Chapter 3 Symmetry Identities for Generalized Hermite-based Apostol-Euler and Apostol-Genocchi Polynomials
3.1 Introduction


The Kampé de Fériet generalization of the Hermite polynomials for 2-variable (refer Bell, (1934) and Dattoli et al. (1999)) subjected as

\[ H_n(x, y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \tag{3.1.1} \]

These polynomials are in common specified by the generating function

\[ e^{xt+y^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}, \tag{3.1.2} \]

and precipitates to the ordinary Hermite polynomials \( H_n(x) \) (Andrews, 1985) when \( y = -1 \) and \( x \) is reset by \( 2x \).

The ordinary Bernoulli polynomials \( B_n(x) \), the ordinary Euler polynomials \( E_n(x) \) and the ordinary Genocchi polynomials \( G_n(x) \), along with their close generalizations \( B_n^{(\alpha)}(x) \), \( E_n^{(\alpha)}(x) \) and \( G_n^{(\alpha)}(x) \) of order \( \alpha \) (real or complex) are defined by means of the following generating functions (for details see: Andrews (1985), Apostol (1951), Khan (2015, p.597-614), Pathan and Khan (2014, p.113-136, 2015, p.679-695, 2016, p.913-928, 2014, p.92-109, 2015, p.53-70, 2015, p.153-170)):

\[ \left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_{n}^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < 2\pi; 1^\alpha = 1). \tag{3.1.3} \]

\[ \left( \frac{2}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_{n}^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < \pi; 1^\alpha = 1). \tag{3.1.4} \]

and

\[ \left( \frac{2t}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} G_{n}^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < \pi; 1^\alpha = 1). \tag{3.1.5} \]
so that
\[
B_n(x) = B_n^{(1)}(x); E_n(x) = E_n^{(1)}(x); G_n(x) = G_n^{(1)}(x), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \quad (3.1.6)
\]
In particular, Luo and Srivastava (2006, p.290-302, 2005, p.631-642 2011), Luo (2009, p.2193-2208, 2009, p.377-391, 2009, p.336-346, 2009, p.1-9, 2009, p.113-125, 2011, p.291-310) established the generalized Apostol-Bernoulli polynomials \( B_n^{(\alpha)}(x; \lambda) \), the generalized Apostol-Euler polynomials \( E_n^{(\alpha)}(x; \lambda) \) and the generalized Apostol-Genocchi polynomials \( G_n^{(\alpha)}(x; \lambda) \) each of order \( \alpha \in \mathbb{C} \) defined as follows:

**Definition 3.1.1.** The generalized Apostol-Bernoulli polynomials \( B_n^{(\alpha)}(x) \) (of order \( \alpha \)) are defined in terms of the generating function
\[
\left( \frac{t}{\lambda e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}; \quad |t| < 2\pi, i\lambda = 1; |t| < |\log \lambda|, i\lambda \neq 1; 1^\alpha = 1.
\]  
(3.1.7)

with
\[
B_n^{(\alpha)}(x) = B_n^{(\alpha)}(x; 1),
\]
and
\[
B_n^{(\alpha)}(\lambda) = B_n^{(\alpha)}(0; \lambda),
\]  
(3.1.8)

where we denote \( B_n^{(\alpha)}(\lambda) \) by the so called Apostol-Bernoulli numbers (of order \( \alpha \)).

**Definition 3.1.2.** The generalized Apostol-Euler polynomials \( E_n^{(\alpha)}(x) \) (of order \( \alpha \)) are defined in terms of the generating function
\[
\left( \frac{2}{\lambda e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}; \quad |t| < |\log(-\lambda)| < \pi, 1^\alpha = 1.
\]  
(3.1.9)

with
\[
E_n^{(\alpha)}(x) = E_n^{(\alpha)}(x; 1),
\]
and
\[
E_n^{(\alpha)}(\lambda) = E_n^{(\alpha)}(0; \lambda),
\]  
(3.1.10)

where we denote \( E_n^{(\alpha)}(\lambda) \) by the so called Apostol-Euler numbers (of order \( \alpha \)).

**Definition 3.1.3.** The generalized Apostol-Genocchi polynomials \( G_n^{(\alpha)}(x) \) (of order \( \alpha \)) are defined in terms of the generating function
\[
\left( \frac{2t}{\lambda e^t + 1} \right)^\alpha = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}; \quad |t| < |\log(-\lambda)| < \pi, 1^\alpha = 1,
\]  
(3.1.11)
with
\[ G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x; 1), \quad G_n^{(\alpha)}(\lambda) = G_n^{(\alpha)}(0; \lambda), \quad (n \in \mathbb{N}). \tag{3.1.12} \]

where we denote \( G_n^{(\alpha)}(\lambda) \) by the so called Apostol-Genocchi numbers (of order \( \alpha \)).

Very recently, Pathan and Khan (2015) studied a new family of generalized Hermite-Apostol-Bernoulli, Hermite-Apostol-Euler and Hermite-Apostol-Genocchi polynomials of order \( \alpha \) in the following form:

**Definition 3.1.4.** For arbitrary parameter \( \alpha \) (real or complex) and for \( a, c \in \mathbb{R}^+ \), the generalized Hermite-Apostol-Bernoulli polynomials \( H_{\alpha}^{[m-1,\alpha]}(x; a, c, \lambda) \) with \( m \in \mathbb{N} \) and \( \lambda \in \mathbb{C} \) are defined in a suitable neighborhood of \( t = 0 \) with \( |t \log(a)| < |\log(-\lambda)| \), in terms of the following generating function:

\[ t^{m\alpha}[A(\lambda, a; t)]^{\alpha} e^{xt+y^2} = \sum_{n=0}^\infty H_{\alpha}^{[m-1,\alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!}, \tag{3.1.13} \]

where
\[ A(\lambda, a; t) = \left( \lambda a^t - \sum_{h=0}^{m-1} \frac{(t \log(a))^h}{h!} \right)^{-1}. \tag{3.1.14} \]

It can be easily seen that if we set \( y = 0 \) in (3.1.13), we reach at a recent result of Tremblay et al. (2012, p.3, (1.8)) including the generalized Apostol-Bernoulli polynomials

\[ t^{m\alpha}[A(\lambda, a; t)]^{\alpha} e^{xt} = \sum_{n=0}^\infty B_{\alpha}^{[m-1,\alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!}, \tag{3.1.15} \]

For \( c = e \) in (3.1.13) gives

\[ t^{m\alpha}[A(\lambda, a; t)]^{\alpha} e^{xt+y^2} = \sum_{n=0}^\infty H_{\alpha}^{[m-1,\alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!}, \tag{3.1.16} \]

Moreover by substituting \( y = 0, \ m = 1, \ a = c = e \) in (3.1.13), we reach at the following result

\[ \left( \frac{t}{\lambda e^t - 1} \right)^{\alpha} e^{xt} = \sum_{n=0}^\infty B_{\alpha}^{(0,\alpha)}(x; e, e, \lambda) \frac{t^n}{n!}, \quad (|t| < 2\pi, \ 1^\alpha = 1), \tag{3.1.17} \]

This is nothing but the generating function for the generalized Apostol-Bernoulli polynomials of order \( \alpha \). Therefore, we have

\[ B_{\alpha}^{(0,\alpha)}(x; e, e, \lambda) = B_{\alpha}^{(\alpha)}(x; \lambda). \tag{3.1.18} \]
Definition 3.1.5. For arbitrary parameter \(\alpha\) (real or complex) and \(a, c \in \mathbb{R}^+\), the generalized Apostol-Hermite-Euler polynomials \(\mathcal{H}_{n}^{[m-1,\alpha]}(x, y; a, c, \lambda)\) with \(m \in \mathbb{N}\) and \(\lambda \in \mathbb{C}\) are defined in a suitable neighborhood of \(t = 0\) with \(|t \log a| < |\log(-\lambda)|\) in terms of the generating function

\[
2^{m\alpha}[B(\lambda, a; t)]^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!},
\]

(3.1.19)

where

\[
B(\lambda, a; t) = \left(\lambda a^t + \sum_{h=0}^{m-1} \frac{(t \log a)^h}{h!}\right)^{-1}.
\]

(3.1.20)

It can be easily seen that if we set \(y = 0\) in (3.1.19), we reach at a recent result of Tremblay et al. (2012, p.3(2.1)) involving the generalized Apostol-Euler polynomials

\[
2^{m\alpha}[B(\lambda, a; t)]^\alpha e^{xt} = \sum_{n=0}^{\infty} E_{n}^{[m-1,\alpha]}(x; a, c, \lambda) \frac{t^n}{n!},
\]

(3.1.21)

For \(c = e\) in (3.1.19) gives

\[
2^{m\alpha}[B(\lambda, a; t)]^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\alpha]}(x; a, e, \lambda) \frac{t^n}{n!},
\]

(3.1.22)

Moreover if we substitute \(y = 0, m = 1, a = c = e\) in (3.1.19), we reach at the following result

\[
\left(\frac{2}{\lambda e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_{n}^{[0,\alpha]}(x; e, e, \lambda) \frac{t^n}{n!} (|t| < \pi, 1^\alpha = 1).
\]

(3.1.23)

This is nothing but the generating function for the generalized Apostol-Euler polynomials of order \(\alpha\). Therefore, we have

\[
E_{n}^{[0,\alpha]}(x; e, e, \lambda) = E_{n}^{[\alpha]}(x; \lambda).
\]

(3.1.24)

Definition 3.1.6. For arbitrary parameter \(\alpha\) (real or complex) and \(a, c \in \mathbb{R}^+\), the generalized Hermite-Apostol-Genocchi polynomials \(\mathcal{H}_{n}^{[m-1,\alpha]}(x, y; a, c, \lambda)\) with \(m \in \mathbb{N}\) and \(\lambda \in \mathbb{C}\) are defined in a suitable neighborhood of \(t = 0\) with \(|t \log a| < |\log(-\lambda)|\) in terms of the generating function

\[
2^{m\alpha}\lambda^{m\alpha}[B(\lambda, a; t)]^\alpha e^{xt+y^2t^2} = \sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!},
\]

(3.1.25)
where $B(\lambda, a; t)$ is given by equation (3.1.20). It can be easily seen that if we substitute $y = 0$ in (3.1.25), we reach at a recent result of Tremblay et al. (2012, p.5(2.4)]) including the generalized Apostol-Genocchi polynomials.

$$2^{m\alpha} t^{m\alpha}[B(\lambda, a; t)]^\alpha e^{xt} = \sum_{n=0}^\infty G_n^{[m-1,\alpha]}(x, a, c, \lambda) \frac{t^n}{n!}, \quad (3.1.26)$$

For $c = e$ in (3.1.25) gives

$$2^{m\alpha} t^{m\alpha}[B(\lambda, a; t)]^\alpha e^{xt+yt^2} = \sum_{n=0}^\infty H_n^{[m-1,\alpha]}(x, y; a, e, \lambda) \frac{t^n}{n!}, \quad (3.1.27)$$

Obviously if we substitute $y = 0$, $m = 1$, $a = c = e$ in (3.1.25), we reach at the following result

$$\left(\frac{2t}{xe^t+1}\right)^\alpha e^{xt} = \sum_{n=0}^\infty G_n^{[0,\alpha]}(x; e, e, \lambda) \frac{t^n}{n!}, \quad (|t| < \pi, 1^\alpha = 1), \quad (3.1.28)$$

This is nothing but the generating function for the generalized Apostol-Genocchi polynomials (of order $\alpha$). Therefore, we have

$$G_n^{[0,\alpha]}(x; e, e, \lambda) = G_n^{[\alpha]}(x; \lambda). \quad (3.1.29)$$

3.2 Symmetry identities for generalized Hermite-Apostol-Euler polynomials

Here, we give general symmetry identities for the generalized Hermite-Apostol-Euler polynomials $H_n^{[\alpha,m-1]}(x, y; a, c, \lambda)$ by utilizing the generating functions (3.1.19) and (3.1.21), where $\alpha$ will be taken as an arbitrary real or complex parameter.

**Theorem 3.2.1.** For all integers $n \geq 0$, $a > 0$, $b > 0$, $c > 0$ with $a \neq b$ and $x, y \in \mathbb{R}$, the below identity holds true:

$$\sum_{k=0}^n \binom{n}{k} a^{n-k} b^k H_{n-k}^{[\alpha,m-1]}(bx, b^2y; c, \lambda) = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} H_{n-k}^{[\alpha,m-1]}(ax, a^2y; c, \lambda) \quad (3.2.1)$$
Proof. Proceeding with

\[ A(t) := \left( \frac{2^{2m}}{\left( \lambda c^{at} + \sum_{h=0}^{m-1} \frac{(t \log a)^h}{h!} \right) \left( \lambda b^{bt} + \sum_{h=0}^{m-1} \frac{(t \log b)^h}{h!} \right)} \right)^\alpha e^{abt+a^2b^2y^2}. \tag{3.2.2} \]

One can see that the expression for \( A(t) \) is symmetric in \( a \) and \( b \) and we can expand \( A(t) \) in the form of two series to obtain

\[ A(t) = \frac{1}{(ab)^\alpha m} \sum_{n=0}^{\infty} \sum_{k=0}^{n} HE_n^{[\alpha,m-1]}(bx,b^2y;c,\lambda) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} HE_k^{[\alpha,m-1]}(ax,a^2y;c,\lambda) \frac{(bt)^k}{k!}, \tag{3.2.3} \]

In the similar pattern we can show that

\[ A(t) := \frac{1}{(ab)^\alpha m} \sum_{n=0}^{\infty} HE_n^{[\alpha,m-1]}(ax,a^2y;c,\lambda) \frac{b^{n-k}}{(n-k)!} \sum_{k=0}^{\infty} HE_k^{[\alpha,m-1]}(bx,b^2y;c,\lambda) \frac{a^k}{k!} t^n. \tag{3.2.4} \]

Comparison of the coefficients of \( \frac{t^n}{n!} \) on the R.H.S. in the last two equations, gives us the desired result.

Remark 3.2.1. Adjustment \( \lambda = 1 \) and \( c = e \), the above result reduces to a recognized result of Pathan and Khan (2014. p.104, Theorem 4.1). Now by taking \( m = 1 \) in Theorem (3.2.1), we deduce the following result.

Corollary 3.2.1. Let \( a > 0, b > 0, c > 0 \) and \( n \geq 0 \), then the following identity holds true:

\[ \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k HE_n^{(\alpha)}(bx,b^2y;c,\lambda) HE_k^{(\alpha)}(ax,a^2y;c,\lambda) \]

\[ = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} HE_{n-k}^{(\alpha)}(ax,a^2y;c,\lambda) HE_k^{(\alpha)}(bx,b^2y;c,\lambda). \tag{3.2.5} \]

Remark 3.2.2. On setting \( b = 1 \) in Theorem (3.2.1), the following corollary is deduced.
Corollary 3.2.2. For all integers \( a > 0, c > 0, n \geq 0 \) and \( m \geq 1 \), the below identity holds true:

\[
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} H E_{n-k}^{[a,m-1]}(x, y; c, \lambda) H E_k^{[a,m-1]}(ax, a^2 y; c, \lambda)
= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} H E_{n-k}^{[a,m-1]}(ax, a^2 y; c, \lambda) H E_k^{[a,m-1]}(x, y; c, \lambda).
\]

(3.2.6)

Theorem 3.2.2. Let \( a, b, c > 0 \) with \( a \neq b \). Then for \( x, y \in \mathbb{R} \) and \( n \geq 0 \), the below identity holds true:

\[
\sum_{k=0}^{n} \binom{n}{k} a^{-n-k} b^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H E_{n-k}^{(a)} \left( bx + \frac{b}{a} i + j, b^2 z; c, \lambda \right) E_k^{(a)}(ay; c, \lambda)
= \sum_{k=0}^{n} \binom{n}{k} b^{-n-k} a^k \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} H E_{n-k}^{(a)} \left( ax + \frac{a}{b} i + j, a^2 z; c, \lambda \right) E_k^{(a)}(by; c, \lambda).
\]

(3.2.7)

Proof. Let

\[
A(t) := \frac{(2a)^{(a)(\lambda c) + 1)}(\lambda c + 1)(\lambda c + 1)^{a+1}}{(\lambda c + 1)(\lambda c + 1)^{a+1}}.
\]

\[
A(t) = \left( \frac{2a}{\lambda c + 1} \right)^a c^{a b x t + a^2 b^2 z t^2} \left( \frac{\lambda c + 1}{\lambda c + 1} \right) \left( \frac{2b}{\lambda c + 1} \right)^b c^{a b y t} \sum_{j=0}^{b-1} (-\lambda)^j c^{a t j},
\]

(3.2.8)

\[
= \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H E_{n}^{(a)} \left( bx + \frac{b}{a} i + j, b^2 z; c, \lambda \right) \frac{(al)^n}{n!} \sum_{k=0}^{\infty} E_k^{(a)}(ay; c, \lambda) \left( \frac{bt}{k} \right)^k,
\]

(3.2.9)

Looking on the other side, we have

\[
A(t) := \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a^{-n-k} b^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H E_{n-k}^{(a)} \left( ax + \frac{a}{b} i + j, a^2 z, c, \lambda \right) E_k^{(a)}(by; c, \lambda) \frac{t^n}{n!}.
\]

(3.2.10)

An identification of the coefficients of \( \frac{t^n}{n!} \) on R.H.S. of the last two equations, gives us the desired result.
Remark 3.2.3. $\lambda = 1$ and $c = e$ in the above result gives us a known result of Pathan and Khan (2014, p.105, Theorem 4.2).

**Theorem 3.2.3.** For a pair of integers $a$ and $b$ and all integers $n \geq 0$, the below identity holds true:

\[
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H_{n-k}^{(\alpha)} \left(bx + \frac{b}{a}i, b^2 z; c, \lambda \right) E_k^{(\alpha)} \left(ay + \frac{a}{b}j; c, \lambda \right)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H_{n-k}^{(\alpha)} \left(ax + \frac{a}{b}i, a^2 z; c, \lambda \right) E_k^{(\alpha)} \left(by + \frac{b}{a}j; c, \lambda \right).
\]

(3.2.11)

**Proof.** The proof is alike to Theorem (3.2.2) but we need to put equation (3.2.8) as

\[
A(t) := \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H_n^{(\alpha)} \left(bx + \frac{b}{a}i, b^2 z; c, \lambda \right) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} E_k^{(\alpha)} \left(ay + \frac{a}{b}j; c, \lambda \right) \frac{(bt)^k}{k!}.
\]

(3.2.12)

On the other hand equation (3.2.8) can be proved equal to

\[
A(t) := \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H_n^{(\alpha)} \left(ax + \frac{a}{b}i, a^2 z; c, \lambda \right) \frac{(bt)^n}{n!} \sum_{k=0}^{\infty} E_k^{(\alpha)} \left(by + \frac{b}{a}j; c, \lambda \right) \frac{(at)^k}{k!}.
\]

(3.2.13)

Some change of index and equation of the coefficients of $t$ to zero in (3.2.12) and (3.2.13), gives us the desired result.

**Remark 3.2.4.** $\lambda = 1$ and $c = e$ in the above result gives us a known result of Pathan and Khan (2014, p.106, Theorem 4.3).

**Remark 3.2.5.** On setting $y = 0$ in Theorem (3.2.3), an immediately corollary is obtained.

**Corollary 3.2.3.** For integers $a > 0, b > 0, c > 0$ and $n \geq 0$, the below identity holds true:

\[
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H_{n-k}^{(\alpha)} \left(bx + \frac{b}{a}i, b^2 z; c, \lambda \right) E_k^{(\alpha)} \left(\frac{a}{b}j; c, \lambda \right)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H_{n-k}^{(\alpha)} \left(ax + \frac{a}{b}i, a^2 z; c, \lambda \right) E_k^{(\alpha)} \left(\frac{b}{a}j; c, \lambda \right).
\]

(3.2.14)
Theorem 3.2.4. Let $a, b, c > 0$ with $a \neq b$. Then for $x, y \in \mathbb{R}$ and $n \geq 0$, the below identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k E_{n-k}^{(a)}(ay; c, \lambda) \sum_{i=0}^{a-1} (-\lambda)^i H E_k^{(a)} \left( bx + \frac{b}{a} i, b^2 z; c, \lambda \right)
= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k E_{n-k}^{(a)}(by; c, \lambda) \sum_{i=0}^{b-1} (-\lambda)^i H E_k^{(a)} \left( ax + \frac{a}{b} i, a^2 z; c, \lambda \right).
$$

(3.2.15)

Proof. Assume

$$
A(t) := \frac{(2a)^a (2b)^b (1 + \lambda (-1)^{a+1} a b t) e^{ab t + a^2 b^2 t^2}}{(\lambda e^{at} + 1)^a (\lambda e^{bt} + 1)^{a+1}}.
$$

(3.2.16)

$$
A(t) := \left( \frac{2a}{\lambda e^{at} + 1} \right)^a e^{ab t + a^2 b^2 t^2} \left( 1 - \lambda (-\lambda e^{bt})^a \right) \left( \frac{2b}{\lambda e^{bt} + 1} \right)^a e^{ab t},
$$

$$
= \sum_{k=0}^{\infty} \sum_{i=0}^{a-1} (-\lambda)^i H E_k^{(a)} \left( bx + \frac{b}{a} i, b^2 z; c, \lambda \right) \frac{(at)^k}{k!} \sum_{n=0}^{\infty} E_n^{(a)}(ay; c, \lambda) \frac{(bt)^n}{(n)!}.
$$

(3.2.17)

On the other side

$$
A(t) := \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{i=0}^{b-1} (-\lambda)^i H E_k^{(a)} \left( ax + \frac{a}{b} i, a^2 z; c, \lambda \right) E_{n-k}^{(a)}(by; c, \lambda) \frac{t^n}{n!}.
$$

(3.2.18)

A simple comparison of the coefficients of $\frac{t^n}{n!}$ on the R.H.S. of the last two equations, yields us the desired result.

Theorem 3.2.5. Let $a, b, c > 0$ with $a \neq b$ and $m \geq 1$. Then for $x, y \in \mathbb{R}$ and $n \geq 0$, the below identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k E_{n-k}^{(a,m)}(ay; c, \lambda) \sum_{i=0}^{a-1} (-\lambda)^i H E_k^{(a,m)} \left( bx + \frac{b}{a} i, b^2 z; c, \lambda \right)
= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k E_{n-k}^{(a,m)}(by; c, \lambda) \sum_{i=0}^{b-1} (-\lambda)^i H E_k^{(a,m)} \left( ax + \frac{a}{b} i, a^2 z; c, \lambda \right).
$$

(3.2.19)

3.3 Symmetry identities for generalized Hermite-Apostol-Genocchi polynomials
In this section, we give general symmetry identities for the generalized Hermite-Apostol-Genocchi polynomials $H^{\alpha,m-1}_n(x,y; a,c, \lambda)$ by applying the generating functions (3.1.25) and (3.1.26). Throughout this section $\alpha$ will be taken as an arbitrary real or complex parameter.

**Theorem 3.3.1.** Let $a > 0, b > 0, c > 0$ with $a \neq b$ then for $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k H^{\alpha,m-1}_{n-k}(bx,b^2y;c,\lambda) H^{\alpha,m-1}_k(ax,a^2y;c,\lambda) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H^{\alpha,m-1}_{n-k}(ax,a^2y;c,\lambda) H^{\alpha,m-1}_k(bx,b^2y;c,\lambda) \quad (3.3.1)
$$

**Proof.** Start with

$$
A(t) := \left( \frac{2^m t^{2m}}{(\lambda e^{at} + \sum_{h=0}^{m-1} \frac{(t \log a)^h}{h!}) (\lambda e^{bt} + \sum_{h=0}^{m-1} \frac{(t \log b)^h}{h!})} \right)^{\alpha} e^{abt+a^2b^2y^2}. \quad (3.3.2)
$$

One can see that the expression for $A(t)$ is symmetric in $a$ and $b$, so we can expand $A(t)$ into series in two ways

$$
A(t) := \frac{1}{(ab)^{\alpha m}} \sum_{n=0}^{\infty} H^{\alpha,m-1}_n(bx,b^2y;c,\lambda) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} H^{\alpha,m-1}_k(ax,a^2y;c,\lambda) \frac{(bt)^k}{k!},
$$

$$
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} H^{\alpha,m-1}_{n-k}(bx,b^2y;c,\lambda) \frac{a^{n-k}}{(n-k)!} H^{\alpha,m-1}_k(ax,a^2y;c,\lambda) \frac{b^k}{k!} t^n. \quad (3.3.3)
$$

With a similar pattern we can show that

$$
A(t) := \frac{1}{(ab)^{\alpha m}} \sum_{n=0}^{\infty} H^{\alpha,m-1}_n(ax,a^2y;c,\lambda) \frac{b^{n-k}}{(n-k)!} H^{\alpha,m-1}_k(bx,b^2y;c,\lambda) \frac{a^k}{k!} t^n. \quad (3.3.4)
$$

On comparison of the coefficients of $\frac{t^n}{n!}$ in the R.H.S. of the last two equations, we arrive at the desired result.

**Remark 3.3.1.** With $m = 1$ in Theorem 3.3.1, we immediately get the following result.
Corollary 3.3.1. Let $a > 0, b > 0$ and $c > 0$. Then for all integers $n \geq 0$, the following identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k H_{n-k}(b x, b^2 y; c, \lambda) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H_{n-k}(a x, a^2 y; c, \lambda). \tag{3.3.5}
$$

Remark 3.3.2. By adjusting $b = 1$ in Theorem 3.3.1, we get the following corollary.

Corollary 3.3.2. Let $a > 0$ and $b > 0$. Then for all integers $n \geq 0$ and $m \geq 1$, the below identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k H_{n-k}^{[\alpha, m-1]}(x, y; c, \lambda) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H_{n-k}^{[\alpha, m-1]}(a x, a^2 y; c, \lambda). \tag{3.3.6}
$$

Theorem 3.3.2. Let $a, b, c > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the below identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H_{n-k}^{[\alpha]}(bx + \frac{b}{a} i + j, b^2 z; c, \lambda) G_{k}^{(\alpha)}(ay; c, \lambda) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H_{n-k}^{[\alpha]}(ax + \frac{a}{b} i + j, a^2 z; c, \lambda) G_{k}^{(\alpha)}(by; c, \lambda). \tag{3.3.7}
$$

Proof. Let

$$
A(t) := \frac{(2at)^{\alpha}(2bt)^{\alpha}(\lambda c^{abt} + 1)^2 c^{ab(x+y)t} + a^2 b^2 z t^2}{(\lambda c^{at} + 1)^{\alpha+1}(\lambda c^{bt} + 1)^{\alpha+1}}.
$$

$$
A(t) := \left(\frac{2at}{\lambda c^{at} + 1}\right)^{\alpha} c^{abxt + a^2 b^2 z t^2} \left(\frac{\lambda c^{abt} + 1}{\lambda c^{bt} + 1}\right)^{\alpha} c^{abjt} \left(\frac{2bt}{\lambda c^{bt} + 1}\right)^{\alpha} c^{a^2zjt} \sum_{i=0}^{a-1} (-\lambda)^i c^{bti} \left(\frac{2bt}{\lambda c^{bt} + 1}\right)^{\alpha} c^{abjt} \sum_{j=0}^{b-1} (-\lambda)^j c^{a^2zjt}, \tag{3.3.8}
$$

$$
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H_{n-k}^{[\alpha]}(ax + \frac{b}{a} i + j, b^2 z; c, \lambda) G_{k}^{(\alpha)}(ay; c, \lambda) \frac{t^n}{n!}. \tag{3.3.9}
$$
On the other hand

\[ A(t) := \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} H G_{n-k}^{(\alpha)} \left( bx + \frac{a}{b} i + j, a^2 z, c, \lambda \right) G_k^{(\alpha)} (by; c, \lambda) \right) \frac{t^n}{n!}. \]  

(3.3.10)

A comparison of the coefficients of \( \frac{t^n}{n!} \) on the R.H.S. of the last two equations, gives us the desired result.

**Theorem 3.3.3.** Let \( a > 0 \) and \( b > 0 \). Then for all integers \( n \geq 0 \), the below identity holds true:

\[ \sum_{k=0}^{n} \binom{n}{k} (a)^{n-k}(b)^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H G_n^{(\alpha)} \left( bx + \frac{a}{b} i + j, a^2 z, c, \lambda \right) G_k^{(\alpha)} (by; c, \lambda) = \sum_{k=0}^{n} \binom{n}{k} (b)^{n-k}(a)^k \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} H G_n^{(\alpha)} \left( ax + \frac{a}{b} i, a^2 z, c, \lambda \right) G_k^{(\alpha)} (by + \frac{b}{a} j; c, \lambda). \]

(3.3.11)

**Proof.** The proof is alike Theorem 3.3.2 but equation (3.3.8) has to be written in the form

\[ H(t) := \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H G_n^{(\alpha)} \left( bx + \frac{a}{b} i, a^2 z, c, \lambda \right) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} G_k^{(\alpha)} (ay + \frac{a}{b} j; c, \lambda) \frac{(bt)^k}{k!}. \]

(3.3.12)

Also equation (3.3.8) can be depicted as

\[ H(t) := \sum_{n=0}^{\infty} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} H G_n^{(\alpha)} \left( ax + \frac{a}{b} i, a^2 z, c, \lambda \right) \frac{(bt)^n}{n!} \sum_{k=0}^{\infty} G_k^{(\alpha)} (by + \frac{b}{a} j; c, \lambda) \frac{(at)^k}{k!}. \]

(3.3.13)

Next with change of index and coefficient comparison of \( \frac{t^n}{n!} \) in (3.3.12) and (3.3.13), we get the expected result.

**Remark 3.3.3.** By adjusting \( y = 0 \) in Theorem 3.3.3, we get the following corollary.

**Corollary 3.3.3.** Let \( a > 0, b > 0 \) and \( c > 0 \). Then for all integers \( n \geq 0 \), the below identity holds true:

\[ \sum_{k=0}^{n} \binom{n}{k} (a)^{n-k}(b)^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} H G_n^{(\alpha)} \left( bx + \frac{a}{b} i, a^2 z, c, \lambda \right) G_k^{(\alpha)} (\frac{b}{a} j; c, \lambda) \]

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On comparing the coefficients of $t$

The other side gives

Proof. Let

$$A(t) := \frac{(2at)\alpha(2bt)\alpha(1 + \lambda(-1)^{a+1}e^{abt})e^{ab}(x+y)t + a^2b^2zt^2}{(\lambda e^{at} + 1)^{\alpha}(\lambda e^{bt} + 1)^{\alpha+1}}.$$ (3.3.16)

$$A(t) := \left(\frac{2at}{\lambda e^{at} + 1}\right)^{\alpha}e^{abxt + a^2b^2zt^2} \left(\frac{1 - \lambda(-b)^{a}}{\lambda e^{bt} + 1}\right)^{\alpha}e^{abt},$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{a-1} (-\lambda)^{i} H_{k}^{(a)} \left(bx + \frac{b}{a}i, b^2z; c, \lambda\right) \left(\frac{a^{k}}{k!} \sum_{n=0}^{\infty} G_{n}^{(a)}(ay; c, \lambda) b^{n} \frac{t^{n+k}}{(n)!}\right) t^{n},$$ (3.3.17)

$$A(t) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) b^{n-k} a^{k} \sum_{i=0}^{a-1} (-\lambda)^{i} H_{k}^{(a)} \left(bx + \frac{b}{a}i, b^2z; c, \lambda\right) G_{n-k}^{(a)}(ay; c, \lambda) \right) \frac{t^{n}}{n!}.$$ (3.3.18)

The other side gives

$$A(t) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) a^{n-k} b^{k} \sum_{i=0}^{b-1} (-\lambda)^{i} H_{k}^{(a)} \left(ax + \frac{a}{b}i, a^2z; c, \lambda\right) G_{n-k}^{(a)}(by; c, \lambda) \right) \frac{t^{n}}{n!}.$$ (3.3.18)

On comparing the coefficients of $\frac{t^{n}}{n!}$ on the R.H.S. of the last two equations, we reach at the desired result.

In view of Theorems (3.3.1) to (3.3.5), we easily obtain the following general symmetry identity

\[ \text{Theorem 3.3.4.} \text{ Let } a, b, c > 0 \text{ and } a \neq b. \text{ Then for } x, y \in \mathbb{R} \text{ and } n \geq 0, \text{ the below identity holds true:} \]

\[ \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) b^{n-k} a^{k} \sum_{i=0}^{a-1} (-\lambda)^{i} H_{k}^{(a)} \left(bx + \frac{b}{a}i, b^2z; c, \lambda\right) \right) G_{n-k}^{(a)}(ay; c, \lambda) \]

\[ = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) a^{n-k} b^{k} \sum_{i=0}^{b-1} (-\lambda)^{i} H_{k}^{(a)} \left(ax + \frac{a}{b}i, a^2z; c, \lambda\right) G_{n-k}^{(a)}(by; c, \lambda). \] (3.3.15)
Theorem 3.3.5. Let $a, b, c > 0$ with $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the below identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k} G_{n-k}^{(\alpha, m)} \left( a y; c, \lambda \right) \sum_{i=0}^{a-1} (-\lambda)^{i} G_{k}^{(\alpha, m)} \left( bx + \frac{b}{a} i, b^{2} z; c, \lambda \right)
$$

$$
= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k} G_{n-k}^{(\alpha, m)} \left( by; c, \lambda \right) \sum_{i=0}^{b-1} (-\lambda)^{i} G_{k}^{(\alpha, m)} \left( ax + \frac{a}{b} i, a^{2} z; c, \lambda \right) \tag{3.3.19}
$$