CHAPTER 1

INTRODUCTION
A summary of ideas on chaos, dynamical systems and fractal dimensions relevant to time series analysis are presented in this chapter.

1.1 MEANING OF CHAOS

The laws of science aim at relating cause and effect and thereby making predictions possible. For example, based on the laws of gravitation, eclipses can be predicted thousands of years in advance. But there are other natural phenomena like weather, bacterial production, flow of a mountain stream, fluid turbulence, Brownian motion, roll of a dice etc. that are unpredictable, though they obey the same laws of Physics. Since there is no clear relation between cause and effect, such phenomena are said to have random elements. It was believed until recently that precise predictability can in principle be achieved, by gathering and processing sufficient amount of information, a powerful idea put forth by French mathematician Laplace in seventeenth century.

The study of deterministic time evolution has been of central interest to science since the days of Isaac Newton. In recent years, it has become clear that simple deterministic systems with only a few elements can generate a random behavior. The randomness is fundamental; gathering more information does not make it go away. Randomness generated in this way has come to be called CHAOS. A seeming paradox is that while randomness is by itself unpredictable, the deterministic behavior of randomness viz. chaos, which do not themselves involve any element of chance is predictable. Underlying chaotic
behavior there are elegant geometric forms that can create randomness in the same way as a card dealer shuffles a deck of cards or a blender mixes cake batter.

The discovery of chaos has created a new paradigm in scientific modeling. On the one hand, it implies new fundamental limits on the ability to make predictions. On the other hand, the determinism inherent in chaos implies that many random phenomena are more predictable than had been thought. Chaos allows us to find order amidst disorder.

One of the characteristics of chaos is the property of extreme sensitive dependence on initial conditions. In principle, the future is completely determined by the past, but in practice arbitrarily small uncertainties in the state of a system are amplified in time so that even though the behavior is predictable in the short term, it is unpredictable in the long term, a view put forth by Poincaré in the 19th century.

1.2 CHAOS AND DYNAMICAL SYSTEMS

Chaos emerges from the larger framework of the theory of dynamical systems. A dynamical system consists of two parts: The idea of a state (the variable(s) used to describe the system) and a dynamic (the rule(s) that describe how the system evolves with time). The evolution of the system may be visualized in a phase or state space, a construct whose coordinates are the components of the state.

A dynamical system's temporal evolution may happen in either continuous time described by a flow (e.g. pendulum) or by a discrete —time
mapping (e.g. number of insects born each year in a specific area, time interval between drops from a dripping faucet).

It is possible to distinguish between two types of dynamical systems, depending on whether or not the system loses energy due to friction, whenever the status of the system changes with time. The presence of friction characterizes dissipative dynamical systems as opposed to conservative systems, where there is no loss of energy. A dissipative system always approaches an asymptotic or limiting state of motion over time.

1.3 ATTRACTORS

The usefulness of the state space picture lies in its ability to represent the behavior of the system in geometric form. Consider, for example the simple pendulum. Only two state variables are needed to determine the motion: The initial position and velocity. Its state is thus a point in a plane whose coordinates are position and velocity. Newton’s law provides the rule expressed as a differential equation that describes how the state evolves. As the pendulum swings back and forth the state moves along an orbit or path in the plane. In the case of a frictionless pendulum, the orbit is a loop; for the pendulum with friction the orbit spirals down to a point as the pendulum comes to rest. This point does not move— it is a fixed point— and since it attracts nearby orbits, it is known as an attractor. If the pendulum is perturbed from its position of rest, it returns to the same fixed-point attractor. Any system that comes to rest with the passage of time can be characterized by a fixed-point in the state space. This is an example of a very general phenomena where losses due to friction causes orbits
to be attracted to a smaller region of state space with a lower dimension. Any such region is called an attractor.

Some systems do not come to rest in the long term but instead cycle periodically through a sequence of states. An example is the pendulum clock in which the energy lost through friction is replaced by a mainspring of weights. The pendulum repeats the same motion over and over again. In the state space such a motion corresponds to a cycle or periodic orbit. No matter how the pendulum is set swinging, the cycle approached in the long term is the same. Such attractors are known as limit cycle attractors.

The next most complicated form of attractor is the torus. The shape describes the motion made up of two independent oscillations, sometimes called quasi-periodic motion. The orbit winds around the torus in state space, one frequency determines how fast the orbit circles the “doughnut” in the short direction, and the other regulates how fast the orbit circles the long way around. Attractors may also be higher dimensional tori since they might represent the combination of more than two frequencies of oscillation.

All these attractors are well behaved; their long-term behavior can be forecast as accurately as possible.

A chaotic or strange attractor corresponds to non-periodic and unpredictable motions over long times and have a more complicated geometric form. This behavior is due to the complex orbits generated by simple stretching and folding operation, which takes place in the state space. A chaotic attractor has a sensitive dependence on initial conditions—nearby orbits on the attractor
separate exponentially fast with time. In 1963, Edward Lorenz observed the first chaotic attractor known as Lorenz attractor.

The following Fig 1(a) shows the different attractor types in a three-dimensional state space.

![Fig 1(a)](image)

**Fixed-point**  **limit cycle**

**torus**  **Strange attractor**

1.4 NONLINEAR MODELS AND CHAOS

Any deterministic model will be potentially capable of describing chaotic behavior only if non-linearity is present in the system.

The following examples motivate this idea.

*Example 1:*

Consider a system, which is observed at distinct points in time, say \( t=0,1, \ldots \) described by a single variable \( x_t \) that takes values in unit interval \([0,1]\), and two successive values of the variable are related by
where \( \omega \) is a constant \( 0 < \omega < 1 \). The graph of this model is a “tent” or “hut” and the time path (trajectory) generated by it appears to be random in the open interval \((0,1)\) for almost any initial point in \((0,1)\) and for any value of \( \omega \). Any periodic time path will be unstable in the sense that even a slightest change in initial point will lead to a drastically different trajectory after some time. Fig 1(b) shows the graph of the tent model and Fig 1(c) the time path with initial value as \(.19\) and \( \omega = .4 \).

![Graph of tent model](image1.png)

![Time path](image2.png)

**Fig 1(b)**

**Fig 1(c)**

**Example 2:**

Consider the logistic model

\[
x_{t+1} = \omega x_t (1-x_t), \quad 0 \leq x_t \leq 1, \quad 0 \leq \omega \leq 4
\]

The graph of this model and the time path generated by it for different values of \( \omega \) are given in the following figures Fig 1(d), Fig 1(e), Fig 1(f), Fig 1(g) and Fig 1(h).
Fig 1(d)

Fig 1(e) \( \omega = 2.3 \)

Fig 1(f) \( \omega = 3 \)

Fig 1(g) \( \omega = 3.2 \)

Fig 1(h) \( \omega = 3.6 \)
The behavior of time path for this model is summarized as follows:

Value of $\omega$ | Nature of time path
--- | ---

i. $\omega < 3$ | single stable point

ii. $\omega = 3$ | period-doubling bifurcation
(Changes from steady state to
period 2)

iii. $\omega > 3$ | oscillates between two
values (period is 2)

iv. $\omega > 3.57$ | chaotic oscillations

A common feature of the above example is that $x_{t+1}$ is nonlinearly related to $x_t$. Any deterministic model is potentially capable of describing chaotic behavior only if non-linearity is present. If $x_t$ is linearly related to $x_{t+1}$, e.g. cobweb model, Harrod Domar growth model then the time path can only exhibit damped, stable or explosive oscillations, or stable or explosive non-oscillations.

In a nutshell, CHAOS is a type of complicated and unpredictable behavior of phenomena over time that can arise in deterministic, nonlinear, dissipative dynamical systems.

1.5 FRACTALS

Mathematical patterns delicately poised between order and chaos are strange attractors. Geometrically a strange attractor is a fractal. Fractals are sets that exhibit self-similarity at all levels of magnification. In other words, they display symmetry across scales. Fractals provide the mathematics necessary to
describe the phase space portrait of chaotic systems. Some well-known fractals are given in Fig 1(d).

Sierpinsky's triangle  
peano curve  
Mandelbrot set

Julia set  
curls  
fern

Fractal mountain  
fractal moon

Fig 1(d)

Fractal geometry is concerned with the description, classification, analysis and observation of subsets of metric spaces. The metric spaces are usually, but not always, of an inherently “simple” geometrical character; the subsets on which fractals live are typically geometrically complicated.
1.6 FRACTAL DIMENSION

A measure of chaos is the **entropy** of the motion, which roughly speaking is the average rate of stretching and folding or the average rate at which information is produced. Another statistic is the **dimension** of the attractor. This is clearly the first level of knowledge necessary to characterize its properties. It attempts to quantify a subjective feeling, which we have about how densely the fractal occupies the metric space in which it lives. It is also a lower bound on the number of essential variables needed to model the dynamics and an invariant quantity of a dynamical system.

Fractal dimension is important because they can be defined in connection with real-world data, and they can be measured approximately by means of experiments. For e.g. one can measure the fractal dimension of the coastline of Great Britain or southwest monsoon. Fractal dimension can be attached to clouds, trees, feathers, and networks of neurons in the body, dust in the air at an instant in time, the distribution of frequencies of light reflected by a flower, and the wrinkled surface of the sea during storm.

Simple attractors like point, limit cycle and torus have fractal dimension 0,1 and 2 respectively. But in between those objects with integer dimension lie complex irregular objects viz. chaotic attractors whose fractal dimension can be thought of as a measure of their irregularity.

1.7 DEFINITIONS OF DIMENSION

The various definitions of dimension are of two types. The capacity and the Hausdorff dimension require only a metric (i.e. a concept of distance) for
their definition and are referred to as “metric dimension” whose generic name is 
fractal dimension. The information dimension and correlation dimension require 
both a metric and a probability measure for their definition, and are referred to as 
“probabilistic dimension”. They are also called dimension of the natural measure. 
The Lyapunov dimension is defined in terms of dynamical properties of an 
attractor.

1.7.1 METRIC DIMENSION

1) CAPACITY DIMENSION

The basic idea is to cover an attractor A with volume 
elements (spheres, cubes etc.) each with diameter \( \varepsilon \). Let \( N(\varepsilon) \) be the 
number of volume elements needed to cover A. As \( \varepsilon \) is made smaller, 
the number of boxes increases, the sum of the volume elements 
approaches the volume A. If A is a D-dimensional manifold, then for \( \varepsilon \) small, \( N(\varepsilon) \) is inversely proportional to \( \varepsilon^D \)

i.e. \( N(\varepsilon) \sim k \varepsilon^{-D} \) for some constant \( k \).

The capacity dimension \( d_{\text{cap}} \) is the given by

\[
d_{\text{cap}} = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \quad \text{when the limit exists.}
\]

For a point, line and an area \( N(\varepsilon)=1 \), \( N(\varepsilon) \sim \varepsilon^{1} \) and \( N(\varepsilon) \sim \varepsilon^{2} \) and 
\( d_{\text{cap}} = 0, 1 \) and 2 respectively. However for fractals \( d_{\text{cap}} \) can be a non-
integer.

The construction and computation of dimension of some 
fractals are illustrated below:
Example 3: CANTOR SET

Remove the middle third of the unit interval leaving the two intervals \([0, 1/3]\) and \([2/3, 1]\). Remove the middle third of each of these intervals leaving four intervals. Repeat this process ad infinitum. The resulting set is called the middle-third cantor set (Fig 1(e)). It consists of an infinite number of points, but no length.

![Diagram of Cantor Set](image)

Fig 1(e)

Choose a covering of intervals with length \(\varepsilon = 1/3^m\). Then

\[ N(\varepsilon) = 2^m \]

and

\[ d_{\text{cap}} = \lim_{\varepsilon \to 0} \frac{\ln(2^m)}{\ln(3^m)} = \frac{m \ln 2}{m \ln 3} = 0.6309 \]

Cantor set is something more than a point but something less than an interval. It has the property of scale invariance. i.e., by the nature of the construction, the set between 0 and 1 will look precisely the same as that part of it between 0 and 1/3, if the latter is examined under a magnifying glass, which magnifies by a factor of three.

EXAMPLE 4: KOCH CURVE

A line segment is divided into three parts. Two line segments replace the middle one-third of the line segment (step 2 of Fig
1(f)). The above operation is now performed on the 4 line segments (step 3 of Fig 1(f)). This process is carried on ad infinitum. At the end of the process we get a fractal object called the Koch curve.

Fig 1(f)

More generally for $\varepsilon=1/3^m$, $N(\varepsilon)=4^m$, the capacity dimension is

$$d_{\text{cap}} = \lim_{\varepsilon \to 0} \frac{\ln(4^m)}{\ln(3^m)} = \frac{m \ln 4}{m \ln 3} = 1.26$$

The Koch curve is neither a line nor a plane but something in between them.
2) HAUSDORFF-BESICOVITCH DIMENSION

Consider a covering of a set lying in a p-dimensional Euclidean space with p-dimensional cubes of variable edge length $\varepsilon_i$. Define the quantity $l_d(\varepsilon) = \inf \sum_{i} \varepsilon_i^d$, where the infimum (i.e. minimum) extends over all possible coverings subject to the constraint that $\varepsilon_i \leq \varepsilon$. Now let

$$l_d = \lim_{\varepsilon \to 0} l_d(\varepsilon).$$

Hausdorff showed that there exists a critical value of $d$ above which $l_d=0$ and below which $l_d=\infty$. This critical value, $d = d_H$, is the Hausdorff dimension. The capacity dimension is an upper bound for Hausdorff dimension.

1.7.2 PROBABILISTIC DIMENSION

1) INFORMATION DIMENSION

This is defined in terms of the relative frequency with which a typical trajectory visits the various parts of the attractor. Consider a covering of an attractor with $N(\varepsilon)$ volume elements each with diameter $\varepsilon$. The information dimension is defined by

$$d_i = \lim_{\varepsilon \to 0} \frac{\ln(S(\varepsilon))}{\ln(1/\varepsilon)}$$

where

$$S(\varepsilon) = -\sum_{i=1}^{N(\varepsilon)} P_i \ln(P_i);$$

$P_i$ is the probability for a point to fall in the $i^{th}$ volume element of the covering; $S(\varepsilon)$ is the information entropy.
2) CORRELATION DIMENSION (Grassberger & Procaccia 1983a)

This is defined by \( d_{\text{corr}} = \lim_{\varepsilon \to 0} \frac{\ln(C(\varepsilon))}{\ln(\varepsilon)} \) where

\[
C(\varepsilon) = \lim_{N \to \infty} \frac{1}{N^2} \{ \text{No. of pairs } (x_i, x_j) \text{ such that } |x_i - x_j| < \varepsilon \} \]

and \( N \) is the total number of observations on the attractor.

An inequality relates three of the dimensions

\[ d_{\text{corr}} \leq d_i \leq d_{\text{cap}} \]

1.7.3 LYAPUNOV DIMENSION

This is defined in terms of the Lyapunov exponents of the process, one of the most important dynamical invariant of an attractor that quantifies its sensitive dependence to initial conditions. There are as many exponents as there are dimension, which correspond with the number of unique variables with time subscripts and the number of equations in the system. The set of exponents for the system is called spectrum of Lyapunov exponents.

The rationale behind this is as follows:

Consider two points in a space \( X_0 \) & \( X_0 + \Delta x_0 \) each of which will generate an orbit in that space using some equation or system of equations. These orbits can be
thought of, as parametric functions of a variable that is something like time. If we use one of the orbits a reference orbit, then the separation between the two orbits will also be a function of time. Because sensitive dependence can arise only in some portions of a system (like the logistic equation), this separation is also a function of the location of the initial value and has the form \( \Delta x (X_0, t) \). In a system with attracting fixed points or attracting periodic points, \( \Delta x (X_0, t) \) diminishes asymptotically with time. If a system is unstable, like pins balanced on their points, then the orbits diverge exponentially for a while, but eventually settle down. For chaotic points, the function \( \Delta x (X_0, t) \) will behave erratically. It is thus useful to study the mean exponential rate of divergence of two initially close orbits using the formula

\[
\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\Delta x(x_0, t)}{\Delta x_0}
\]

This number, called the Lyapunov exponent "\( \lambda \)" [lambda], is useful for distinguishing among the various types of orbits. It works for discrete as well as continuous systems.

**\( \lambda < 0 \):** The orbit attracts to a stable fixed point or stable periodic orbit. Negative Lyapunov exponents are characteristic of dissipative or non-conservative systems. Such systems exhibit asymptotic stability.

**\( \lambda = 0 \):** The orbit is neutral fixed point. A Lyapunov exponent of zero indicated that the system is in some sort of steady state mode. A physical system with this exponent is conservative. Such systems exhibit Lyapunov stability.
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1.9 PLAN OF THE THESIS

Our specific interest in this thesis is to expose the information in the estimated fractal dimension on the structure and the parameters of the time series models generating the given time series. Keeping in view the vastness of time series models, we have restricted our attention to first order stochastic difference equations generating the time series.

Our reason for restricting our study to first order stochastic difference equation is that even under this simple situation, there are inferential problems relating to them, which are not conclusively solved in the literature. With reference to the model \( X(t+1) = \rho \ast X(t) + \varepsilon(t+1) \), the generated time series \( (X(t), t=1,2,\ldots,N) \) is said to be autoregressive (non-explosive) if \(|\rho| < 1\) and explosive if \(|\rho| > 1\). The unit root case occurs when \(|\rho| = 1\).

It is well known in the literature that, the least square estimator of the slope parameter \( \rho \) is not asymptotically normal, under explosive situation. In fact they are asymptotically distributed as ratio of two random variables whose distribution is not available for further inference, like testing of hypothesis or constructing confidence intervals.

The next crucial problem is to identify the 'unit root case'. The available test procedures tests for unit root against the root being less than unity. One big question is “How can one decide the explosive or non-explosive nature of the time series given the least squares estimate of the slope, especially when the estimate is in the neighborhood of 1? This question has not been resolved conclusively. Towards this end, we would like to explore the fractal approach and
examine if the information thus obtained can throw more light on these problems from an empirical point of view.

We are aware of the complexities involved due to (i) wide choice of models (ii) wide choice of distribution of innovative errors and (iii) multiple choices of methods of finding the fractal dimension. So, we have restricted our attention to a simple situation, and if we succeed in this effort the same can be extended to other situations as well.

Under this backdrop, we have tried two-fractal approaches (a) Higuchi’s approach (1988) and (b) the popular Grassberger-Procaccia approach (1983) (which has had wide applications, in the empirical analysis of time series).

Time series of various sizes, under various specifications of the distribution of the error term have been simulated, and their fractal dimension has been identified using the above two approaches.

It may be noted that the fractal dimension is usually obtained as the slope parameter between two quantities. There is also an intercept term in its calculation. We have tried to study this intercept estimates, as an academic curiosity.

Similarly, a time series can also be arranged in an increasing order, and this series is also simultaneously studied, so that any useful information in analyzing such ordered sequences can be interpreted to improve our knowledge on the structure of the underlying model, from an empirical angle.

Finally, we propose to consolidate our findings in a manner that would expose the utility of fractal approach in exploring the general features of
To sum up, the thesis essentially contains

(i) The results related to the investigation of fractal dimension on simulated time series, under various assumptions, using the two popular approaches.

(ii) Extracting information, if any, that would identify the general features of the underlying model.

(iii) Augmenting the information so obtained towards assessing the validity of certain assumptions in the classical time series analysis.