Chapter 1

Introduction

1.1 Fractal Analysis

Mathematics has been concerned largely with sets and functions to which the methods of classical Euclidean geometry and calculus can be applied. Methods of classical Euclidean geometry and calculus are unsuited to study the irregular sets and natural objects, and we need alternative techniques. Euclidean geometry describes the objects in the universe which has integer dimension. But as far as the real life applications are concerned, there are objects which has non-integer dimension. Moreover, irregular sets provide a much better representation of many natural phenomena than do the figures of classical geometry. Fractal geometry provides a general framework for the study of such irregular sets. Fractal geometry is an extension of classical geometry. It can be used to make precise models of physical structures from ferns to galaxies. Fractal geometry is a new language to represent the irregular objects in Mathematics. The main tool of fractal geometry is dimension in its many forms and dimension of the fractal objects is also a non-integer. A fractal is a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole, a property called self-similarity [13, 27, 28, 29, 55].
The mathematical term “Fractal” was introduced by the Professor of Mathematics, Benoit Mandelbrot in 1975. The fractal was derived from the Latin word, “fractus” means broken or fractured to describe objects that were too irregular or complex to fit into a traditional geometrical setting. Mandelbrot defined a fractal mathematically as a set with Hausdorff dimension strictly exceeds its topological dimension [54, 55, 56, 61].

A fractal is an object which appears self-similar under varying degrees of magnification. Many fractals have some degree of self-similarity – they are made up of parts that resemble the whole in some way. Sometimes, the resemblance may be weaker than strict geometrical similarity; for example, the similarity may be approximation or statistical. Fractal sets are mathematical models of non-integer dimensional sets satisfying certain scaling properties. Roughly speaking, a fractal is a set that is more “irregular” than the sets considered in classical geometry [2, 13, 14, 29, 48]. Fractal analysis has been popularized by various mathematicians [16, 17, 25, 30, 45, 73] in different aspects.

A fractal has all or most of the following properties:

- It has a fine structure at arbitrarily small scales.
- It is too irregular to be described in traditional Euclidean geometrical language, both locally and globally.
- It is self-similar (at least approximately or statistically).
- It has a fractal dimension which is greater than its topological dimension.
- It has a simple and recursive definition.
Hutchinson [45] and Barnsley [13] initiated and developed the theory, called as Hutchinson-Barnsley theory (HB theory), in order to define and construct the fractal as a compact invariant subset of a complete metric space generated by the Iterated Function System (IFS) of contractions, by using the Banach fixed point theorem. That is, fractal set is a unique fixed point of the Hutchinson-Barnsley operator (HB operator), which is defined by the dynamical system of contraction mappings. An IFS was introduced as an application of the theory of discrete dynamical systems and is a useful tool to build fractal and other similar sets. So, IFS have proven to be a very useful way of constructing self-similar or fractal objects on metric spaces.

In literature, there are numerous contractions and multivalued contractions are defined as well as the corresponding fixed point theorems are proved, and the interesting subsequent applications are also discussed [64, 74, 78, 83, 92]. Several authors have generalized and extended the HB theory by replacing the classical fixed point theorem by other metric and topological fixed point theorems in order to define metric fractals, topological fractals, Tarski’s fractals, semifractals, multivalued fractals, etc. [12, 15, 34, 43, 52, 77, 79, 80, 88, 94, 95]. Especially in [5, 6, 7, 8, 9, 18, 23, 24, 93], multivalued fractals were developed in metric space, $b$-metric space and topological space.

In 1976, Fisher defined a contraction namely Fisher contraction and proved the fixed point theorems based on the Fisher contraction [33]. Here we intend to initiate the fractals by replacing the Banach fixed point theorem in the HB theory by the Fisher fixed point theorem. In continuation of the above generalizations, we have generalized the HB theory for Fisher contractions to define Fisher Fractals by using Fisher fixed point theorem.
1.2 Fuzzy Mathematics

Fuzzy set theory was introduced by Zadeh in 1965 [99]. The notion of intuitionistic fuzzy sets was defined by Atanassov in 1986 as a generalization of fuzzy sets [11]. Many authors have introduced and discussed several notions of fuzzy metric space in different ways [37, 49] and also proved fixed point theorems with interesting consequent results in fuzzy metric spaces [40, 97]. Then the concept of intuitionistic fuzzy metric space was given by Park [70] and the subsequent fixed point results in the intuitionistic fuzzy metric spaces were investigated by Alaca and et al. [4] and Mohamad [62].

Rodriguez-Lopez and Romaguera [86] provided a new contribution to the development of the theory of fuzzy metrics in a potentially interesting direction due to the undoubted importance of Hausdorff distance not only in general topology but also in other areas of Mathematics and Computer Science, such as fractals, image processing, etc.

Recently, the concepts of fuzzy metrics are explicitly used in the image processing as a main tool. The peer group concept in the impulsive noise detection in color images, is redefined by means of a certain fuzzy metric. This concept is employed for the fast detection of noisy pixels by taking advantage of the fuzzy metric properties.

Moreover, fractal concepts also are widely used in image processing, image denoising, image compression and etc. In the literature, some important theorems related with metric fractals such as Collage theorem and Falling Leaves theorem are directly used in the distance concepts between the image pixels while processing the image. Therefore, fractals and fuzzy spaces play a significant role in all scientific areas especially, in non-linear analysis and image processing.
Hence, the above studies motivate our direction to investigate the fractal concepts and multivalued fractal concepts in fuzzy metric spaces, intuitionistic fuzzy metric spaces by using the corresponding fixed point theorems. To satisfy these motivations, we have constructed the fractals and multivalued fractals by generalizing the HB theory for IFS of fuzzy contractions in fuzzy metric space and intuitionistic fuzzy contractions, intuitionistic fuzzy B-contractions and intuitionistic fuzzy Edelstein contractions in intuitionistic fuzzy metric space. Also we have proved the existence and uniqueness theorems of fractals for all the above generalizations of HB theory. Moreover, we have discussed and analyzed many interesting results and properties for HB operator on fuzzy and intuitionistic fuzzy hyperspaces.

1.3 Multifractal Analysis

Fractals have broad applications in non-linear dynamical systems, computer graphics, biomedicine and other applied sciences. The complexity and irregularity that can be found in many physical and biological non-linear systems naturally and which has been analyzed by the tools of fractal theory and computed by the measure called fractal dimension.

In the literature, when fractal technique has been applied to the complex signals and images, the dimensional measure has mainly been used to analyze the chaotic nature in different conditions [1]. Among all the non-linear techniques the correlation dimension measurement [65] is more accessible in dealing with experimental signals and images. This single dimensional quantity is insufficient to characterize the nonuniformity or inhomogeneity of the system. Generally, chaotic attractors are inhomogeneous.
Such an inhomogeneous set is called a Multifractal and is characterized by Generalized Fractal Dimensions (GFD) or Renyi Fractal Dimensions [39, 44, 51, 57, 58, 59, 60, 84, 85]. Abnormal signals as well as noisy images are essentially multi scale fractal, i.e. multifractal. Multifractal signals and images are intrinsically more complex and inhomogeneous than monofractals. Therefore, quantifying the complexity of the fractal signals and corrupted images requires estimation of the fractal dimension spectrum where the complexity means higher variability in general fractal dimension spectrum. The non-linear measure, correlation dimension belongs to an infinite family of Generalized Fractal Dimensions [44]. The usage of the whole family of fractal dimensions should be very useful in comparison with using only some of the dimensions. Unlike the Fourier spectra, the fractal spectra consists of a family of fractal dimensions that characterize the multifractal objects from both the amplitude and the frequency point of view [50]. So generalized fractal spectra is very efficient technique to quantify the chaotic nature and irregularity of the complex signals and corrupted images.

The human brain is the center of the human nervous system and is a highly complex organ. Diagnosing the diseases related to the human brain is very challenging task for the physicians. In 1875, the famous Physician, Richard Canton discovered the electrical currents in our brain. Hans Berger, the psychiatrist in Germany, first recorded these currents and called it as Electro-encephalo-gram (EEG). Electroencephalogram is an electrical signal recorded from the scalp or intracranial, and is thought to reflect the mass activity of neurons and their interactions. EEG is the measurement of electrical activity produced in the human brain as recorded from electrodes placed on the scalp. Depending upon the nature of the diseases, electrodes are attached on the cortical site to get the EEG signals from the human brain [66, 67].
EEG signal is very effective tool to know useful information as much as possible about the human brain dynamics in the different physiological states of the patients. Also it is very important for a deeper understanding of the complex dynamical behaviour of the brain. The Electroencephalogram signal has been the most utilized signal to clinically assess brain activities. EEG has more fluctuations recorded from the human brain due to the spontaneous electrical activity [35, 47, 66, 67, 68, 69, 72, 96]. Karl Weierstrass gave an example of a function with the non-intuitive property of being everywhere continuous but nowhere differentiable, called Weierstrass Function, whose graph would today be considered as fractal [29]. Since the fractal curve such as Weierstrass function characterizes the biomedical waveforms and complex signals, hence EEG signals are represented as fractal time series.

One of the most fatal diseases in the human brain is Epilepsy. Epilepsy is a common chronic neurological disorder, which is characterized by recurrent unprovoked seizures. Epileptic seizures are the outcome of the transient and unexpected electrical disturbance of the brain. EEG was used by the physicians to compute the neurological disorders especially the epilepsy. EEG is used in epileptic seizure to detect, to predict and to diagnosis the seizure onset. The classification of normal & abnormal subjects and detection of epileptic seizures in the EEG Signals is an important technique in the diagnosis of epilepsy [35, 46, 47, 66, 67, 68, 69, 72, 81, 96]. So we made an attempt to analyze the EEG in depth for knowing the mystery of human consciousness.

Recently, various signal processing techniques are used to process the EEG signals. Fourier Transforms and Wavelet Transforms are widely used in the analysis of EEG time series. The Fourier transform of a signal compromises the time localization information to achieve the frequency localization since it requires large time analysis window; and also vice-versa [66, 67].
The Short-Time Fourier transform represents a sort of compromise between the time and frequency based views of a signal and contains both time and frequency information. Short-Time Fourier transform has a limited frequency resolution determined by the size of the analysis window. This frequency resolution is fixed for the entire frequency band. Contrary to Short-Time Fourier transform, Wavelet Transform (WT) provides a more flexible way of time-frequency representation of a signal by allowing the use of variable sized windows. In WT, long time windows are used to get a finer low frequency resolution and short time windows are used to get high frequency information. Thus, WT gives precise frequency information at low frequencies and precise time information at high frequencies [19, 66, 67].

The informative cortically generated signals are contaminated by extra-cerebral artifact sources such as ocular movements, eye blinks, and muscular artifacts. Generally the mixture between brain signals and artifact signals is present in all sensors, although not necessarily in the same proportions (depending on the spatial distribution). Moreover, the EEG recordings are also affected by other unknown basically random signals (instrumentation noise, other physiological generators, external electromagnetic activity, etc) which can be modeled as additive random noise. These phenomena make difficult to analyze the actual interpretation of EEGs, and a first important processing step would be the elimination of the artifacts and noise [3, 98].

Several methods for artifact and noise elimination were proposed for denoising the signals in the literature [3, 20, 26, 36, 75, 76, 98]. Recently discrete wavelet transform (DWT) has been used for enhancing artifact suppression in EEG signals. WT has been proved to be a powerful tool for analysis of biological signals because of its good localization properties in time and frequency domain. Different wavelet based methods are used for denoising biological signals.
In these methods, noisy biological signal is decomposed into wavelet coefficients by applying wavelet transform. Denoised signal estimate is obtained by the inverse wavelet transform [76]. A major advantage of this method is that a large class of well-behaved functions can be sparsely represented in wavelet space and WT transforms independent, identically distributed (i.i.d.) noise to i.i.d. wavelet coefficients [26, 75].

The noise will still be spread evenly among the coefficients. Hence, this method transforms the noisy signal into wavelet space, threshold small coefficients, and performs the inverse wavelet transform for noise free reconstruction. Since the true signal (noise free) will be characterized by sparse representation in wavelet space, the signal will be concentrated in a small number of large coefficients while the noise will be distributed as smaller coefficients [26, 75]. So wavelet analysis is a suitable means of analyzing these non-stationary signals. This makes the wavelet transform suitable for the analysis of irregular data patterns, such as impulses occurring at various time instances. One area in which the DWT has been particularly successful in the epileptic seizure detection because it captures transient features and localizes them in both time and frequency content accurately [35, 36, 66, 67, 96].

The traditional linear analysis, both in time and frequency domains, has been used for Epileptic Seizure detection but it has its more limits [72]. But our Human Brain is a highly complex and a non-linear system. So the non-linear time series analysis methods have been effectively applied to the studies of brain functions and pathological changes in Epileptic EEG Signals [46, 68, 69, 81]. The non-linear measures used, including Correlation Dimension, Largest Lyapunov Exponent and Approximate Entropy quantify the degree of complexity and irregularity in a Fractal Time Series [47, 67].
So far many authors have used the concepts of Generalized Fractal Dimensions on finding out the chaotic nature of the brain dynamics. Classification, detection or recognition of the Epilepticform in EEG was a vast research area in the brain dynamics. The Multifractal techniques in Signal Processing and Non-linear Analysis are devoted in the development of Epileptic research. So we concentrate on the multifractal measure named GFD in signal classification.

As an application part of the thesis, we have designed three new forms of GFD to classify the healthy and epileptic EEG signals graphically and decomposed the EEG signals of different types by using discrete wavelet transform for denoising the signals as a pre-processing step in the proposed epileptic seizure detection method. Mathematical modeling of epileptic EEG signals using multifractal and wavelet techniques helps the physicians as an additional tool to analyze, classify, detect or predict the epileptic seizure before the onset time among the affected peoples, so that the patient can be cautious about further disturbances, and lives of affected people will be protected from the unexpected disasters. Also we have introduced the fuzzy GFD to classify and analyze the fractal Weierstrass waveforms.

Images are obtained by photo electronic or photochemical methods. Transmission process of acquired objects tends to corrupt the quality of the digital images by introducing noise. The existence of noise in an image may be a drawback in any subsequent processing to be done over the noisy image such as edge detection, image segmentation or pattern recognition. As a consequence, restoring the image to reduce or remove the noise without degrading its quality is a major step in any computer vision application [21, 22, 42]. Because the corrupted images have high complexity and irregularity in nature or in its pixel values, it is very difficult to identify and quantify the restoring images or noise free images by using quantitative measures.
Also natural images, especially color or multi component images, are complex information-carrying signals. To contribute to the characterization of this complexity, we have to investigate the possibility of multiscale organization in the colorimetric structure of natural images. This is realized by means of a multifractal analysis using GFD applied to the gray scale and the color images. Fractal and multifractal concepts so far have been applied essentially to the spatial organization of gray scale and color images [21].

In this thesis, we have invoked the multifractal analysis by using GFD in the recognition of noise free as well as noisy gray scale and RGB color images. We have shown that GFD measure discriminate the noisy and noise free images very accurately in both gray scale and RGB color images; and also demonstrated by the graphical methods for the standard gray scale and RGB color images. Also we have presented the fuzzy multifractal theory in order to define the Fuzzy Generalized Fractal Dimensions by introducing fuzzy membership function in classical GFD method for both gray scale and RGB color images.

1.4 Preliminaries

1.4.1 Metric Fractals

In this section, we recall the Hutchinson-Barnsley theory (HB theory) to define and construct the IFS fractals in the complete metric spaces.

**Definition 1.1** ([13, 25, 29]). Let \((X, d)\) be a metric space and \(\mathcal{K}_0(X)\) be the collection of all non-empty compact subsets of \(X\) or hyperspace of compact subsets of \(X\).

Define, \(d(x_o, B) := \inf_{y \in B} d(x_o, y)\) and \(d(A, B) := \sup_{x \in A} d(x, B)\) for all \(x_o \in X\) and \(A, B \in \mathcal{K}_0(X)\).
The Hausdorff metric or Hausdorff distance \((H_d)\) is a function 
\[H_d : K_o(X) \times K_o(X) \rightarrow \mathbb{R}\]
defined by 
\[H_d(A, B) = \max\{d(A, B), d(B, A)\}.
\]
Then \(H_d\) is a metric on the hyperspace of compact sets \(K_o(X)\) and hence \((K_o(X), H_d)\) is called a Hausdorff metric space.

**Theorem 1.1** ([13, 25, 29]). If \((X, d)\) is a complete metric space, then \((K_o(X), H_d)\) is also a complete metric space.

**Definition 1.2** ([13, 25, 29]). Let \((X, d)\) be a metric space. We note that, 
\[\left(\mathcal{K}_o\left(\mathcal{K}_o(X)\right), \mathcal{H}_{H_d}\right)\] is also a metric space, where \(\mathcal{K}_o(X)\) is the hyperspace of all non-empty compact subsets of \((K_o(X), H_d)\) and \(\mathcal{H}_{H_d}\) is the Hausdorff metric on \(\mathcal{K}_o(X)\) implied by the Hausdorff metric \(H_d\) on \(K_o(X)\). That is, for all \(A, B \in \mathcal{K}_o(X)\), 
\[\mathcal{H}_{H_d}(A, B) = \max\{H_d(A, B), H_d(B, A)\},\]
where 
\[H_d(A, B) := \sup_{A \in \mathcal{A}} H_d(A, B)\] and 
\[H_d(A_o, B) := \inf_{B \in \mathcal{B}} H_d(A_o, B)\] for all \(A_o \in \mathcal{K}_o(X)\) and \(A, B \in \mathcal{K}_o(X)\).

**Theorem 1.2** ([13, 14]). If \((X, d)\) is a complete metric space, then \(\left(\mathcal{K}_o\left(\mathcal{K}_o(X)\right), \mathcal{H}_{H_d}\right)\) is also a complete metric space.

**Definition 1.3** ([13]). We say that the function \(f : X \rightarrow X\) is a contraction or Banach contraction mapping on a metric space \((X, d)\), if there exists \(k \in [0, 1)\) such that 
\[d(f(x), f(y)) \leq kd(x, y), \quad \forall x, y \in X.\]
Here \(k\) is called a contractivity ratio of \(f\).

**Theorem 1.3** (Banach Contraction Theorem [13]). A Banach contraction mapping on a non-empty complete metric space has a unique fixed point.
**Theorem 1.4** ([13]). A Banach contraction mapping on a metric space is a continuous function.

**Definition 1.4** ([13, 45]). Let \((X, d)\) be a metric space and \(f_n : X \rightarrow X, \ n = 1, 2, 3, ..., N_o (N_o \in \mathbb{N})\) be \(N_o\) - Banach contraction mappings with the corresponding contractivity ratios \(k_n, \ n = 1, 2, 3, ..., N_o\). Then the system \(\{X; f_n, \ n = 1, 2, 3, ..., N_o\}\) is called an Iterated Function System (IFS) or Hyperbolic Iterated Function System of Banach contractions with the ratio \(k = \max_{n=1}^{N_o} k_n\).

**Definition 1.5** ([13, 45]). Let \((X, d)\) be a metric space. Let \(\{X; f_n, \ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}\) be an IFS of Banach contractions. Then the Hutchinson-Barnsley operator (HB operator) of the IFS of Banach contractions is a function \(F : \mathcal{K}_o(X) \rightarrow \mathcal{K}_o(X)\) defined by

\[
F(B) = \bigcup_{n=1}^{N_o} f_n(B), \quad \text{for all } B \in \mathcal{K}_o(X).
\]

**Theorem 1.5** ([13, 45]). Let \((X, d)\) be a metric space. Let \(\{X; f_n, \ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}\) be an IFS of Banach contractions. Then, the HB operator \((F)\) is a contraction mapping on \((\mathcal{K}_o(X), H_d)\).

**Theorem 1.6** (HB Theorem for Metric IFS [13, 45]). Let \((X, d)\) be a complete metric space and \(\{X; f_n, \ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}\) be an IFS of Banach contractions. Then, there exists only one compact invariant set \(A_\infty \in \mathcal{K}_o(X)\) of the HB operator \((F)\) or, equivalently, \(F\) has a unique fixed point namely \(A_\infty \in \mathcal{K}_o(X)\).

**Definition 1.6** (Metric Fractals [13]). The fixed point \(A_\infty \in \mathcal{K}_o(X)\) of the HB operator \(F\) described in the Theorem 1.6 is called the Attractor (Fractal) of the IFS of Banach contractions. Sometimes \(A_\infty \in \mathcal{K}_o(X)\) is called as Metric Fractal generated by the IFS of Banach contractions.
**Theorem 1.7** ([14]). Let \((X, d)\) be a metric space. Let \(\mathcal{A}, \mathcal{B} \in \mathcal{K}_o(\mathcal{K}_o(X))\) be such that

\[
\{a \in A : A \in \mathcal{A}\}, \{b \in B : B \in \mathcal{B}\} \in \mathcal{K}_o(X).
\]

Then

\[
H_d\left(\{a \in A : A \in \mathcal{A}\}, \{b \in B : B \in \mathcal{B}\}\right) \leq \mathcal{H}_{H_d}(\mathcal{A}, \mathcal{B}).
\]

**Theorem 1.8** (Collage Theorem for Metric IFS [13, 29]). Let \((X, d)\) be a complete metric space. Let \((\mathcal{K}_o(X), H_d)\) be the corresponding Hausdorff metric space and \(\{X; f_n, n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}\) be an IFS of Banach contractions with the contractivity ratio \(k\). If \(B \in \mathcal{K}_o(X)\), then

\[
H_d(A_\infty, B) \leq \frac{H_d(B, F(B))}{1 - k},
\]

where \(F\) is the \(HB\) operator of Banach contractions and \(A_\infty\) is the Attractor (Fractal) of the IFS of Banach contractions.

**Theorem 1.9** (Falling Leaves Theorem in Metric Space [14]). Let \((X, d)\) be a metric space. Let \((\mathcal{K}_o(X), H_d)\) be the corresponding Hausdorff metric space. If \(A, B, C \in \mathcal{K}_o(X)\) are disjoint and \(A', B', C' \in \mathcal{K}_o(X)\) are such that \(H_d(A, A'), H_d(B, B')\) and \(H_d(C, C')\) are all sufficiently small then

\[
\mathcal{H}_{H_d}\left(\{A, B, C\}, \{A', B', C'\}\right) = H_d\left(A \cup B \cup C, A' \cup B' \cup C'\right).
\]

### 1.4.2 Examples of Metric Fractals

**Example 1.1** (Cantor Set [13, 25, 29]). Let \(X = [0, 1]\) and consider the IFS on \(X = [0, 1]\) consists of the following two contractions,

\[
f_1(x) = \frac{x}{3}; \quad f_2(x) = \frac{x + 2}{3}.
\]

All the above contractions have contractivity factor \(1/3\). By Theorem 1.6, the resulting fractal is called as “Cantor Set”, as shown in Figure 1.1.
Example 1.2 (Sierpinski Gasket [13, 25, 29]). Let $X$ be an equilateral $\triangle ABC \subseteq [0, 1]^2$ with the vertices $A = (0, 0), B = (1, 0)$ and $C = (1/2, \sqrt{3}/2)$, and consider the IFS on $X$ consists of the following three contractions,

$$f_1(x, y) = \left( \frac{1}{2} x, \frac{1}{2} y \right);$$

$$f_2(x, y) = \left( \frac{1}{2} x + \frac{1}{2}, \frac{1}{2} y \right);$$

$$f_3(x, y) = \left( \frac{1}{2} x + \frac{1}{4}, \frac{1}{2} y + \frac{\sqrt{3}}{4} \right).$$

All the above contractions have contractivity factor $1/2$. By Theorem 1.6, the resulting fractal is called as “Sierpinski Gasket”, as shown in Figure 1.2.
Figure 1.2: Construction of the Sierpinski Gasket
Figure 1.3: Construction of the Koch Curve
**Example 1.3** (Koch Curve [13, 25, 29]). Let $X = [-1, 1] \times \{0\} \subseteq [-1, 1]^2$ and consider the IFS on $X$ consists of the following four contractions,

$$
\begin{align*}
  f_1(x, y) &= \left( \frac{1}{3} x - \frac{2}{3}, \frac{1}{3} y \right); \\
  f_2(x, y) &= \left( \frac{1}{6} x - \frac{\sqrt{3}}{6} y - \frac{1}{6}, \frac{\sqrt{3}}{6} x + \frac{1}{6} y + \frac{\sqrt{3}}{6} \right); \\
  f_3(x, y) &= \left( \frac{1}{6} x + \frac{\sqrt{3}}{6} y + \frac{1}{6}, -\frac{\sqrt{3}}{6} x + \frac{1}{6} y + \frac{\sqrt{3}}{6} \right); \\
  f_4(x, y) &= \left( \frac{1}{3} x + \frac{2}{3}, \frac{1}{3} y \right).
\end{align*}
$$

All the above contractions have contractivity factor $1/3$. By Theorem 1.6, the resulting fractal is called as “Koch Curve”, as shown in Figure 1.3.

The Koch curve is an example of a continuous curve which is nowhere differentiable. It is also a curve of infinite length.

### 1.4.3 Multivalued Metric Fractals

In this section, we recall the iterated multifunction system to define and construct the multivalued fractals in the complete metric spaces.

**Definition 1.7** ([7, 8, 9, 32, 93]). Let $(X, d)$ be a metric space and $f_n : X \rightarrow \mathcal{K}_c(X)$, $n = 1, 2, 3, ..., N_o$ $(N_o \in \mathbb{N})$ be $N_o$ - multivalued contractions or multivalued Banach contractions (with non-empty compact values) with the corresponding contractivity ratios $k_n$, $n = 1, 2, 3, ..., N_o$, i.e.,

$$H_d(f_n(x), f_n(y)) \leq k_n d(x, y)$$

for all $x, y \in X$ and $k_n \in [0, 1); n = 1, 2, 3, ..., N_o$.

The system $\{X; f_n, n = 1, 2, 3, ..., N_o\}$ is called an Iterated Multifunction System (IMS) of multivalued Banach contractions with the ratio $k = \max_{n=1}^{N_o} k_n$. 

18
Definition 1.8 ([7, 8, 9, 32, 93]). Let \((X, d)\) be a metric space. Let 
\(\{X; f_n, \ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}\) be an IMS of multivalued Banach contractions.

The associated Hutchinson-Barnsley map (HB map) of the IMS of multivalued Banach contractions is a function \(F : X \rightarrow \mathcal{K}_o(X)\) defined by

\[
F(x) = \bigcup_{n=1}^{N_o} f_n(x), \quad \text{for all} \ x \in X.
\]

Then the induced Hutchinson-Barnsley operator (HB operator) of the IMS of multivalued Banach contractions is a function \(F^* : \mathcal{K}_o(X) \rightarrow \mathcal{K}_o(X)\) defined by

\[
F^*(B) = \bigcup_{x \in B} F(x) = \bigcup_{x \in B} F(x), \quad \text{for all} \ B \in \mathcal{K}_o(X).
\]

Theorem 1.10 (HB Theorem for Metric IMS [7, 8, 9, 32, 93]). Let \((X, d)\) be a complete metric space and \(\{X; f_n, \ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}\) be an IMS of multivalued Banach contractions on \(X\). Then, there exists exactly one compact invariant set \(A^* \in \mathcal{K}_o(X)\) of the HB operator \((F)\) or, equivalently, \(F^*\) has a unique fixed point namely \(A^* \in \mathcal{K}_o(X)\).

Definition 1.9 ([7, 8, 9, 32, 93]). The fixed point \(A^* \in \mathcal{K}_o(X)\) of the HB operator \(F^*\) described in the Theorem 1.10 is called the Multivalued Attractor (Multivalued Fractal) of the IMS of multivalued Banach contractions. Sometimes \(A^* \in \mathcal{K}_o(X)\) is called as Multivalued Metric Fractal generated by the IMS of multivalued Banach contractions.

Theorem 1.11 (Collage Theorem for Metric IMS [32]). Let \((X, d)\) be a complete metric space. Let \((\mathcal{K}_o(X), H_d)\) be the corresponding Hausdorff metric space and 
\(\{X; f_n, \ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}\) be an IMS of multivalued Banach contractions on \(X\) with the contractivity ratio \(k\). If \(B \in \mathcal{K}_o(X)\), then

\[
H_d(A^*, B) \leq \frac{H_d(B, F^*(B))}{1 - k},
\]

where \(F^*\) is the HB operator of multivalued Banach contractions and \(A^*\) is the attractor of the IMS of multivalued Banach contractions.
1.4.4 Fuzzy Metric Space

In [99], Zadeh defined a fuzzy set on $X$ as a function $f : X \rightarrow [0, 1]$. In order to develop some interesting results on Fuzzy IFS Fractals and Multivalued Fuzzy IFS Fractals, we have to state the required concepts of fuzzy metric spaces as follows:

**Definition 1.10 ([90]).** A binary operation $\ast : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous $t$-norm, if $\ast$ satisfies the following conditions:

(a) $\ast$ is commutative and associative;

(b) $\ast$ is continuous;

(c) $a \ast 1 = a$ for all $a \in [0, 1]$;

(d) $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

**Definition 1.11** (Kramosil and Michalek [49]). The 3-tuple $(X, M, \ast)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $\ast$ is a continuous $t$-norm and $M$ is a fuzzy set on $X \times X \times [0, \infty)$ satisfies the following conditions:

1. $M(x, y, 0) = 0$,

2. $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

3. $M(x, y, t) = M(y, x, t)$,

4. $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$,

5. $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous,

$x, y, z \in X$ and $t, s > 0$.

In order to introduce a Hausdorff topology on the fuzzy metric space, George and Veeramani [37] modified the above definition and gave the following.
Definition 1.12 (George and Veeramani [37]). The 3-tuple \((X, M, *)\) is said to be a fuzzy metric space if \(X\) is an arbitrary set, \(*\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X \times X \times (0, \infty)\) satisfies the following conditions:

1. \(M(x, y, t) > 0\),

2. \(M(x, y, t) = 1\) if and only if \(x = y\),

3. \(M(x, y, t) = M(y, x, t)\),

4. \(M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)\),

5. \(M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]\) is continuous,

\(x, y, z \in X\) and \(t, s > 0\).

The function \(M(x, y, t)\) represents the degree of nearness between \(x\) and \(y\) with respect to \(t\). We identify \(x = y\) with \(M(x, y, t) = 1\), for \(t > 0\) and \(M(x, y, t) = 0\) as \(t \rightarrow \infty\). Here afterwards, a fuzzy metric space refers a fuzzy metric space of George and Veeramani [37], unless mentioned specifically.

Lemma 1.1 ([37]). If \((X, M, *)\) be a fuzzy metric space, then \(M(x, y, \cdot)\) is non-decreasing for all \(x, y \in X\).

Definition 1.13 ([37]). Let \((X, d)\) be a metric space. Define \(a \ast b = a \cdot b\), the usual multiplication, for all \(a, b \in [0, 1]\), and let \(M_d\) be the function defined on \(X \times X \times (0, \infty)\) by

\[M_d(x, y, t) = \frac{t}{t + d(x, y)},\]

for all \(x, y \in X\) and \(t > 0\). Then \((X, M_d, \ast)\) is a fuzzy metric space called standard fuzzy metric space, and \(M_d\) is called as the standard fuzzy metric induced by the metric \(d\).

Definition 1.14 ([37]). Let \((X, M, \ast)\) be a fuzzy metric space. The open ball \(B(x, r, t)\) for \(t > 0\) with centre \(x \in X\) and radius \(r, 0 < r < 1\), is defined as

\[B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\} .\]
Define

\[
\tau_M = \left\{ A \subset X : x \in A \iff \text{there exists } t > 0 \text{ and } r, 0 < r < 1, \text{ such that } B(x, r, t) \subset A \right\}.
\]

Then \( \tau_M \) is a topology on \( X \) induced by a fuzzy metric \( M \).

The topologies induced by the metric and the corresponding standard fuzzy metric are the same. That is, if \((X, d)\) is a metric space, then the topology \( \tau_d \) induced by the metric \( d \) coincides with the topology \( \tau_{M_d} \) induced by the standard fuzzy metric \( M_d \).

**Definition 1.15** ([37]). Let \((X, M, \ast)\) be a fuzzy metric space. A sequence \( \{x_n\} \) in \( X \) converges to \( x \in X \) if and only if \( M(x_n, x, t) \to 1 \) as \( n \to \infty \), for all \( t > 0 \).

**Definition 1.16** ([37]). A sequence \( \{x_n\} \) in a fuzzy metric space \((X, M, \ast)\) is a Cauchy sequence if and only if for each \( \varepsilon \in (0, 1) \) and each \( t > 0 \) there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x_m, t) > 1 - \varepsilon \) for all \( n, m \geq n_0 \).

**Definition 1.17** ([37]). A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

**Proposition 1.1** ([38]). The metric space \((X, d)\) is complete if and only if the standard fuzzy metric space \((X, M_d, \ast)\) is complete.

**Definition 1.18** ([40]). Let \((X, M, \ast)\) be a fuzzy metric space. We say that the mapping \( f : X \to X \) is \( t \)-uniformly continuous if for each \( \varepsilon > 0 \), with \( 0 < \varepsilon < 1 \), there exists \( r > 0 \), with \( 0 < r < 1 \) such that \( M(x, y, t) \geq 1 - r \) implies \( M(f(x), f(y), t) \geq 1 - \varepsilon \), for each \( x, y \in X \) and \( t > 0 \).

**Definition 1.19** ([40]). Let \((X, M, \ast)\) be a fuzzy metric space. We say that the mapping \( f : X \to X \) is fuzzy contractive if there exists \( k \in (0, 1) \) such that

\[
\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M(x, y, t)} - 1 \right)
\]

for each \( x, y \in X \) and \( t > 0 \). Here, \( k \) is called the fuzzy contractivity ratio of \( f \).
**Proposition 1.2 ([40]).** A fuzzy contractive mapping on a fuzzy metric space is a \( t \)-uniformly continuous function.

**Proposition 1.3 ([40]).** Let \((X, d)\) be a metric space. The mapping \( f : X \rightarrow X \) is a contraction on the metric space \((X, d)\) with contractivity ratio \( k \) if and only if \( f \) is fuzzy contractive, with fuzzy contractivity ratio \( k \), on the standard fuzzy metric space \((X, M_d, \ast)\), induced by \( d \).

**Definition 1.20 ([40]).** Let \((X, M, \ast)\) be a fuzzy metric space. We say that the sequence \( \{x_n\} \) in \( X \) is fuzzy contractive if there exists \( k \in (0, 1) \) such that

\[
\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k \left( \frac{1}{M(x_n, x_{n+1}, t)} - 1 \right)
\]

for each \( t > 0 \) and \( n \in \mathbb{N} \).

**Proposition 1.4 ([40]).** Let \((X, M_d, \ast)\) be the standard fuzzy metric space induced by the metric \( d \) on \( X \). Then, the sequence \( \{x_n\} \) in \( X \) is contractive in \((X, d)\) if and only if \( \{x_n\} \) is fuzzy contractive in \((X, M_d, \ast)\).

**Theorem 1.12** (Fuzzy Banach Contraction Theorem [40]). Let \((X, M, \ast)\) be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let \( f : X \rightarrow X \) be a fuzzy contractive mapping being \( k \) the contractivity ratio. Then \( f \) has a unique fixed point.

**Corollary 1.1** (Fuzzy Version of the Classical Banach Contraction Theorem [40]). Let \((X, M_d, \ast)\) be a complete standard fuzzy metric space induced by the metric \( d \) on \( X \) and let \( f : X \rightarrow X \) be a fuzzy contractive mapping. Then \( f \) has a unique fixed point.

**Definition 1.21 ([31, 89]).** A fuzzy B-contraction (Sehgal contraction) on a fuzzy metric space \((X, M, \ast)\) is a self-mapping \( f \) on \( X \) for which

\[
M(f(x), f(y), kt) \geq M(x, y, t),
\]

for all \( x, y \in X \) and \( t > 0 \), where \( k \) is a fixed constant in \((0, 1)\).
1.4.5 Intuitionistic Fuzzy Metric Space

In order to define the Intuitionistic Fuzzy HB operator, Intuitionistic Fuzzy Fractals, Multivalued Intuitionistic Fuzzy HB operator and Multivalued Intuitionistic Fuzzy Fractals, we have to state the required concepts of intuitionistic fuzzy metric spaces as follows:

**Definition 1.22 ([90]).** A binary operation \( \bigtriangleup : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is a continuous \( t \)-conorm, if \( \bigtriangleup \) satisfies the following conditions:

(a) \( \bigtriangleup \) is commutative and associative;

(b) \( \bigtriangleup \) is continuous;

(c) \( a \bigtriangleup 0 = a \) for all \( a \in [0, 1] \);

(d) \( a \bigtriangleup b \leq c \bigtriangleup d \) whenever \( a \leq c \) and \( b \leq d \), and \( a, b, c, d \in [0, 1] \).

**Remark 1.1.** If \( * \) is a continuous \( t \)-norm, it follows from the Definition 1.10 that for every \( a \in [0, 1], 0 * a \leq 0 * 1 = 0 \) and so \( 0 * a = a * 0 = 0 \).

**Remark 1.2.** If \( \bigtriangleup \) is a continuous \( t \)-conorm, it follows from the Definition 1.22 that for every \( a \in [0, 1], 1 = 0 \bigtriangleup 1 \leq a \bigtriangleup 1 \) and so \( a \bigtriangleup 1 = 1 \bigtriangleup a = 1 \).

**Definition 1.23 ([4, 41, 70]).** A 5-tuple \((X, M, N, *, \bigtriangleup)\) is said to be an intuitionistic fuzzy metric space if \( X \) is an arbitrary (non-empty) set, \( * \) is a continuous \( t \)-norm, \( \bigtriangleup \) is a continuous \( t \)-conorm and \( M, N \) are fuzzy sets on \( X^2 \times (0, \infty) \) satisfies the following conditions:

(a) \( M(x, y, t) + N(x, y, t) \leq 1 \);

(b) \( M(x, y, t) > 0 \);

(c) \( M(x, y, t) = 1 \) if and only if \( x = y \);

(d) \( M(x, y, t) = M(y, x, t) \);
(e) \( M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s); \)

(f) \( M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1] \) is continuous;

(g) \( N(x, y, t) < 1; \)

(h) \( N(x, y, t) = 0 \) if and only if \( x = y; \)

(i) \( N(x, y, t) = N(y, x, t); \)

(j) \( N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s); \)

for all \( x, y, z \in X \) and \( t, s > 0. \)

Then \((M, N, \ast, \diamond)\) or simply \((M, N)\) is called an intuitionistic fuzzy metric on \( X. \)

The functions \( M(x, y, t) \) and \( N(x, y, t) \) represent the degree of nearness and the
degree of non-nearness between \( x \) and \( y \) in \( X \) with respect to \( t, \) respectively.

**Definition 1.24** ([70]). Let \((X, d)\) be a metric space. Let \( M_d \) and \( N_d \) be the functions
defined on \( X^2 \times (0, \infty) \) by
\[
M_d(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{and} \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)},
\]
for all \( x, y \in X \) and \( t > 0. \) Then \((X, M_d, N_d, \ast, \diamond)\) is an intuitionistic fuzzy metric space,
which is called standard intuitionistic fuzzy metric space, and \((M_d, N_d)\) is called as the
standard intuitionistic fuzzy metric induced by the metric \( d. \)

**Definition 1.25** ([70]). Let \((X, M, N, \ast, \diamond)\) be an intuitionistic fuzzy metric space. The
open ball \( B(x, r, t) \) with center \( x \in X \) and radius \( r, 0 < r < 1, \) with respect to \( t > 0, \) is
defined as
\[
B(x, r, t) = \left\{ y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r \right\}.
\]

Define
\[
\tau(M, N) = \{A \subset X : \text{for each } x \in A, \exists t > 0 \text{ and } r \in (0, 1) \text{ such that } B(x, r, t) \subset A\}.
\]
Then \( \tau_{(M,N)} \) is a topology on \( X \) induced by an intuitionistic fuzzy metric \( (M,N) \).

The topologies induced by the metric and the corresponding standard intuitionistic fuzzy metric are the same.

**Definition 1.26** ([4, 71, 87]). Let \( (X, M, N, *, \diamond) \) be an intuitionistic fuzzy metric space. We say that \( X \) is complete if every Cauchy sequence is convergent.

**Definition 1.27** ([4, 71, 87]). Let \( (X, M, N, *, \diamond) \) be an intuitionistic fuzzy metric space. We say that \( X \) is compact if every sequence contains a convergent subsequence.

**Definition 1.28** ([4, 71, 87]). Let \( (X, M, N, *, \diamond) \) be an intuitionistic fuzzy metric space. We say that \( X \) is precompact if for each \( 0 < r < 1 \) and \( t > 0 \) there exists a finite subset \( A \) of \( X \) such that \( X = \bigcup_{a \in A} B(a, r, t) \).

**Proposition 1.5** ([62]). The metric space \( (X, d) \) is complete if and only if the standard intuitionistic fuzzy metric space \( (X, M_d, N_d, *, \diamond) \) is complete.

**Theorem 1.13** ([87]). Let \( (X, M, N, *, \diamond) \) be an intuitionistic fuzzy metric space and \( A \subset X \). Then \( A \) is a precompact set if and only if every sequence in \( A \) has a Cauchy subsequence.

**Theorem 1.14** ([87]). A subset \( A \) of an intuitionistic fuzzy metric space \( (X, M, N, *, \diamond) \) is compact if and only if it is precompact and complete.

**Definition 1.29** ([62]). Let \( (X, M, N, *, \diamond) \) be an intuitionistic fuzzy metric space. We say that the mapping \( f : X \to X \) is intuitionistic fuzzy contractive if there exists \( k \in (0, 1) \) such that

\[
\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M(x, y, t)} - 1 \right)
\]

and

\[
\frac{1}{N(f(x), f(y), t)} - 1 \geq \frac{1}{k} \left( \frac{1}{N(x, y, t)} - 1 \right)
\]

for each \( x, y \in X \) and \( t > 0 \).

Here, \( k \) is called the intuitionistic fuzzy contractivity ratio of \( f \).
Proposition 1.6 ([62]). Let \((X, d)\) be a metric space. The mapping \(f : X \to X\) is contraction on the metric space \((X, d)\) with contractivity ratio \(k\) if and only if \(f\) is intuitionistic fuzzy contractive, with intuitionistic fuzzy contractivity ratio \(k\), on the standard intuitionistic fuzzy metric space \((X, M_d, N_d, *, \bowtie)\), induced by \(d\).

Theorem 1.15 (Intuitionistic Fuzzy Version of the Classical Banach Contraction Theorem [62]). Let \((X, M_d, N_d, *, \bowtie)\) be a complete standard intuitionistic fuzzy metric space induced by the metric \(d\) on \(X\) and let \(f : X \to X\) be an intuitionistic fuzzy contractive mapping. Then \(f\) has a unique fixed point.

Definition 1.30 ([4]). Let \((X, M, N, *, \bowtie)\) be an intuitionistic fuzzy metric space. We say that the mapping \(f : X \to X\) is intuitionistic fuzzy B-contraction (intuitionistic fuzzy Sehgal contraction) if there exists \(k \in (0, 1)\) such that

\[
M(f(x), f(y), kt) \geq M(x, y, t) \quad \text{and} \quad N(f(x), f(y), kt) \leq N(x, y, t)
\]

for all \(x, y \in X\) and \(t > 0\). Here, \(k\) is called the intuitionistic fuzzy B-contractivity ratio of \(f\).

Definition 1.31 ([4]). Let \((X, M, N, *, \bowtie)\) be an intuitionistic fuzzy metric space. We say that the mapping \(f : X \to X\) is intuitionistic fuzzy Edelstein contraction if

\[
M(f(x), f(y), \cdot) > M(x, y, \cdot) \quad \text{and} \quad N(f(x), f(y), \cdot) < N(x, y, \cdot)
\]

for all \(x, y \in X\) such that \(x \neq y\).

Theorem 1.16 (Intuitionistic Fuzzy Banach Contraction Theorem [4]). Let \((X, M, N, *, \bowtie)\) be a complete intuitionistic fuzzy metric space. Let \(f : X \to X\) be an intuitionistic fuzzy B-contraction. Then \(f\) has a unique fixed point.

Theorem 1.17 (Intuitionistic Fuzzy Edelstein Contraction Theorem [4]). Let \((X, M, N, *, \bowtie)\) be a compact intuitionistic fuzzy metric space. Let \(f : X \to X\) be an intuitionistic fuzzy Edelstein contraction. Then \(f\) has a unique fixed point.
1.4.6 Monofractal Dimensions

A dimension is a measure of the prominence of irregularities of a set when viewed at very small scales. It contains much information about the geometrical properties of a set.

**Definition 1.32** (Topological Dimension [29]). The Topological Dimension of a set is always an integer and is 0, if it is totally disconnected; 1, if each point has arbitrarily small neighbourhoods with boundary of dimension 0; and so on.

There are various numbers, associated with fractals, generally referred to as fractal dimensions, which can be used to compare them. It helps us to have a subjective feeling of how densely the fractal occupies the metric space in which it lies. Fractal dimensions are objective means for comparing fractals. Fractal dimensions are important because they can be defined in connection with real-word data, and they can be measured approximately by means of experiments. It is possible to define the ‘fractal dimension’ of a set in many ways, some satisfactory and others less. Usually fractal dimension is referred as the Hausdorff Dimension, which is defined as follows.

**Definition 1.33** (Hausdorff Dimension [13, 29]). If $U$ is any non-empty subset of $n$-dimensional Euclidean space, $\mathbb{R}^n$, the **diameter** of $U$ is defined as $|U| = \sup\{|x - y| : x, y \in U\}$, i.e., the greatest distance apart of any pair of points in $U$. If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most $\delta$ that cover $F$, i.e., $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 < |U_i| \leq \delta$ for each $i$, we say that $\{U_i\}$ is a $\delta$-cover of $F$.

Suppose that $F$ is a subset of $\mathbb{R}^n$ and $s$ is a non-negative number. For any $\delta > 0$ we define

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta \text{- cover of } F \right\}.$$  

As $\delta$ decreases, the class of permissible covers of $F$ is reduced. Therefore, the infimum $\mathcal{H}_\delta^s(F)$ increases, and so approaches a limit as $\delta \to 0$. Thus,

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F).$$
This limit exists for any subset $F$ for $\mathbb{R}^n$, though the limiting value can be 0 or $\infty$. We call $\mathcal{H}^s(F)$ as the $s$-dimensional Hausdorff measure of $F$.

Then the Hausdorff Dimension or Hausdorff-Besicovitch Dimension of $F$ is defined as,

$$
dim_H(F) := \inf \left\{ s : \mathcal{H}^s(F) = 0 \right\} = \sup \left\{ s : \mathcal{H}^s(F) = \infty \right\}
$$

so that

$$
\mathcal{H}^s(F) = \begin{cases} 
\infty & \text{if } s < \dim_H(F) \\
0 & \text{if } s > \dim_H(F)
\end{cases}
$$

If $s = \dim_H(F)$, then $\mathcal{H}^s(F)$ may be zero or infinite, or may $0 < \mathcal{H}^s(F) < \infty$.

Hausdorff dimension has the advantage of being defined for any set, and is mathematically convenient, as it is based on measures, which are relatively easy to manipulate. A major disadvantage is that in many cases it is difficult to calculate the dimension by computational methods. In order to make the computation easier, the fractal dimension is defined below by applying the Box-Counting method.

**Definition 1.34** (Box-Counting Dimension [13, 29]). Let $F$ be any non-empty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most $\delta$ which can cover $F$. The lower and upper box-counting dimensions of $F$ respectively are defined as

$$
\underline{\dim}_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta};
$$

$$
\overline{\dim}_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.
$$

If these are equal we refer to the common value as the **Box-Counting Dimension or Box Dimension** of $F$,

$$
\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.
$$
1.4.7 Multifractal Dimensions

Renyi Entropy

Renyi Entropy [82, 91] played a significant role in the Information theory. Renyi Entropy, a generalization of Shannon entropy, is one of the family of functionals for quantifying the diversity, uncertainty or randomness of a given system. It was introduced by Alfred Renyi [82]. Renyi Entropy is also known as generalized entropy of a given probability distribution.

The Renyi Entropy of order $q$, where $q \geq 0$ and $q \neq 1$, of the given probability distribution is defined as

$$RE_q = \frac{1}{1-q} \log_2 \left( \sum_{i=1}^{N} p_i^q \right)$$  \hspace{1cm} (1.1)$$

where $p_i \in [0, 1]$ are the probabilities of the random variable which takes the values $x_1, x_2, ..., x_N$.

If the probabilities are all the same then all the Renyi Entropies of the distribution are equal, with $RE_q = \log_2 N$. Otherwise the entropies are decreasing as a function of $q$.

Some Particular Cases of Renyi Entropy

- If $q = 0$, then

$$RE_0 = \log_2 N$$

which is called the Hartley entropy of the given probability distribution.
• Note that if \( q \) approaches 1, it can be shown that \( RE_q \) converges to \( RE_1 \), which is defined as

\[
RE_1 = - \sum_{i=1}^{N} p_i \log_2 p_i
\]

which is the well-known entropy called Shannon entropy of a discrete probability distribution.

• Sometimes Renyi Entropy refers only to the case \( q = 2 \),

\[
RE_2 = -\log_2 \left( \sum_{i=1}^{N} p_i^2 \right)
\]

• As \( q \rightarrow -\infty \), the limit exists as

\[
RE_{-\infty} = -\log_2 \left( \min_{i=1,2,...,N} p_i \right)
\]

which is called Max-entropy, because it is the largest value of \( RE_q \).

• Similarly, as \( q \rightarrow \infty \), the limit exists as

\[
RE_{\infty} = -\log_2 \left( \max_{i=1,2,...,N} p_i \right)
\]

and this is called Min-entropy, because it is the smallest value of \( RE_q \).

Generalized Fractal Dimensions for Fractal Signals

The Renyi Entropies are important in Non-linear Analysis and Statistics as indices of uncertainty or randomness. They also lead to a spectrum of indices of Fractal Dimension (Renyi Fractal Dimensions or Generalized Fractal Dimensions). Grassberger [39] and Hentschel et al. [44] systematically developed the multifractal theory, which is based upon Generalized Fractal Dimensions (GFD). In this section, we describe the GFD Method for fractal signals [39, 44].

Now we define a probability distribution of a given fractal signals by the following construction.
The total range of the signal is divided into $N$ intervals (bins) such that

$$N = \frac{V_{\text{max}} - V_{\text{min}}}{r}$$

where $V_{\text{max}}$ and $V_{\text{min}}$ are the maximum & the minimum values of the signal received in the experiments, respectively; and $r$ is the uncertainty factor, that may be depend on the measuring device used to record the experimental signal.

Now the probability that the signal passes through the $i^{\text{th}}$ interval of length $r$ is given by

$$p_i = \lim_{N \to \infty} \frac{N_i}{N}, \quad i = 1, 2, ..., N$$

where $N_i$ is the number of times the signal passes through the $i^{\text{th}}$ interval of length $r$.

Then, the Renyi Fractal Dimensions or Generalized Fractal Dimensions (GFD) of order $q \in (-\infty, \infty)$ such that $q \neq 1$ for the known probability distribution, denoted by $D_q$, can be defined as

$$D_q = \lim_{r \to 0} \frac{1}{q-1} \log_2 \sum_{i=1}^{N} p_i^q \log_2 r.$$  \hspace{1cm} (1.2)

Here $D_q$ is defined in terms of generalized Renyi Entropy. Note that $D_q = D_0$, for all $q$ for a Self-similar signal with probabilities $p_i = 1/N$, for all $i$. Also observe that $D_q = D_0 = 0$, for all $q$ for a constant signal because all probabilities except one equal to zero, whereas the exceptional probability value is one. For all $q$, we have $D_q > 0$. It can be shown that if $q_1 < q_2$, $D_{q_1} \geq D_{q_2}$ such that $D_q$ is a monotone decreasing function of $q$. In particular $D_{-\infty} \geq D_{\infty}$. Here $D_{\infty}$ corresponds to the region in the time series where the measure is most concentrated, while $D_{-\infty}$ corresponds to the measure which is most ratifies. In the sense $D_q$ describes the nonuniformity property of a fractal object [51].
For two probability distributions, the signal with a larger value of Fractal Dimension of a given order corresponds to the presence of sharper spikes (less expected values of the signal) than in the signal for which the value of the Fractal Dimension of the same order is less. Signals with a wider range of Fractal Dimensions can be named as more Fractal than signals with range of Fractal Dimensions is narrower. Note that signals with the zero range of Fractal Dimensions are Self-similar Fractals. On the other hand, the range of Fractal Dimensions is a measure associated with the range of frequencies in the signal. The Steeper spectra represent the signals in which more unexpected values occur, whereas Flatter spectra represent those signals in which less unexpectedness occurs [50].

Some Special Cases of Generalized Fractal Dimensions

- If \( q = 0 \), then
  \[
  D_0 = -\frac{\log_2 N}{\log_2 r}
  \]
  which is nothing but the Fractal Dimension.

- As \( q \to 1 \), \( D_q \) converges to \( D_1 \), which is given by
  \[
  D_1 = \lim_{r \to 0} \sum_{i=1}^{N} p_i \frac{\log_2 p_i}{\log_2 r}.
  \]
  This is called as Information Dimension.

- If \( q = 2 \), then \( D_q \) is called the Correlation Dimension.

- There are two limit cases when \( q = -\infty \) and \( q = \infty \), which is given as
  \[
  D_{-\infty} = \lim_{r \to 0} \frac{\log_2 (p_{\min})}{\log_2 r}
  \]
  \[
  D_{\infty} = \lim_{r \to 0} \frac{\log_2 (p_{\max})}{\log_2 r}
  \]
  where
  \[
  p_{\min} = \min\{p_1, p_2, ..., p_N\},
  \]
  \[
  p_{\max} = \max\{p_1, p_2, ..., p_N\}.
  \]
Range of Generalized Fractal Dimensions

The two limit cases, $D_{-\infty}$ and $D_{\infty}$, define the Range of Generalized Fractal Dimensions of a given Fractal Time Series as

$$ R_{GFD} = D_{-\infty} - D_{\infty}. $$

(1.3)

1.5 Outline of the Thesis

The thesis is organized into seven different chapters including this introduction chapter. The chapter-wise outline of the thesis is as follows:

Chapter 1

This chapter is introductory in nature. A brief description about fractal analysis, fuzzy metric spaces, intuitionistic fuzzy metric spaces, fractal dimensions and multifractal dimensions are given. Also it highlights the necessary definitions, results and techniques which are used in the thesis.

Chapter 2

The purpose of this chapter is to initiate and construct the Fisher fractals, fuzzy fractals and multivalued fuzzy fractals by generalizing the Hutchinson-Barnsley theory generated by the Fisher contractions, fuzzy contractions and multivalued fuzzy contractions respectively. For that, the existence and uniqueness theorem of fractals and multivalued fractals are proved with respect to Fisher contractions in the metric space, fuzzy contractions and multivalued fuzzy contractions in the standard fuzzy metric spaces. In addition to that, the results such as Collage theorem, Falling Leaves theorem in the standard fuzzy metric spaces and the fuzzy B-contraction properties of the Hutchinson-Barnsley operator on the fuzzy hyperspace are discussed.
Chapter 3

This chapter introduces the concepts of fractals and multivalued fractals in intuitionistic fuzzy metric spaces generated by the dynamical system of intuitionistic fuzzy contractions and multivalued intuitionistic fuzzy contractions. The existence and uniqueness theorem of fractals and multivalued fractals are proved in the standard intuitionistic fuzzy metric spaces. Besides that, the results such as Collage theorem, Falling Leaves theorem in the standard intuitionistic fuzzy metric spaces are analyzed.

Chapter 4

In this chapter, we prove the intuitionistic fuzzy B-contraction and intuitionistic fuzzy Edelstein contraction properties of the HB operator on the intuitionistic fuzzy hyperspace with respect to the Hausdorff intuitionistic fuzzy metrics. Also we discuss about the relationships between the Hausdorff intuitionistic fuzzy metrics on the intuitionistic fuzzy hyperspaces. The proposed theorems generalize and extend some recent results related with Hutchinson-Barnsley operator in the metric spaces. Also we propose a generalization of Hutchinson-Barnsley theory in order to define and construct the intuitionistic fuzzy fractals and multivalued intuitionistic fuzzy fractals through iterated function system of finite number of intuitionistic fuzzy B-contractions and intuitionistic fuzzy Edelstein contractions. The existence and uniqueness theorem of fractals and multivalued fractals in a complete intuitionistic fuzzy metric space and a compact intuitionistic fuzzy metric space are proved by using the intuitionistic fuzzy Banach contraction theorem and the intuitionistic fuzzy Edelstein contraction theorem, respectively. Some interesting results are exhibited based on the fractals and multivalued fractals in the sense of intuitionistic fuzzy B-contractions and intuitionistic fuzzy Edelstein contractions.
Chapter 5

Identification of abnormality in EEG signals is the vast area of research in the neuroscience. In this chapter, the three forms of GFD namely Modified GFD, Improved GFD and Advanced GFD are newly framed in order to discriminate the healthy and epileptic EEG time signals. Also a novel method based on wavelet denoising is proposed to improve the pre-processing analysis of EEG signals and investigated the identification of healthy and epileptic EEG signals by using multifractal measures.

The designed multifractal measures showed the significant differences among normal, interictal and ictal EEGs. The proposed idea is demonstrated with high accuracy through the suitable graphical methods and statistical tools called one-way Analysis of Variance (ANOVA) test with Box Plot. It is shown that the designed methods perform significantly in the detection of epileptic seizures in EEG signals. The EEG data are further tested for linearity by using Normal Probability Plot and proved that Epileptic EEG has significant nonlinearity whereas Healthy EEG distributed normally and similar to Gaussian Linear Process. Finally the fuzzy multifractal theory for signals is developed in order to define the Fuzzy Generalized Fractal Dimensions by introducing fuzzy membership function in classical GFD method and it is used for the classification of chaotic behaviours in the fractal waveforms.

Chapter 6

One of the most eye-catching problems in image processing is the recognition of noise free and noisy images. GFD is the measure of complexity, irregularity and chaotic nature in the fractal or corrupted images. This chapter invokes the multifractal analysis by using the GFD in the recognition of noise less & noisy gray scale images and color images with RGB components.
In this technique, median filter is used to denoise the images corrupted by the salt and pepper noise. It is shown that GFD measure discriminate the noisy and noise free images very accurately in both gray scale and RGB color images; and also demonstrated the classification by the graphical methods for the standard images. Also the fuzzy multifractal theory for gray scale and RGB color images is developed in order to define the Fuzzy Generalized Fractal Dimensions by introducing fuzzy membership function in the classical GFD method.

Chapter 7

Finally, the fuzzy fractal analysis and its applications are summarized in this chapter. Several directions for future research are also mentioned.