Chapter 6

On Complex Lagrange Spaces with \((\gamma, |\beta|)\)-Metric

6.1 Introduction

R. Miron [216] and B. Nicolaescu [16, 17] studied Lagrange spaces with \((\alpha, \beta)\)-metric. T. N. Pandey and V. K. Chaubey [260] introduced the concept of \((\gamma, \beta)\)-metric in a Lagrange space, where \(\gamma\) is a cubic-root metric and \(\beta\) is a 1-form defined by \(\gamma = \sqrt[3]{a_{ijk}(x)y^iy^jy^k}\) and \(\beta = b_i(x)y^i\) respectively. In 2013, S. K. Shukla and P. N. Pandey [253] further extended the theory of Lagrange spaces with \((\gamma, \beta)\)-metric. N. Aldea and G. Munteanu [138] introduced and worked on complex Finsler spaces with \((\alpha, \beta)\) metric. The authors [242] of the present chapter further studied complex Randers spaces. G. Munteanu [45] initiated the study of a complex Lagrange space in 1998. Later on, in 2002, various analysis of complex Lagrange space was done by G. Munteanu [46].

In the present chapter, the notion of \((\gamma, |\beta|)\)-metric in a complex Lagrange space, where \(\gamma\) is a cubic-root metric and \(\beta\) is a differential \((1, 0)\)-form, is being introduced. We determine the fundamental metric tensor, its inverse, Euler-Lagrange equations (1.18.2), complex semis-
pray coefficients (1.18.3), complex nonlinear connection (1.18.4) and Chern - Lagrange connections (1.18.5) for a complex Lagrange space with \((\gamma, |\beta|)\)-metric. Further the relation between these two connections are established (1.18.6).

In the present chapter, we have developed the theory of complex Lagrange spaces with \((\gamma, |\beta|)\)-metric (1.18.7) together with (1.18.8) and (1.18.9). It plays a significant role in the expansion of the earlier works of G. Munteanu [45]- [46]. For some geometric objects, expressions are obtained which are further useful in the development of the current space. The results obtained are useful in the study of connections, holomorphic curvature, complex nonlinear connections and torsions in such spaces.

### 6.2 Fundamental metric tensor of a complex Lagrange space with \((\gamma, |\beta|)\)-metric

Differentiating (1.18.8) partially with respect to \(\eta^l\) and \(\bar{\eta}^{\mu}\) and using the symmetry of \(a_{ij\bar{k}}\) in its indices, we get

\[
\begin{align*}
(a) \quad \dot{\partial}_l \gamma &= \frac{a_l}{3\gamma^2}, \\
(b) \quad \dot{\partial}_{\bar{m}} \gamma &= \frac{2a_{\bar{m}}}{3\gamma^2}
\end{align*}
\]

where \(a_l = a_{ij\bar{k}} \bar{\eta}^j \bar{\eta}^k\) and \(a_{\bar{m}} = a_{ij\bar{m}} \eta^i \bar{\eta}^j\).

Again, differentiating (6.2.1)(a) partially with respect to \(\bar{\eta}^{\mu}\), we obtain

\[
\dot{\partial}_l \dot{\partial}_{\bar{\mu}} \gamma = \frac{2a_{l\bar{\mu}}}{3\gamma^2} - \frac{4a_l a_{\bar{\mu}}}{9\gamma^5}.
\]

where \(a_{l\bar{\mu}} = a_{l\bar{\mu}\bar{k}} \bar{\eta}^k\).
Differentiation of (1.18.9) with respect to \( \eta^l \) and \( \eta^m \) gives

\[
(6.2.3) \quad \begin{cases}
(a) \quad \dot{\hat{\mathbf{b}}} |\beta| = \frac{\beta l}{2|\beta|}; \\
(b) \quad \dot{\hat{\mathbf{b}}} |\beta| = \frac{\beta m}{2|\beta|}.
\end{cases}
\]

Further differentiating (6.2.3)(a) partially with respect to \( \eta^p \), we have

\[
(6.2.4) \quad \partial \dot{\hat{\mathbf{b}}} |\beta| = \frac{\beta l b b^p l}{4|\beta|}.
\]

This leads to

**Proposition 6.2.1:** In a complex Lagrange space with \((\gamma, |\beta|)\)-metric, (6.2.1), (6.2.2), (6.2.3) and (6.2.4) hold.

The moments of Lagrangian \( L(z, \eta) \) are defined as

\[
(6.2.5) \quad p_i = \frac{1}{2} \dot{\Theta}_i L.
\]

Since the Lagrangian \( L(z, \eta) \) is a function of \( \gamma \) and \( |\beta| \), (6.2.5) implies

\[
(6.2.6) \quad p_i = \frac{1}{2} (L_{\gamma} \dot{\Theta}_i \gamma + L_{|\beta|} \dot{\Theta}_i |\beta|),
\]

where \( L_{\gamma} = \partial_\gamma L, L_{|\beta|} = \partial_{|\beta|} L, \partial_\gamma \equiv \partial/\partial \gamma \) and \( \partial_{|\beta|} \equiv \partial/\partial |\beta| \).

Using (6.2.1)(a) and (6.2.3)(a) in (6.2.6), we have

\[
(6.2.7) \quad p_i = \left( \frac{1}{6} \gamma^{-2} L_{\gamma} a_i + \frac{1}{4} |\beta|^{-1} L_{|\beta|} b_i \right).
\]

Thus, we have

**Theorem 6.2.1:** In a complex Lagrange space \( L^n \) with \((\gamma, |\beta|)\)-metric, the moments of Lagrangian \( L(z, \eta) \) are given by

\[
(6.2.8) \quad p_i = \rho a_i + \rho_1 b_i,
\]

where

\[
(6.2.9) \quad \rho = \frac{1}{6} \gamma^{-2} L_{\gamma}
\]

and

\[
(6.2.10) \quad \rho_1 = \frac{1}{4} |\beta|^{-1} L_{|\beta|}.
\]
The scalars \( \rho \) and \( \rho_1 \) appearing in theorem 6.2.1 are called the principal invariants of the space \( L^n \). Differentiating (6.2.9) and (6.2.10) partially with respect to \( \eta^j \) and \( \bar{\eta}^j \), we respectively have

(6.2.11)

\[
\begin{align*}
(a) \partial_j \rho &= \frac{1}{18} \gamma^{-4} (L_{\gamma \gamma} - 2\gamma^{-1} L_{\gamma}) \, a_j + \frac{1}{12} \beta |\beta|^{-1} \gamma^{-2} L_{\gamma|\beta} b_j, \\
(b) \partial_j \rho &= \frac{1}{9} \gamma^{-4} (L_{\gamma \gamma} - 2\gamma^{-1} L_{\gamma}) \, a_j + \frac{1}{12} \beta |\beta|^{-1} \gamma^{-2} L_{\gamma|\beta} b_j, \\
(c) \partial_j \rho_1 &= \frac{1}{12} \beta |\beta|^{-1} \gamma^{-2} L_{\gamma|\beta} a_j + \frac{1}{8} \beta \beta^{-1} (L_{|\beta||\beta|} + |\beta|^{-1} L_{|\beta|}) b_j, \\
(d) \partial_j \rho_1 &= \frac{1}{6} \beta |\beta|^{-1} \gamma^{-2} L_{\gamma|\beta} a_j + \frac{1}{8} |\beta|^{-1} (L_{|\beta||\beta|} + |\beta|^{-1} L_{|\beta|}) b_j,
\end{align*}
\]

where

\[
L_{\gamma \gamma} = \frac{\partial^2 L}{\partial \gamma^2}, \quad L_{\gamma|\beta} = \frac{\partial^2 L}{\partial \gamma |\beta|}, \quad \frac{\partial^2 L}{\partial |\beta| \partial \gamma} = L_{|\beta| \gamma}, \quad L_{|\beta||\beta|} = \frac{\partial^2 L}{\partial |\beta|^2}.
\]

Thus, we have

**Theorem 6.2.2:** The derivatives of the principal invariants of a complex Lagrange space \( L^n \) with \((\gamma, |\beta|)\)-metric are given by

(6.2.12)

\[
\begin{align*}
(a) \quad \partial_j \rho &= \frac{1}{2} \rho_{-2} a_j + \overline{\beta} \beta^{-1} \rho_{-1} b_j, \\
(b) \quad \partial_j \rho &= \rho_{-2} a_j + \rho_{-1} b_j,
\end{align*}
\]

and

(6.2.13)

\[
\begin{align*}
(a) \quad \partial_j \rho_1 &= \overline{\beta} \beta^{-1} (\rho_{-1} a_j + \rho_0 b_j), \\
(b) \quad \partial_j \rho_1 &= 2 \overline{\beta} \beta^{-1} \rho_{-1} a_j + \rho_0 b_j
\end{align*}
\]

with

(6.2.14)

\[
\begin{align*}
(a) \rho_{-2} &= \frac{1}{9} \gamma^{-4} (L_{\gamma \gamma} - 2\gamma^{-1} L_{\gamma}), \\
(b) \rho_{-1} &= \frac{1}{12} \beta |\beta|^{-1} \gamma^{-2} L_{\gamma|\beta|}, \\
(c) \rho_0 &= \frac{1}{8} (L_{|\beta||\beta|} + |\beta|^{-1} L_{|\beta|})
\end{align*}
\]

The energy of the complex Lagrangian \( L(z, \eta) \) is defined as

(6.2.15)

\[ E_L = \eta^i \partial_i L - L. \]
Using (1.18.7) in (6.2.15), we have

\[(6.2.16) \quad E_L = \eta^i (L_\gamma \dot{\gamma}_i + L_{|\beta|} \dot{\beta}_i |\beta|) - L.\]

Since \(\gamma\) and \(|\beta|\) are positively homogeneous of degree one in \(\eta^i\), in view of Euler’s theorem on homogeneous functions, we conclude

\[(6.2.17) \quad \eta^i \dot{\gamma}_i = \frac{\gamma}{3}, \quad \eta^i \dot{\beta}_i |\beta| = \frac{|\beta|}{2}.\]

Using (6.2.17) in (6.2.16), we get

\[(6.2.18) \quad E_L = \frac{\gamma}{3} L_\gamma + \frac{|\beta|}{2} L_{|\beta|} - L.\]

This leads to

**Theorem 6.2.3:** The energy of the Lagrangian \(L(z, \eta)\) in a complex Lagrange space with \((\gamma, |\beta|)-metric\), is given by (6.2.18).

Next, we calculate the fundamental metric tensor \(g_{ij}(z, \eta)\) of a complex Lagrange space with \((\gamma, |\beta|)-metric\). In view of (1.18.7), (1.18.1) implies

\[(6.2.19) \quad g_{ij} = \frac{1}{2} \left[ (L_{\gamma \gamma} \dot{\gamma}_i \dot{\gamma}_j + L_{\gamma |\beta|} \dot{\beta}_i \dot{\beta}_j) \dot{\gamma}_i \dot{\gamma}_j + L_{\gamma \gamma} \dot{\dot{\gamma}}_i \dot{\gamma}_j + L_{|\beta| \gamma} \dot{\beta}_i \dot{\gamma}_j + L_{|\beta| |\beta|} \dot{\beta}_i \dot{\beta}_j |\beta| \right].\]

On using (6.2.9) and (6.2.14) in (6.2.19), we have

\[(6.2.20) \quad g_{ij}(z, \eta) = 4\rho a_{ij} + 2\rho a_i a_j + 2\rho - 1 |\beta|^{-1} (2\beta a_i b_j + \beta a_i b_j) + 2\rho_0 b_i b_j.\]

A simple calculation shows that

\[(6.2.21) \quad (2\beta a_i b_j + \beta a_i b_j) = \frac{3\gamma^2 |\beta|}{2L} \eta_i \overline{\eta}_j - \frac{3\gamma^2 |\beta|}{2} b_i b_j - \frac{4|\beta|}{3\gamma^2} a_i a_j,\]

where \(\eta^i = \dot{\gamma}_i L\) and \(\overline{\eta}^j = \dot{\beta}_j L\).
In view of (6.21), (6.20) reduces to

\[(6.2.22)\quad g_{ij}(z, \eta) = 4\rho a_{ij} + q_{-2} a_i a_j + q_{-1} \eta_i \eta_j + q_0 b_i b_j.\]

with

\[(6.2.23)\quad q_{-2} = 2 \left( \rho_{-2} - \frac{4|\beta|}{3\beta\gamma_2} \right), \quad q_{-1} = \frac{3\gamma^2|\beta|}{\beta L} \rho_{-1} \quad \text{and} \quad q_0 = 2\rho_0 - \frac{3\gamma^2|\beta|}{\beta} \rho_{-1}.\]

Further (6.2.22) can be written as

\[(6.2.24)\quad g_{ij}(z, \eta) = 4\rho a_{ij} + c_i c_j,\]

where

\[(6.2.25)\quad c_i = r_{-1} a_i + r_0 b_i\]

such that

\[(6.2.26)\quad (a) \quad r_0 r_{-1} = q_{-1}, \quad (b) \quad (r_{-1})^2 = q_{-2}, \quad (c) \quad r_0^2 = q_0.\]

Thus, we have

**Theorem 6.2.4:** The expression for the fundamental metric tensor \(g_{ij}\) of a complex Lagrange space with \((\gamma, |\beta|)\)-metric is given by (6.2.24).

Using a Proposition [27] given by D. Bao, S. S. Chern and Z. Shen, the inverse \(g^{\tilde{j}i}\) of the fundamental tensor \(g_{ij}\) is given by

\[(6.2.27)\quad g^{\tilde{j}i} = \frac{1}{4\rho} \left( a^{\tilde{j}i} - \frac{1}{4\rho + c^2 c^{\tilde{j}}} c^i c^{\tilde{j}} \right),\]

where

\[(6.2.28)\quad (a) \quad c^i = a^{\tilde{j}i} c_{\tilde{j}}, \quad (b) \quad c^2 = a^{\tilde{j}i} c_i c_{\tilde{j}}.\]

This leads to

**Theorem 6.2.5:** The inverse \(g^{\tilde{j}i}\) of the fundamental tensor \(g_{ij}\) of a complex Lagrange space with \((\gamma, |\beta|)\)-metric is given by (6.2.27).
6.3 Euler-Lagrange equations

In view of (1.18.7), (1.18.2) reduces to

\[
E_i(L) \equiv L_\gamma E_i(\gamma) + L_{|\beta|} E_i(|\beta|) - \left( L_{\gamma \gamma} \frac{d\gamma}{dt} + L_{\gamma |\beta|} \frac{d|\beta|}{dt} \right) \frac{\partial \gamma}{\partial \eta^i} \]

\[
- \left( L_{|\beta| \gamma} \frac{d\gamma}{dt} + L_{|\beta| |\beta|} \frac{d|\beta|}{dt} \right) \frac{\partial |\beta|}{\partial \eta^i} = 0. \tag{6.3.1}
\]

Also

\[
E_i(\gamma^3) = 3\gamma^2 E_i(\gamma) - 3 \frac{\partial \gamma}{\partial \eta^i} \frac{d\gamma^2}{dt}, \tag{6.3.2}
\]

and

\[
E_i(|\beta|^2) = 2|\beta| E_i(|\beta|) - 2 \frac{\partial |\beta|}{\partial \eta^i} \frac{d|\beta|}{dt}. \tag{6.3.3}
\]

Substituting values of \(E_i(\gamma)\) and \(E_i(|\beta|)\) from (6.3.2) and (6.3.3) in (6.3.1), we obtain

\[
E_i(L) \equiv 2\rho E_i(\gamma^3) + \frac{2}{\beta} \rho_1 E_i(|\beta|^2) + 6\rho \frac{\partial \gamma}{\partial \eta^i} \frac{d\gamma^2}{dt} + 4 \frac{\partial |\beta|}{\partial \eta^i} \frac{d|\beta|}{dt} \]

\[
- \frac{\partial \gamma}{\partial \eta^i} \left( L_{\gamma \gamma} \frac{d\gamma}{dt} + L_{\gamma |\beta|} \frac{d|\beta|}{dt} \right) \]

\[
- \frac{\partial |\beta|}{\partial \eta^i} \left( L_{|\beta| \gamma} \frac{d\gamma}{dt} + L_{|\beta| |\beta|} \frac{d|\beta|}{dt} \right). \tag{6.3.4}
\]

This leads to

**Theorem 6.3.1:** The Euler-Lagrange equations of a complex Lagrange space with \((\gamma, |\beta|)\)-metric are given by (6.3.4).

For the natural parametrization of the curve \(c : t \in [0, 1] \mapsto z^i(t) \in M\) with respect to the cubic-root metric \(\gamma, \gamma(z, \frac{dz}{dt}) = 1\). Thus, we have
Theorem 6.3.2: In the natural parametrization, the Euler-Lagrange equations of a complex Lagrange space with \((\gamma, |\beta|)\)-metric are

\[
E_i(L) \equiv 2\rho E_i(\gamma^3) + \frac{2}{\beta} \rho_1 E_i(|\beta|^2) + \frac{4}{\beta} \rho_1 \frac{\partial |\beta|}{\partial \eta^i} \frac{d|\beta|}{dt} - L_{|\beta|} \frac{\partial \gamma}{\partial \eta^i} \frac{d|\beta|}{dt} - L_{|\beta|} |\beta| \frac{\partial |\beta|}{\partial \eta^i} \frac{d|\beta|}{dt} = 0.
\]

(6.3.5)

If \(|\beta|\) is constant along the integral curve of the Euler-Lagrange equations with natural parametrization, then the Euler-Lagrange equations of the complex Lagrange space with \((\gamma, |\beta|)\)-metric are given by

\[
E_i(L) \equiv 2\rho E_i(\gamma^3) + \frac{2}{\beta} \rho_1 E_i(|\beta|^2) = 0.
\]

(6.3.6)

This leads to

Theorem 6.3.3: If \(|\beta|\) is constant along the integral curve of the Euler-Lagrange equations with natural parametrization, then the Euler-Lagrange equations of the complex Lagrange space with \((\gamma, |\beta|)\)-metric are given by (6.3.6).

6.4 Complex canonical semispray

The coefficients of the complex canonical semispray of a complex Lagrange space with \((\gamma, |\beta|)\)-metric is given by (1.18.3) together with (1.18.7).

Differentiating (1.18.8) and (1.18.9) partially with respect to \(z^h\), we have

\[
\partial_h \gamma = A_h \gamma^{-2}, \quad \partial_h |\beta| = \frac{\beta}{2|\beta|} B_h + \frac{\beta}{2|\beta|} C_h,
\]

(6.4.1)

where

\[
A_h = \frac{1}{3} (\partial_h a_{ijk}) \eta^j \bar{\eta}^i \bar{\eta}^k, \quad B_h = (\partial_h b_i) \eta^i \text{ and } C_h = (\partial_h b_\bar{j}) \bar{\eta}^j.
\]

(6.4.2)
Substituting (6.4.1), (6.2.9) and (6.2.10) in $\partial_k L = L_\gamma \partial_k \gamma + L_{|\beta|} \partial_k |\beta|$, we get

\begin{equation}
\partial_k L = 6\rho A_k + 2\rho_1 \left( B_k + \frac{\beta}{\beta} C_k \right). \tag{6.4.3}
\end{equation}

Differentiating (6.4.3) partially with respect to $\eta^h$, we have

\begin{align*}
\dot{\partial}_k L &= \left( 6\rho_2 A_k + \frac{4\beta}{\beta} \rho_{-1} B_k + 4\rho_{-1} C_k \right) a_{k h} \\
&+ \left( 6\rho_1 A_k + 2\rho_0 B_k + 2\rho_0 \frac{2}{\beta} C_k - 2\rho_1 \frac{\beta}{\beta^2} C_k \right) b_{k h} \\
&+ \left( 6\rho A_{k h} + 2\rho_1 B_{k h} + 2\rho_1 \frac{\beta}{\beta} C_{k h} \right), \tag{6.4.4}
\end{align*}

where

\begin{align*}
(a) & \quad A_{k h} = \dot{\partial}_k A_k, \quad (b) \quad B_{k h} = \dot{\partial}_k B_k \quad \text{and} \quad (c) C_{k h} = \dot{\partial}_k C_k.
\end{align*}

Contracting (6.4.4) with $\eta^k$, we obtain

\begin{align*}
(\dot{\partial}_k L)\eta^k &= \left( 6\rho_2 A_0 + \frac{4\beta}{\beta} \rho_{-1} B_0 + 4\rho_{-1} C_0 \right) a_{0 h} \\
&+ \left( 6\rho_1 A_0 + 2\rho_0 B_0 + 2\rho_0 \frac{2}{\beta} C_0 - 2\rho_1 \frac{\beta}{\beta^2} C_0 \right) b_{0 h} \\
&+ \left( 6\rho A_{0 h} + 2\rho_1 B_{0 h} + 2\rho_1 \frac{\beta}{\beta} C_{0 h} \right). \tag{6.4.6}
\end{align*}

where

\begin{align*}
\begin{cases}
(a) & \quad A_0 = A_k(z, \eta)\eta^k, \quad (b) \quad B_0 = B_k(z, \eta)\eta^k, \\
(c) & \quad C_0 = C_k(z, \eta)\eta^k, \quad (d) \quad A_{0 h} = A_{k h}(z, \eta)\eta^k, \\
(e) & \quad B_{0 h} = B_{k h}(z, \eta)\eta^k, \quad (f) \quad C_{0 h} = C_{k h}(z, \eta)\eta^k.
\end{cases}
\end{align*}

Putting (6.4.6) in (1.18.3), we have

\begin{align*}
G^i = & g_{\tilde{\eta} i} \left[ \left( 3\rho_{-2} A_0 + \frac{2\beta}{\beta} \rho_{-1} B_0 + 2\rho_{-1} C_0 \right) a_{0 h} \\
&+ \left( 3\rho_{-1} A_0 + \rho_0 B_0 + \rho_0 \frac{2}{\beta} C_0 - \rho_1 \frac{\beta}{\beta^2} C_0 \right) b_{0 h} \\
&+ \left( 3\rho A_{0 h} + \rho_1 B_{0 h} + \rho_1 \frac{\beta}{\beta} C_{0 h} \right) \right]. \tag{6.4.8}
\end{align*}
This leads to

**Theorem 6.4.1:** The coefficients of the complex canonical semispray of a complex Lagrange space with \((\gamma, |\beta|)-\text{metric}\) are given by (6.4.8).

### 6.5 Canonical complex nonlinear connection and Chern - Lagrange connection

In this section, we find out the coefficients of the complex nonlinear connection \(N^c_k_j\) and Chern - Lagrange connection \(CL^k_j\) of a complex Lagrange space with \((\gamma, |\beta|)-\text{metric}\).

Partial differentiation of \(g^{\bar{h}i}g_{\bar{h}j} = \delta^i_j\), with respect to \(\eta^j\), yields

\[(6.5.1) \quad \dot{\eta}^i_j = -g^{\bar{r}i}C^i_{rj}.\]

Partial differentiation of the quantities appearing in (6.2.14) and (6.4.7) with respect to \(\eta^j\), we get

\[(6.5.2) \quad \begin{cases}
\dot{\eta}^i_j = \mu_{-2}a_j + \mu_{-1}b_j, \\
\dot{\eta}^i_0 = \mu_{-1}a_j + \mu_0 b_j, \\
\dot{\eta}^i_0 h = \mu_{-1}a_j h, \\
\dot{\eta}^i_0 \bar{h} = \mu_{-1}a_j \bar{h}.
\end{cases}\]

where

\[(6.5.3) \quad \begin{cases}
\mu_{-3} = \frac{1}{27} \gamma^{-8} (\gamma^2 L_{\gamma\gamma\gamma} - 6\gamma L_{\gamma\gamma} + 10L_{\gamma}), \\
\mu_{-2} = \frac{1}{18} \gamma^{-4} |\beta|^{-1} (L_{\gamma|\beta|} - 2\gamma^{-1} L_{|\beta|}), \\
\mu_{-1} = \frac{1}{24} \gamma^{-2} |\beta|^{-1} (|\beta| L_{|\beta|\beta} + L_{|\beta|}), \\
\mu_0 = \frac{1}{16} |\beta|^{-3} (|\beta|^2 L_{|\beta|\beta|\beta} + |\beta| L_{|\beta|\beta} - L_{|\beta|}), \\
A_{0\bar{h}j} = A_{r\bar{h}j} \eta^r, \\
A_{r\bar{h}j} = \partial_r a_{\bar{h}j}.
\end{cases}\]
and $\mathcal{G}_{(kj)}$ denotes the interchange of the indices $k$ & $j$ and addition.

Now, applying (6.4.8) in (1.18.4), we get

$$N^j_i = \frac{1}{2} \left( \partial_j g^{\bar{i}i} \right) \left[ \left( 3\rho_{-2}A_0 + \frac{2\beta}{\beta} \rho_{-1}B_0 + 2\rho_{-1}C_0 \right) a_{\bar{n}} \right.$$ 

$$+ \left( 3\rho_{-1}A_0 + \rho_0 B_0 + \rho_0 \frac{\beta}{\beta} C_0 - \rho_1 \frac{\beta}{\beta} C_0 \right) b_{\bar{n}} $$

$$+ \left( 3\rho A_{0\bar{n}} + \rho_1 B_{0\bar{n}} + \rho_1 \frac{\beta}{\beta} C_{0\bar{n}} \right) \right]$$

Using (6.5.4) and $g_{\bar{i}i} \partial_j$ 

$$+ g^{\bar{i}i} \partial_j \left[ \left( 3\rho_{-2}A_0 + \frac{2\beta}{\beta} \rho_{-1}B_0 + 2\rho_{-1}C_0 \right) a_{\bar{n}} \right.$$ 

$$+ \left( 3\rho_{-1}A_0 + \rho_0 B_0 + \rho_0 \frac{\beta}{\beta} C_0 - \rho_1 \frac{\beta}{\beta} C_0 \right) b_{\bar{n}} $$

$$+ \left( 3\rho A_{0\bar{n}} + \rho_1 B_{0\bar{n}} + \rho_1 \frac{\beta}{\beta} C_{0\bar{n}} \right) \right].$$

Using (6.2.12), (6.2.13), (6.4.5), (6.4.7), (6.5.1) and (6.5.2) in (6.5.4) and
simplifying, we have

\[ N^i_j = -C^i_{rij} G^r + g^{\pi \bar{\eta}} \left[ \rho_{-2} \left( 3A_0j + A_j a_{\pi} \right) + 6A_0a_{\bar{\pi}} \right] + \rho_{-1} \left\{ \left( 3A_0j + 3A_j - a_j \bar{\beta}^{-1} C_0 \right) b_{\pi} + 4(\bar{\beta} \beta^{-1} B_0 + C_0) a_{j\bar{\pi}} \right. \\
+ (3\beta \beta^{-1} A_0\bar{\pi} - 2\beta \beta^{-2} B_0 a_{\bar{\pi}}) b_j + 2(\beta \beta^{-1} C_0) a_{j\bar{\pi}} \right\} + \rho_0 \left\{ \left( \mathcal{S}_{(kj)} \left( \partial_k b_j \right) \eta^k + C_j a_{\pi} \right) \right. \\
+ \bar{\beta} \beta^{-1} (B_0\bar{\pi} + \bar{\beta}^{-1} B_0 a_{\bar{\pi}}) a_j \right\} + \rho_1 \left\{ \left( \mathcal{S}_{(kj)} \left( \partial_k b_j \right) \eta^k + \beta \beta^{-1} C_j \right) b_{\pi} \right. \\
+ \bar{\beta} \beta^{-1} (B_0\bar{\pi} + \bar{\beta}^{-1} B_0 a_{\bar{\pi}}) b_j \right\} + \mu_0 \left\{ \left( \mathcal{S}_{(kj)} \left( \partial_k b_j \right) \eta^k + \beta \beta^{-1} C_j \right) b_{\pi} \right. \\
+ (3\beta \beta^{-1} B_0 + C_0) b_j a_{\pi} + 3A_0b_j a_{\bar{\pi}} \right\} + \mu_{-1} \left\{ \left( \mathcal{S}_{(kj)} \left( \partial_k b_j \right) \eta^k + \beta \beta^{-1} C_j \right) b_{\pi} \right. \\
+ (3\beta \beta^{-1} B_0 + C_0) b_j a_{\pi} + 3A_0b_j a_{\bar{\pi}} \right\} + \mu_0 (B_0 + \bar{\beta}^{-1} B_0) b_{\pi} \bar{\eta} \right]\]

Also, on using (6.4.4) in (1.18.5), we obtain

\[ \bar{N}^k_j = 2g^{\pi \bar{\eta}} \left[ \left( 3\rho_{-2} A_j + \frac{2\beta}{\beta} \rho_{-1} B_j + 2\rho_{-1} C_0 \right) a_{\pi} \right. \\
+ \left( 3\rho_{-1} A_j + \rho_0 B_j + \rho_0 \frac{\beta}{\beta} C_j - \rho_1 \frac{\beta}{\beta^2} C_j \right) b_{\pi} \right. \\
+ \left( 3\rho A_{ji} + \rho_1 B_{ji} + \rho_1 \frac{\beta}{\beta} C_{ji} \right) \right]. \]

Thus, we have

**Theorem 6.5.1:** The coefficients of the complex nonlinear connection and Chern-Lagrange connection of a complex Lagrange space with \((\gamma, |\beta|)\)-metric are given by (6.5.5) and (6.5.6) respectively.