CHAPTER 1

INTRODUCTION
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1.1. GENERAL INTRODUCTION:

Model refers to a set of functional or structural relationships between two or more characteristics. These characteristics may be either measuremental or non-measuremental in nature. The measuremental characteristics which assume different values in a specified range are known as variables. Generally, a set of functional relationships between two or more variables may be expressed in terms of mathematical equations, which is called a mathematical model. This model may be either in the form of a set of linear equations (linear model) or in the form of a set of nonlinear equations (nonlinear model). By introducing a random error variable or a random disturbance term, the mathematical model becomes a statistical model or a regression model. Hence one may have either linear regression model or nonlinear regression model.

A great deal of research in mathematical modelling has been directed to the nonlinear modelling and establishing functional relationships among different variables. Nonlinear models have a wide number of applications in physical, biological and social sciences, business, economics, engineering and management sciences. Now-a-days, nonlinear model building is new and very fascinating filed of research in Applied mathematical sciences. In Mathematics, or in any other scientific discipline, a research worker is certainly facing the problem of formulation of a nonlinear model. A large number of nonlinear models have been specified in the literature and successfully applied to different situations in the real world relating to several research problems in the various fields of Applied mathematics. However, there are a large number of situations, which have not yet been nonlinearly modelled, because of the situations may be complex or they are mathematically or statistically intractable.

1.2. STATEMENT OF PRESENT RESEARCH STUDY:

Nonlinear model building has become an increasing important powerful tool in the subjects of Applied mathematics and Statistics. In recent years, the popularity of applications of nonlinear models has dramatically been rising up. To apply the nonlinear models effectively, mathematicians and statisticians need the awareness and usage of several diagnostic measures for detecting the violations of the assumptions about the model; specification of the nonlinear model; detecting the presence of outliers; performance and validation of the nonlinear models.

A large number of problems in the nonlinear model building are concerned with the inferential aspects including estimating the parameters and testing the hypothesis about the parameters of the nonlinear regression models. Now-a-days, efficient estimation of the nonlinear regression models has received little attention. Several researchers in Applied mathematics are very often interested in inferential aspects of the nonlinear regression models. These inferential aspects have been studied intensively for the last three decades. The nonlinear inferential methods and the error assumptions are generally analogous to those made for the linear regression models. The literature on nonlinear methods of estimation has been grown enormously for the past four decades.
Though a considerable amount of research has been developed on theoretical and applications relating to inferential aspects of nonlinear models by Americans as well as British Mathematicians and Statisticians, not much research work is carried out in the context of India.

In the present research work, an attempt has been made to develop some new inferential methods for nonlinear regression models.

1.3. TYPES OF NONLINEAR REGRESSION MODELS:

Nonlinear regression analysis is a powerful method for analyzing data described by models which are nonlinear in parameters. Generally, a researcher has a mathematical expression which relates the dependent variable to the independent variables and these models are nonlinear in parameters. Under these cases, usually the linear regression analysis can be extended, which introduces considerable complexity.

Generally, a nonlinear model refers to regression function which is nonlinear either in predictor variables or in the unknown parameters (regression coefficients) or in both predictor variables and parameters.

Nonlinear models can be broadly classified into two parts namely:

(A) Nonlinear models which are nonlinear in regressors but linear in parameters

(B) Nonlinear models which are nonlinear in parameters

(A) NONLINEAR MODELS WHICH ARE NONLINEAR IN REGRESSORS BUT LINEAR IN PARAMETERS.

A general form of nonlinear model which is linear in parameters but nonlinear in predictor variables is specified by

\[ Y_i = \beta_0 + \beta_1 Z_{u1} + \beta_2 Z_{u2} + \ldots + \beta_p Z_{up} + \varepsilon_i; \quad i = 1, 2, \ldots, n \]  

(1.3.1)

Here, \( Z_{uj}, j = 1, 2, \ldots, p \) refers to any function of the predictor variables say \( X_1, X_2, \ldots, X_k \).
(B) NONLINEAR MODELS WHICH ARE NONLINEAR IN PARAMETERS.

The nonlinear model whose form differs with the form (1.3.1) is called nonlinear model with nonlinear in the parameters. These models may be again two types:

(i) Nonlinear models that are intrinsically linear
(ii) Nonlinear models that are intrinsically nonlinear

The nonlinear model which can be expressed in the form (1.3.1) by using suitable transformation of the variables is called nonlinear model that is intrinsically linear.

For instance, \( Y_i = \beta_0 X_i^a X_{i1}^{a_1} \ldots X_{in}^{a_n} e_i; \ i = 1, 2, \ldots, n \)

is a nonlinear model that is intrinsically linear.

The nonlinear model which cannot be expressed in the form (1.3.1) by taking any transformation is called nonlinear model that is intrinsically nonlinear.

For instance,
\[
Y_i = \beta_0 X_i^a X_{i1}^{a_1} \ldots X_{in}^{a_n} + e_i, i = 1, 2, \ldots, n
\]

is a nonlinear model that is intrinsically nonlinear

1.4. SPECIFICATION OF NONLINEAR REGRESSION MODEL

Generally, the class of nonlinear models that are nonlinear in parameters allows the mean of these explained variable to be expressed in terms of any function of the explanatory variables and the parameters.

The model becomes
\[
Y_i = f(X_{i1}, X_{i2}, \ldots, X_{in}; \beta_0, \beta_1, \ldots, \beta_p) + e_i; i = 1, 2, \ldots, n
\]
where

\( Y \) is explained variable

\( X_1, X_2, \ldots, X_k \) are \( k \) explanatory variables

\( \beta_1, \beta_2, \ldots, \beta_p \) are \( p \) parameters

\( \varepsilon \) is random error variable

\( n \) is the number of observations on each variable

‘\( f \)’ is the nonlinear function relating \( E(Y) \) to the explanatory variables

The nonlinear regression model in matrix notation is specified by

\[
Y = f(X, \beta) + \varepsilon
\]

where \( Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \), \( X = \begin{pmatrix} (X_{ik})_{i=1}^{n} \end{pmatrix} \), \( \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} \)

\[ \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} \text{ and } f(X, \beta) = \begin{bmatrix} f(X_{11}, \beta) \\ f(X_{12}, \beta) \\ \vdots \\ f(X_{nk}, \beta) \end{bmatrix} \quad j = 1, 2, \ldots, k.\]

Assumptions on Nonlinear Regression Model:

The random errors \( \varepsilon_i \)'s are i. i. d's with \( \varepsilon \sim (0, \sigma^2 I_n) \) but the exact form of distribution is unknown.

1.5 ESTIMATION OF PARAMETERS OF NONLINEAR MODELS:

Generally, optimal estimators for the parameters of nonlinear model that is intrinsically linear, can be obtained by applying Ordinary Least Squares (OLS) estimation method to the transformed model.
The OLS estimation fails to give optimal estimators for the parameters of nonlinear model that is intrinsically nonlinear. However, iterative OLS estimation method can be applied to estimate parameters of this model.

The various mathematical methods in the numerical analysis can be applied to study the inferential aspects of estimators for the parameters in the nonlinear regression models. Some of the inferential questions with regard to the nonlinear models are still unanswered and offered a good research opportunity for the theoretical mathematicians and statisticians.

Another important problem commonly seen with data that is best fitted by a nonlinear model than with data that can be fitted by a linear model. It is known as the problem of Mis-specification of the model.

The literature on numerical techniques for fitting nonlinear models has grown enormously in the past three decades.

There are mainly three important subclasses of model-fitting situations.

(a) Predictive Models: For each of n observations, the model suggests a predicted value, depending on the parameter $\beta$ and possibly on other data values. The most general means of expressing this is in terms of n functions $f_1(\beta), f_2(\beta), \ldots, f_n(\beta)$ such that $f_i(\beta)$ predicts $y_i$.

(b) Probability Models: The model assigns, given values for $Y$ and $\beta$, a probability element, say in terms of a probability density function $P(Y, \beta)$.

(c) Transformation Models: In this case, the data as a whole are replaced by transformed values $U = (u_1, u_2, \ldots, u_n)$, where the transformation depends on a parameter $\beta$, and the objective is to find $U$ with certain desired properties.
One may fit a nonlinear regression model to a set of data, which involves three main steps:

(i) to specify an appropriate model for data and a criterion for choosing good estimators for the parameters of the model;
(ii) to choose or write a fitting algorithm, write a subprogram which defines the model to the algorithm, and run the algorithm with the model and data;
(iii) to assess the results of run, both computationally in terms of the numerical results and statistically in terms of the implications of the fit.

Computational methods for fitting nonlinear models have developed considerably in the last two decades.

1.6. NONLINEAR METHODS OF ESTIMATION:

Some important estimation methods available in the literature for fitting nonlinear models are given by:

(i) Nonlinear least squares estimation method.
(ii) Taylor series expansion method or linear approximation method of nonlinear estimation.
(iii) Maximum likelihood method of nonlinear estimation.
(iv) Newton-Raphson method of nonlinear estimation.
(v) Steepest Descent method of nonlinear estimation.
(vi) Steepest Ascent method of nonlinear estimation.
(vii) Gauss-Newton method of nonlinear estimation.
(viii) Method of scoring for nonlinear estimation.
(ix) Quadratic Hill-Climbing Method of nonlinear estimation.
(x) Conjugate Gradient methods of nonlinear estimation.
1.7 INFERENTIAL ASPECTS OF NONLINEAR MODELS:

Some important inferential aspects of nonlinear models given in the
literature are:

(i)    Theil's test for linearity of regression.
(ii)   Test for the specification of error in nonlinear model.
(iii)  Goldfeld and Quandt likelihood ratio test.
(iv)   Luch test based on Box and Cox Transformations.
(v)    Specification of Seemingly unrelated nonlinear regression equations
        model.
(vi)   Fitting of Gompertz growth model with additive error.
(vii)  Joint confidence regions on a vector of parameters of nonlinear models.
(viii) Nonlinear models involving correlated residuals.
(ix)   Nonlinear models involving Autocorrelated errors.
(x)    Likelihood ratio test for specification of nonlinear model.
(xi)   $R^2$ measure based on likelihood ratio statistic for nonlinear model.
(xii)  Test for autocorrelation in nonlinear model.
(xiii) White's approximate test for nonlinear model.
(xiv)  Robust tests for nonlinear models.
(xv)   Nonlinear dynamic structures.
(xvi)  Cameron and Windmeijer $R^2$ measure of goodness of fit for nonlinear
        models.
(xvii) Improved Likelihood Ratio Statistics for Exponential family nonlinear
        models.
(xviii) Prediction for a class of Dynamic nonlinear models.
(xix)  Bias in the maximum likelihood estimation of nonlinear models.
1.8 OBJECTIVES OF THE PRESENT RESEARCH STUDY

The main aim of the present research study is to develop some new procedures with reference to the mathematical and statistical aspects of nonlinear regression models.

The specific objectives of the present research work are:

(i) to describe the various mathematical and statistical aspects of nonlinear models;

(ii) to brief about the existing literature on inferential aspects of nonlinear regression models;

(iii) to brief about the existing literature on applications of nonlinear regression models;

and (iv) to develop some new inferential procedures of estimation and testing the hypotheses on parameters of nonlinear regression models.

1.9 ORGANIZATION OF PRESENT RESEARCH STUDY

The organization of the present research work itself shows how the objectives of the research study are achieved within the described framework.

Chapter-1 is an introductory one. It contains the general introduction about the mathematical modeling and nonlinear regression models. It states clearly the problem besides the objectives of the present research study. It also brings out various types of nonlinear regression models and methods of estimation of their parameters together with organization of the study and chapter scheme.

Chapter -2 deals with the mathematical and statistical aspects of linear models. It explains different types of linear mathematical model such as simultaneous, piecewise and transformed linear model besides the fitting of transformed linear model. The linear statistical model has been specified along with its crucial assumptions and least squares estimation of parameters. Gauss-Markoff theorem and Aitken’s theorem for linear estimation have been explained. Various problems of linear statistical models by violating the assumptions have been given in this chapter.
Chapter -3 briefs about the mathematical aspects of nonlinear models by discussing some important techniques in Numerical analysis. Solutions of Algebraic, Transcendental, Simultaneous nonlinear equations have been presented in this chapter.

Chapter - 4 gives a brief about the statistical aspects of nonlinear models. The general nonlinear regression model has been specified along with its assumptions. Also, three kinds of nonlinear regression models have been specified besides the presentation of various nonlinear methods of estimation such as nonlinear least squares, Maximum likelihood and Numerical methods of estimation.

Chapter 5 describes some applications of nonlinear regression models. Sampling distribution of nonlinear least squares estimator has been given for testing general nonlinear hypothesis on parameters of nonlinear regression model. Some important nonlinear production function models namely, Cobb-Douglas, Constant Elasticity of Substitution (CES), Generalized Cobb-Douglas production function models have been specified along with their properties. The methods for estimating the parameters of aforementioned production functional models have been explained in this chapter.

Chapter -6 proposes some new inferential procedures pertaining the both estimation and testing of hypothesis aspects of parameters of nonlinear regression models. Iterative method of estimation of nonlinear regression model has been developed by using nonlinear least squares estimator. Nonlinear studentized and predicted residuals have been defined for testing of nonlinear hypothesis on parameters of nonlinear regression model.

The problem of heteroscedasticity with reference to nonlinear regression model has been discussed and a test for heteroscedasticity has been developed by using Iterative Nonlinear Least Squares Internally Studentized Residuals.
1.10. **CHAPTER SCHEME**

The contents of the present research work have been arranged under the following heads:

- **Chapter - 1**: Introduction
- **Chapter - 2**: Mathematical and Statistical Aspects of Linear Models
- **Chapter - 3**: A Brief about Mathematical Aspects of Nonlinear Regression Models
- **Chapter - 4**: A Brief about Statistical Aspects of Nonlinear Regression Models
- **Chapter - 5**: Some Applications of Nonlinear Models
- **Chapter - 6**: New Inferential Procedures for Nonlinear Models
- **Chapter - 7**: Conclusions

**Bibliography**