Chapter 8

On Integral Operator Involving Mittag-Leffler Function
8.1 Introduction

The chief object in the chapter is to put forward an interesting double integral involving generalized Mittag-Leffler function, which is expressed in form of generalized (Wright) hypergeometric function. Also we consider few special cases as an application of main result.

The well known Mittag-Leffler function is presented as (Mittag-Leffler, 1903):

\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \]  
(8.1.1)

where \( \alpha \in \mathbb{C}, \Re(\alpha) > 0 \) with \( z \in \mathbb{C} \) and its general form is given by

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \]  
(8.1.2)

where \( \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, z \in \mathbb{C} \) with \( \mathbb{C} \) being the set of complex numbers which is known as Wiman function (see, Wiman, 1905).

In 1971, Prabhakar gave the function \( E^\gamma_{\alpha,\beta}(z) \) as follows:

\[ E^\gamma_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta)} n!, \]  
(8.1.3)

where \( \alpha, \beta, \gamma \in \mathbb{C} \) with \( \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0 \) and \( z \in \mathbb{C} \). \( (\gamma)_n \) is the general Pochhammer’s symbol (see, Rainville, 1960).

In sequence of his work, Shukla and Prajapati (2007) introduced the following extension of Mittag-Leffler function:

\[ E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta)} n!, \]  
(8.1.4)

again here \( \alpha, \beta, \gamma \in \mathbb{C} \) with \( \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0 \) and \( q \in (0,1) \cup \mathbb{N} \).

The function \( E^{\gamma,q}_{\alpha,\beta}(z) \) is most natural generalization of the exponential function \( \exp(z) \), Mittag-Leffler function \( E_\alpha(z) \) and Wiman function \( E_{\alpha,\beta}(z) \). Furthermore, the function \( E^{\gamma,q}_{\alpha,\beta}(z) \) has the following special cases (see, Haubold, 2011):

\[ E^{\gamma,1}_{\alpha,\beta}(z) = E^\gamma_{\alpha,\beta}(z), \quad E^{1,1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z), \quad E_{\alpha,1}(z) = E_\alpha(z), \]

\[ E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E^{1,1}_{1,1}(z) = E_{1,1}(z) = E_1(z) = \exp(z), \quad z \in \mathbb{C}. \]  
(8.1.5)
The generalization of the (generalized) hypergeometric series \( pF_q \) is due to Fox (1928) and Wright (1935, p.286-293, 1940, p.423-451, 1940, p.389-408) who investigated the asymptotic expansion of the generalized (Wright) hypergeometric function stated by (Srivastava and Karlsson, 1985, see also, Rathie, 1997):

\[
\begin{align*}
    p_{\Psi_q} & \left[ \begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\
                          (\beta_1, B_1), \ldots, (\beta_q, B_q); \\
                          z \end{array} \right] \\
    & = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k) \Gamma(\beta_j + B_j k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + A_j k) \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}, \\
    \text{(8.1.6)}
\end{align*}
\]

where the coefficients \( A_1, \ldots, A_p \) and \( B_1, \ldots, B_q \) are positive real numbers such that

(i) \( 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0 \) and \( 0 < |z| < \infty; \ z \neq 0. \)  

(ii) \( 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j = 0 \) and \( 0 < |z| < A_1^{-A_1} \ldots A_p^{-A_p} B_1 B_1 \ldots B_q B_q. \)

\( \text{(8.1.6a, 8.1.6b)} \)

A special case of (8.1.6) is

\[
\begin{align*}
    p_{\Psi_q} & \left[ \begin{array}{c} (\alpha_1, 1), \ldots, (\alpha_p, 1); \\
                          (\beta_1, 1), \ldots, (\beta_q, 1); \\
                          z \end{array} \right] \\
    & = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} p_{F_q} \left[ \begin{array}{c} \alpha_1, \ldots, \alpha_p; \\
                          \beta_1, \ldots, \beta_q; \\
                          z \end{array} \right], \\
    \text{(8.1.7)}
\end{align*}
\]

where \( p_{F_q} \) is the generalized hypergeometric series stated by (Rainville, 1960)

\[
\begin{align*}
    p_{F_q} & \left[ \begin{array}{c} \alpha_1, \ldots, \alpha_p; \\
                          \beta_1, \ldots, \beta_q; \\
                          z \end{array} \right] \\
    & = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n \beta_1)_n \ldots (\beta_q)_n}{n!} \frac{z^n}{n!} \\
    & = p_{F_q}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots \beta_q; z), \\
    \text{(8.1.8)}
\end{align*}
\]

where \( (\lambda)_n \) is the Pochhammer’s symbol (see, Rainville, 1960).

For our present investigation, the following interesting and useful result due to Edward (1922, p.445) will be required:

\[
\begin{align*}
    \int_{0}^{1} \int_{0}^{1} y^x (1-x)^{a-1} (1-y)^{b-1} (1-xy)^{1-a-b} \ dx \ dy & = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}, \\
    \text{(8.1.9)}
\end{align*}
\]

provided \( \Re(\alpha) > 0 \) and \( \Re(\beta) > 0. \)
8.2 Integral involving Mittag-Leffler function

We establish here the double integral involving Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\nu}$, expressed in terms of Wright hypergeometric function:

$$
\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta}^{\gamma,\nu} \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] \, dx \, dy
$$

$$
= \frac{1}{\Gamma(\gamma)} 3\Psi_2 \left[ \begin{array}{ccc} (\gamma, \nu), & (\lambda, 1), & (\mu, 1) ; & a \\ (\beta, \alpha), & (\lambda + \mu, 2) & \end{array} \right],
$$

where $a$ is nonzero constant, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\lambda) > 0$, $\Re(\mu) > 0$, $E_{\alpha,\beta}^{\gamma,\nu}$ and $3\Psi_2$ are the Mittag-Leffler and Wright hypergeometric functions defined by (8.1.4) and (8.1.6), respectively.

**Proof:** To establish our main result (8.2.1), we denote the L.H.S. of (8.2.1) by $I$ and then using (8.1.4), we have

$$
I = \int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} \sum_{n=0}^\infty \frac{(\gamma)_n a^n y(1-x)(1-y)(1-xy)^2}{(1-xy)^n} \, dx \, dy.
$$

Now due to the uniform convergence of the series in the interval (0,1) we can change the order of integration and summation and then by using (8.1.9), we arrive at

$$
I = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^\infty \frac{\Gamma(\gamma+\nu n) \Gamma(\lambda+n) \Gamma(\mu+n)}{\Gamma(\alpha n+\beta) \Gamma(\lambda+\mu+2n)} \frac{(a)^n}{n!}.
$$

Finally, summing up the above series with the help of (8.1.6), we reach at the R.H.S. of (8.2.1). This winds the proof of our chief result.

Next, we consider other variation of (8.2.1). In short, we establish an integral formula for the Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\nu}$, which is expressed in terms of the generalized hypergeometric function $\pFq$.

8.3 Variation of (8.2.1)

Let the conditions of our main result be satisfied, then the below integral formula
exists:

\[
\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta}^\gamma \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] \, dx \, dy
\]

\[
= \frac{\Gamma(\lambda) \Gamma(\mu)}{\Gamma(\beta) \Gamma(\lambda + \mu)} \nu + 2 F_{\alpha + 2} \left[ \Delta(\nu; \gamma), \lambda, \mu; a \nu \frac{\alpha \nu}{4 \alpha^2} \right], \quad \text{(8.3.1)}
\]

where \( \Delta(m; l) \) abbreviates the array of \( m \) parameters \( l, l + 1, \ldots, l + m - 1 \), \( m \geq 1 \).

**Proof:** To prove the result (8.3.1), using the results

\[
\Gamma(\alpha + n) = \Gamma(\alpha) \Gamma(\alpha)_n
\]

and

\[
(l)_kn = \kappa^k \left( \frac{l}{k} \right)_n \left( \frac{l + 1}{k} \right)_n \cdots \left( \frac{l + k - 1}{k} \right)_n.
\]

(Gauss multiplication theorem) in (8.2.3) and summing up the given series with the help of (8.1.8), we easily arrive at our required result (8.3.1).

**8.4 Special cases**

(i). On taking \( \nu = 1 \) in (8.2.1) and by using \( E_{\alpha,\beta}^\gamma(z) = E_{\alpha,\beta}^\gamma(z) \), we get

\[
\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta}^\gamma \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] \, dx \, dy
\]

\[
= \frac{1}{\Gamma(\gamma)} 3 \Psi_2 \left[ \begin{array}{ccc}
(\gamma, 1), & (\lambda, 1), & (\mu, 1) \\
(\beta, \alpha), & (\lambda + \mu, 2)
\end{array} \right] a, \quad \text{(8.4.1)}
\]

where \( \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\mu) > 0, \Re(\lambda) > 0 \) and \( E_{\alpha,\beta}^\gamma \) is the Mittag-Leffler function defined by (8.1.3).

(ii). On setting \( \gamma = 1 \) in (8.4.1) and by using \( E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z) \), we get

\[
\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta} \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] \, dx \, dy
\]

\[
= 3 \Psi_2 \left[ \begin{array}{ccc}
(1, 1), & (\lambda, 1), & (\mu, 1) \\
(\beta, \alpha), & (\lambda + \mu, 2)
\end{array} \right] a, \quad \text{(8.4.2)}
\]
where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ and $E_{\alpha,\beta}$ is the Mittag-Leffler function stated by (8.1.2).

(iii). On putting $\beta = 1$ in (8.4.2) and by using $E_{\alpha,1}(z) = E_{\alpha}(z)$, we get

$$
\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha} \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] dx \, dy
$$

$$
= 3 \Psi_2 \left[ \begin{array}{ccc} (1, 1), & (\lambda, 1), & (\mu, 1) ; \\ (1, \alpha), & (\lambda + \mu, 2) ; & a \end{array} \right],
$$

where $\Re(\alpha) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ and $E_{\alpha}$ is the Mittag-Leffler function given by (8.1.1).

(iv). On taking $\alpha = 1$ in (8.4.3) and by using $E_1(z) = \exp(z)$, we get

$$
\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} \exp \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] dx \, dy
$$

$$
= 2 \Psi_1 \left[ \begin{array}{ccc} (\lambda, 1), & (\mu, 1) ; \\ (\lambda + \nu, 2) ; & a \end{array} \right],
$$

where $\Re(\gamma) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$.

(v). On setting $\alpha = 1, \beta = 2$ in (8.4.2) and by using $E_{1,2}(z) = \frac{e^z-1}{z}$, we get

$$
\int_0^1 \int_0^1 y^{\lambda-1} (1-x)^{\lambda-2} (1-y)^{\mu-2} (1-xy)^{3-\lambda-\mu} \exp \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] - 1 \right] \ dx \, dy
$$

$$
= 3 \Psi_2 \left[ \begin{array}{ccc} (1, 1), & (\lambda, 1), & (\mu, 1) ; \\ (2, 1), & (\lambda + \mu, 2) ; & a \end{array} \right],
$$

where $\Re(\mu) > 0, \Re(\lambda) > 0$.

(vi). On taking $\nu = 1$ in (8.3.1), we get

$$
\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta} \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] dx \, dy
$$

$$
= \frac{\Gamma(\lambda) \Gamma(\mu)}{\Gamma(\beta) \Gamma(\lambda + \mu)} \, _3F_{\alpha+2} \left[ \begin{array}{ccc} \gamma & \lambda & \mu ; \\ \Delta(\alpha; \beta), & \Delta(2; \lambda + \mu) ; & a \end{array} \right],
$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$. 
where $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\mu) > 0$, $\Re(\lambda) > 0$.

(vii). On putting $\gamma = 1$ in (8.4.6), we get

$$\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta} \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] \, dx \, dy$$

$$= \frac{\Gamma(\lambda) \Gamma(\mu)}{\Gamma(\beta) \Gamma(\lambda + \mu)} 3F_{\alpha+2} \left[ \begin{array}{c} 1 \lambda \mu \frac{a}{4\alpha^\alpha} \\ \Delta(\alpha; \beta), \Delta(2; \lambda + \mu); \end{array} \right],$$

(8.4.7)

where $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\mu) > 0$, $\Re(\lambda) > 0$.

(viii). Further, on setting $\beta = 1$ in (8.4.7), we get

$$\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta} \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] \, dx \, dy$$

$$= \frac{\Gamma(\lambda) \Gamma(\mu)}{\Gamma(\lambda + \mu)} 3F_{\alpha+2} \left[ \begin{array}{c} 1 \lambda \mu \frac{a}{4\alpha^\alpha} \\ \Delta(\alpha; 1), \Delta(2; \lambda + \mu); \end{array} \right],$$

(8.4.8)

where $\Re(\alpha) > 0$, $\Re(\mu) > 0$, $\Re(\lambda) > 0$.

(ix). On taking $\alpha = 1$ in (8.4.8), we get

$$\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} \exp \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] \, dx \, dy$$

$$= \frac{\Gamma(\lambda) \Gamma(\mu)}{\Gamma(\lambda + \mu)} 2F_2 \left[ \begin{array}{c} \lambda \mu \frac{a}{4} \\ \Delta(2; \lambda + \mu); \end{array} \right],$$

(8.4.9)

where $\Re(\mu) > 0$, $\Re(\lambda) > 0$.

(x). On putting $\alpha = 1$, $\beta = 2$ in (8.4.7), we get

$$\int_0^1 \int_0^1 y^\lambda (1-x)^{\lambda-2} (1-y)^{\mu-2} (1-xy)^{3-\lambda-\mu} \left[ \exp \left[ \frac{ay(1-x)(1-y)}{(1-xy)^2} \right] - 1 \right] \, dx \, dy$$

$$= \frac{\Gamma(\lambda) \Gamma(\mu)}{\Gamma(\lambda + \mu)} 3F_{3} \left[ \begin{array}{c} 1 \lambda \mu \frac{a}{4} \\ 2 \Delta(2; \lambda + \mu); \end{array} \right],$$

(8.4.10)

where $\Re(\mu) > 0$, $\Re(\lambda) > 0$. 

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Concluding Remarks: For the present investigation, we have derived an interesting double integral involving Mittag-Leffler function, expressed in terms of (Wright) hypergeometric function. Also, we have established some other integrals as special cases of our chief result. As the Mittag-Leffler functions, have some interesting connections with Bessel-Maitland functions. So, the results in this chapter are easily converted for the Bessel-Maitland functions.