Filter

Figure 6.18: Generalized Fractal Spectra for Original, Noisy and Denoised Color Peppers Image
Figure 6.19: (a). Noisy Color Pills Image Corrupted by Salt and Pepper Noise; (b). Denoised Color Pills Image using Median Filter

Figure 6.20: Generalized Fractal Spectra for Original, Noisy and Denoised Color Pills Image
Table 6.2: Comparison of the Denoising Performance of Tested RGB Color Images in Terms of MAE and PSNR Measures

<table>
<thead>
<tr>
<th>Image</th>
<th>Noisy Image Corrupted by Salt and Pepper Noise with Density 0.09</th>
<th>Denoised Image using Median Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MAE</td>
<td>PSNR</td>
</tr>
<tr>
<td>Lena</td>
<td>11.4918</td>
<td>15.6276</td>
</tr>
<tr>
<td>Barbara</td>
<td>11.6035</td>
<td>15.5658</td>
</tr>
<tr>
<td>Mandril</td>
<td>11.5465</td>
<td>15.6556</td>
</tr>
<tr>
<td>Peppers</td>
<td>11.5144</td>
<td>15.3647</td>
</tr>
<tr>
<td>Pills</td>
<td>11.4586</td>
<td>15.4363</td>
</tr>
</tbody>
</table>

The comparison of the performance of denoising process for all tested RGB color images are analyzed and depicted in Table 6.2 by using the quality measures such as MAE and PSNR. The quality measures in Table 6.2 also showed that the denoising process is good for all categories of tested RGB color images.

In Figures 6.12, 6.14, 6.16, 6.18 and 6.20, the GFD values of Original and Denoised Images are significantly differed from the GFD Values of Noisy images in all categories of tested natural RGB color images. Also the GFD values of Original and Denoised RGB color images are coincided in some cases. It is observed by graphically in Figures 6.12, 6.14, 6.16, 6.18 and 6.20.

Hence, Figures 6.12, 6.14, 6.16, 6.18 and 6.20 are the evidence that the multifractal analysis plays an efficient role in the discrimination of Noisy and Noise free RGB color images by using Generalized Fractal Dimensions.
6.5 Fuzzy Multifractal Analysis for Gray Scale Images

In this section, we define the Fuzzy Generalized Fractal Dimensions (FGFD) for quantifying the fuzziness of multifractal gray scale images.

Now we define the Fuzzy Generalized Fractal Dimensions for an experimental fractal gray scale images by the following construction.

Let \( N \) be the number of boxes to cover the tested gray scale image with box size \( r \). Define a fuzzy membership function, \( \mu : S \rightarrow [0, 1] \), on the gray scale image \( S \), partitioned by \( N \) boxes such as \( S_1, S_2, ..., S_N \).

Then, the Fuzzy Renyi Fractal Dimensions or Fuzzy Generalized Fractal Dimensions (FGFD) of order \( q \in (-\infty, \infty) \) such that \( q \neq 1 \) of the tested gray scale image for the given fuzzy membership function on the set of image pixel values \( S \), denoted by \( FD_q \), can be defined as

\[
FD_q = \lim_{r \to 0} \frac{1}{q - 1} \log_2 \left( \frac{\sum_{i=1}^{N} (\sum_{x \in S_i} \mu(x))^q}{\log_2 r} \right). \tag{6.3}
\]

Here \( FD_q \) is defined in terms of generalized Fuzzy Renyi Entropy.

6.6 Fuzzy Multifractal Analysis for RGB Color Images

In this section, we define the Fuzzy Generalized Fractal Dimensions (FGFD) for quantifying the fuzziness of multifractal RGB color images.

Now we define the Fuzzy Generalized Fractal Dimensions for an experimental fractal RGB color images by the following construction. The three-dimensional space of color coordinates (R,G,B) possesses \( 256^3 \) distinct colors.
Let \( N \) be the number of boxes to cover the tested RGB color image with box size \( r \). Define a fuzzy membership function for \( k^{th} \) component of RGB color image as
\[
\mu_k : S_k \rightarrow [0, 1]
\]
on \( S_k \), the \( k^{th} \) component of RGB color image \( S \), partitioned by \( N \) boxes such as \( S_{1k}, S_{2k}, \ldots, S_{Nk} \), where \( k \in \{R, G, B\} \).

Then, the Fuzzy Renyi Fractal Dimensions or Fuzzy Generalized Fractal Dimensions (FGFD) of order \( q \in (-\infty, \infty) \) such that \( q \neq 1 \) of the tested RGB color image, denoted by \( FD_q \), can be defined as
\[
FD_q = \lim_{r \to 0} \frac{1}{q-1} \log_2 \left( \frac{\sum_{k \in \{R,G,B\}} \left( \sum_{i=1}^{N} \left( \sum_{x \in S_{ik}} \mu_k(x) \right)^q \right)}{\log_2 r} \right). \tag{6.4}
\]

Here \( FD_q \) is defined in terms of generalized Fuzzy Renyi Entropy.

### 6.7 Concluding Remarks

In this chapter, we have presented the multifractal measure such as Generalized Fractal Dimensions (GFD) for gray scale and RGB color images, and invoked the GFD in the recognition of noisy and noise free gray scale images as well as color images with triplet of (R,G,B) components. The performances of GFD are tested with various natural gray scale and RGB color images. We have proved that GFD discriminates the noisy and denoised images with high accuracy, and illustrated the classification rate through graphical methods by comparing GFD values for Original, Noisy and Denoised gray scale and color images. From these observations we have suggested that the Multifractal Analysis using GFD plays a significant role in the image analysis to detect the noise level in all type of experimental images such as biomedical images, biometric images, satellite images, agricultural images and so on. Further, the fuzzy multifractal dimensions are defined for both gray scale and RGB color images.