Chapter 2
Several Identities for the Generalized Apostol-Euler and Apostol-Genocchi Polynomials
2.1 Introduction

In this chapter, the families of special functions have been produced in a unified and generalized form. Different elementary properties and some symmetric identities for the generalized Apostol-Euler and also the Apostol-Genocchi polynomials are established with the support of different analytical means on their respective generating functions. In recent times, Zhang et al. (2008), Yang (2008), Yang et al. (2010) and Pathan and Khan (2012) also pointed out symmetric identities for the generalized Bernoulli and Apostol-Bernoulli polynomials. Most of these properties are appropriate extension to generalized Euler and Genocchi polynomials.

The ordinary Bernoulli polynomials $B_n(x)$, the ordinary Euler polynomials $E_n(x)$ and the ordinary Genocchi polynomials $G_n(x)$, along with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of order $\alpha$ (real or complex) are commonly defined through following generating functions (see details in Sandor and Cristei, (2004) and Srivastava and Choi, (2001); see also Srivastava and Pinter, (2004)):

\[
\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < 2\pi; 1^\alpha = 1), \quad (2.1.1)
\]

\[
\left( \frac{2}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < \pi; 1^\alpha = 1), \quad (2.1.2)
\]

and

\[
\left( \frac{2t}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < \pi; 1^\alpha = 1), \quad (2.1.3)
\]

so that obviously

\[
B_n(x) = B_n^1(x), \quad E_n(x) = E_n^1(x) \quad \text{and} \quad G_n(x) = G_n^1(x), \quad (n \in \mathbb{N}), \quad (2.1.4)
\]

where $\mathbb{N}_0$ is a set of natural numbers ($\mathbb{N} = 1, 2, 3, \cdots$).

We readily find from (2.1.1) to (2.1.4) that

\[
B_n^1(0) = B_n(0) = B_n, \quad E_n^1(0) = E_n(0) = E_n \quad \text{and} \quad G_n^1(0) = G_n(0) = G_n, \quad (n \in \mathbb{N}), \quad (2.1.5)
\]
A number of affecting analogous of the classical Bernoulli numbers and polynomials were initially investigated by Apostol (1951) and (more recently) by Srivastava (2000). We begin here with Apostol’s definition as follows:

**Definition 2.1.1.** The Apostol-Bernoulli polynomials $B_n(x; \lambda), (\lambda \in \mathbb{C})$ are given in terms of the generating function

$$\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x; \lambda) \frac{t^n}{n!}, \quad (|t| < 2\pi; \text{when } \lambda = 1; t < |\log \lambda| \text{ when } \lambda \neq 1).$$

(2.1.6)

with

$$B_n(x) = B_n(x; 1) \text{ and } B_n(\lambda) = B_n(0; \lambda),$$

(2.1.7)

here $B_n(\lambda)$ specifies the so called Apostol-Bernoulli numbers.

In addition to this, a further extended of the Apostol-Bernoulli polynomials was done by Luo and Srivastava (2006) as the so called Apostol-Bernoulli polynomials (of order $\alpha$) as follows:

**Definition 2.1.2.** The Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x; \lambda), (\text{of order } \alpha \in \mathbb{N}_0)$ with $(\lambda \in \mathbb{C})$ are given in terms of the following generating function

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t| < 2\pi; \text{when } \lambda = 1; t < |\log \lambda| \text{ when } \lambda \neq 1).$$

(2.1.8)

with

$$B_n^{(\alpha)}(x) = B_n^{(\alpha)}(x; 1) \text{ and } B_n^{(\alpha)}(\lambda) = B_n^{(\alpha)}(0; \lambda),$$

(2.1.9)

here $B_n^{(\alpha)}(x; \lambda)$ specifies the so called Apostol-Bernoulli numbers of order $\alpha$.

Another analogous extension of the generalized Euler polynomials was done by Luo (2006) as the so called Apostol-Euler polynomials of order $\alpha$.

**Definition 2.1.3.** The Apostol-Euler polynomials $E_n^{(\alpha)}(x; \lambda), (\text{of order } \alpha \in \mathbb{N}_0)$ with
\(\lambda \in \mathbb{C}\) are given in the form of generating function
\[
\left(\frac{2}{\lambda e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t| < \log(-\lambda)).
\] (2.1.10)
with
\[
E_n^{(\alpha)}(x) = E_n^{(\alpha)}(x; 1) \text{ and } E_n^{(\alpha)}(\lambda) = E_n^{(\alpha)}(0; \lambda),
\] (2.1.11)
here \(E_n^{(\alpha)}(x; \lambda)\) specifies the so called Apostol-Euler numbers of order \(\alpha\).

The topic of the Genocchi polynomials \(G_n(x)\) and their various extensions finds a remarkably large number of research which have appeared in the literature (for example Horadam (1991, p.145-166; 1992, p.21-34; 1992, p.239-243), Jang and Kim (2008), Luo (2009, p.1-9; 2009, p.113-125; 2011)). Moreover, Luo (2009, p.1-9; 2009, p.113-125; 2011), Lu and Luo (2013) brought up and explored the Apostol-Genocchi polynomials of order \(\alpha\) (real or complex) which are defined as follows:

**Definition 2.1.4.** The Apostol-Genocchi polynomials \(G_n^{(\alpha)}(x; \lambda)\) (of order \(\alpha \in \mathbb{N}_0\)) with \((\lambda \in \mathbb{C})\) are given in terms of the following generating function
\[
\left(\frac{2t}{\lambda e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t| < \log(-\lambda)).
\] (2.1.12)
with
\[
G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x; 1) \text{ and } G_n^{(\alpha)}(\lambda) = G_n^{(\alpha)}(0; \lambda)
\]
\[
G_n(x; \lambda) = G_n^{(1)}(x; \lambda) \text{ and } G_n(\lambda) = G_n^{(1)}(\lambda),
\] (2.1.13)
where \(G_n(\lambda), \ G_n^{(\alpha)}(\lambda)\) and \(G_n(x; \lambda)\) is the so called Apostol-Genocchi numbers (of order \(\alpha\)) and the Apostol-Genocchi polynomials respectively.

For every non-negative integer \(k \geq 0\), \(M_k(n) = \sum_{i=0}^{n} (-1)^i t^k\) is called the alternative integer power sum. \(M_k(n)\) is given by the exponential generating function as
\[
\sum_{k=0}^{\infty} M_k(n) \frac{t^k}{k!} = 1 - e^t + e^{2t} + \ldots + (-1)^n e^{nt} = \frac{1 - (-e^t)^{n+1}}{e^t + 1}.
\] (2.1.14)

**Definition 2.1.5.** For an arbitrary parameter \(\lambda\) (real or complex), the generalized sum of alternative integer powers \(M_k(n; \lambda)\) is given by the following generating functions
\[
\sum_{k=0}^{\infty} M_k(n; \lambda) \frac{t^k}{k!} = \frac{1 - \lambda(-e^t)^{n+1}}{\lambda e^t + 1}.
\] (2.1.15)
2.2 Some symmetric identities for the Apostol-Euler polynomials

Here we give general symmetry identities for the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x; \lambda)$ by utilizing generating functions (2.1.10) and (2.1.15). In the entire section $\alpha$ will be taken as an arbitrary real or complex parameter.

**Theorem 2.2.1.** For all integers $a > 0, b > 0$ and $n \geq 0, \alpha \geq 1, \lambda \in \mathbb{C}$, the following symmetry identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} E_{n-k}^{(\alpha)}(bx; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(a-1; \lambda) E_{k-i}^{(\alpha-1)}(ay; \lambda)
= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} E_{n-k}^{(\alpha)}(ax; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(b-1; \lambda) E_{k-i}^{(\alpha-1)}(by; \lambda).
$$

(2.2.1)

**Proof.** Let

$$
A(t) = \frac{2^{2a-1}e^{ab(x+y)t}(1 - \lambda(-e^{abt}))}{(\lambda e^{at} + 1)^{\alpha}(\lambda e^{bt} + 1)^{\alpha}}.
$$

(2.2.2)

$$
= \frac{1}{a^{\alpha} b^{\alpha-1}} \left( \frac{2a}{\lambda e^{at} + 1} \right)^{\alpha} e^{abxt} \left( \frac{1 - \lambda(-e^{at})}{\lambda e^{bt} + 1} \right)^{\alpha} e^{abyt},
$$

$$
= \frac{1}{a^{\alpha} b^{\alpha-1}} \left( \sum_{n=0}^{\infty} E_n^{(\alpha)}(bx; \lambda) \frac{(at)^n}{n!} \right) \left( \sum_{n=0}^{\infty} M_n(a-1; \lambda) \frac{(bt)^n}{n!} \right)
\times \left( \sum_{n=0}^{\infty} E_n^{(\alpha-1)}(ay; \lambda) \frac{(bt)^n}{n!} \right).
$$

(2.2.3)

Using same process, we have

$$
A(t) = \frac{1}{a^{\alpha} b^{\alpha}} \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} E_{n-k}^{(\alpha)}(bx; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(a-1; \lambda) E_{k-i}^{(\alpha-1)}(ay; \lambda) \right) \frac{t^n}{n!}.
$$

(2.2.4)

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Comparing the coefficients of $t^n$ in the last two equations (2.2.3) and (2.2.4), we get the result (2.2.1).

**Remark 2.2.1.** By adjusting $\lambda=1$ in Theorem 2.2.1, the result precipitates to a similar known result of Yang (2008, (9)).

**Corollary 2.2.1.** Let $a > 0, b > 0$ and $n \geq 0, \alpha \geq 1, \lambda \in \mathbb{C}$, then

$$
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} E_{n-k}^{(\alpha)}(bx) \sum_{i=0}^{k} \binom{k}{i} M_{i}(a-1; \lambda) E_{k-i}^{(\alpha-1)}(ay)
$$

$$=
\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} E_{n-k}^{(\alpha)}(ax) \sum_{i=0}^{k} \binom{k}{i} M_{i}(b-1) E_{k-i}^{(\alpha-1)}(by). \tag{2.2.5}
$$

**Remark 2.2.2.** Again adjusting $y=0$ and $\alpha=1$ in Theorem (2.2.1), we find the relation

**Corollary 2.2.2.** Let $a > 0, b > 0$ and $n \geq 0, \lambda \in \mathbb{C}$, then

$$
\sum_{i=0}^{n} \binom{n}{i} a^{i-1} b^{n-i} E_{i}(bx; \lambda) M_{n-i}(a-1; \lambda) = \sum_{i=0}^{n} \binom{n}{i} b^{i-1} a^{n-i} E_{i}(ax; \lambda) M_{n-i}(b-1; \lambda).
$$

**Theorem 2.2.2.** For all positive integers $a$ and $b$ and all integers $n \geq 0, \alpha \geq 1, (\lambda \in \mathbb{C})$, the below identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (\lambda)^{i+j} a^{k} b^{n-k} E_{k}^{(\alpha)}(bx + \frac{b}{a} i; \lambda) E_{n-k}^{(\alpha)}(ay + \frac{a}{b} j; \lambda)
$$

$$=
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\lambda)^{i+j} b^{k} a^{n-k} E_{k}^{(\alpha)}(ax + \frac{a}{b} i; \lambda) E_{n-k}^{(\alpha)}(by + \frac{b}{a} j; \lambda). \tag{2.2.7}
$$

**Proof.** Let $A(t) = \frac{(2a)^{\alpha}(2b)^{\alpha} e^{abt(x+y)} (\lambda^{a} e^{abt+1})(\lambda^{b} e^{abt+1})}{(\lambda e^{at+1})^{|a|+1}(\lambda e^{bt+1})^{|b|+1}}$, one can see that $A(t)$ is symmetric in $a$ and $b$, so $A(t)$ can be expanded into series in two forms

$$A(t) = \left(\frac{2a}{\lambda e^{at}+1}\right)^{\alpha} e^{abt} \left(\frac{\lambda^{a} e^{abt+1}}{\lambda e^{bt+1}+1}\right) \left(\frac{2b}{\lambda e^{bt}+1}\right)^{\alpha} e^{abt} \left(\frac{\lambda^{b} e^{abt+1}}{\lambda e^{at}+1}\right),$$
\[
\left(\frac{2a}{\lambda e^{at} + 1}\right)^{\alpha} e^{abt} \sum_{i=0}^{a-1} \lambda^i e^{bi} \left(\frac{2b}{\lambda e^{bt} + 1}\right)^{\alpha} e^{abjt} \sum_{j=0}^{b-1} \lambda^j e^{atj},
\]

\[
= \sum_{i=0}^{a-1} \lambda^i \left(\frac{2a}{\lambda e^{at} + 1}\right)^{\alpha} e^{(bx + \frac{b}{a}i)t} \sum_{j=0}^{b-1} \lambda^j \left(\frac{2b}{\lambda e^{bt} + 1}\right)^{\alpha} e^{(ax + \frac{a}{b}j)t},
\]

\[
= \left(\sum_{i=0}^{a-1} \lambda^i \sum_{k=0}^{\infty} E_k^{(\alpha)}(bx + \frac{b}{a}i; \lambda) \frac{(at)^k}{k!}\right) \left(\sum_{j=0}^{b-1} \lambda^j \sum_{n=0}^{\infty} E_n^{(\alpha)}(ay + \frac{a}{b}j; \lambda) \frac{(bt)^n}{n!}\right),
\]

\[
A(t) = \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \lambda^{i+j} E_k^{(\alpha)}(bx + \frac{b}{a}i; \lambda) E_n^{(\alpha)}(ay + \frac{a}{b}j; \lambda) \frac{t^{n+k}}{n!k!}.
\]

On the other side

\[
A(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \lambda^{k} b^{n-k} E_k^{(\alpha)}(bx + \frac{b}{a}i; \lambda) E_{n-k}^{(\alpha)}(ay + \frac{a}{b}j; \lambda)\right) \frac{t^n}{n!}.
\]

(2.2.8)

Comparing the coefficients of \(\frac{t^n}{n!}\) in the last two expressions for \(A(t)\), yields us the desired result.

Remark 2.2.3. After setting \(\lambda = 1\) in Theorem (2.2.2), the following result is proved as a special case

Corollary 2.2.3. Let \(a > 0, b > 0\) and \(n \geq 0\), then for \(\alpha \geq 1\), we have

\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \lambda^{k} b^{n-k} E_k^{(\alpha)}(bx + \frac{b}{a}i; \lambda) E_{n-k}^{(\alpha)}(ay + \frac{a}{b}j; \lambda)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \lambda^{k} b^{n-k} E_k^{(\alpha)}(ax + \frac{a}{b}i; \lambda) E_{n-k}^{(\alpha)}(by + \frac{b}{a}j; \lambda).\]

(2.2.10)

Remark 2.2.4. Adjusting \(y = 0, \alpha = 1\) in Theorem (2.2.2), we have

Corollary 2.2.4. Let \(a > 0, b > 0\) and \(n \geq 0\), then for \(\alpha \in \mathbb{C}\), we have

\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (\lambda)^{i+j} a^{k} b^{n-k} E_k(bx + \frac{b}{a}i; \lambda) E_{n-k}(\frac{a}{b}j; \lambda)
\]
Proof. The proof is alike Theorem (2.2.2), but requires a little change in the order of the summation of series. Suppose

\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\lambda)^{i+j} a^{n-k} b^{n-\lambda} E_k(a x + \frac{a}{b} i; \lambda) E_{n-k}(\frac{b}{a} j; \lambda).
\]

(2.2.11)

Theorem 2.2.3. Let for each pair of positive integers \(a\) and \(b\) and all integers \(n \geq 0, \alpha \geq 1, (\lambda \in \mathbb{C})\), the below identity holds true:

\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\lambda)^{i+j} a^{n-k} b^{\alpha} E_k^{(\alpha)}(bx + \frac{b}{a} i + j; \lambda) E_{n-k}^{(\alpha)}(ay; \lambda)
\]

= \[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\lambda)^{i+j} b^{n-k} E_k^{(\alpha)}(ax + \frac{a}{b} i + j; \lambda) E_{n-k}^{(\alpha)}(by; \lambda).
\]

(2.2.12)

Proof. The proof is alike Theorem (2.2.2), but requires a little change in the order of the summation of series. Suppose

\[
A(t) = \frac{(2a)^\alpha (2b)^\alpha} {\lambda e^{at} + 1} \left( \frac{e^{\lambda e^{bt} + 1}} {\lambda e^{bt} + 1} \right) \left( \frac{e^{\lambda^\alpha e^{bt} + 1}} {\lambda e^{bt} + 1} \right),
\]

(2.2.13)

\[
= \frac{2a} {\lambda e^{at} + 1} \left( \sum_{j=0}^{b-1} \sum_{i=0}^{a-1} \lambda^{i+j} \left( \frac{2b} {\lambda e^{bt} + 1} \right) e^{(bx + \frac{b}{a} i + j) at} \right) e^{\lambda e^{at} j},
\]

(2.2.14)
Identifying the coefficients of \( t^n \) in the above two expressions for \( A(t) \) provides us the desired result.

**Remark 2.2.5.** After setting \( \lambda = 1 \) in Theorem (2.2.3), following result is deduced as a special case.

**Corollary 2.2.5.** Let for all integers \( a > 0, b > 0 \) and \( n \geq 0 \) with \( \alpha \geq 1 \), we have

\[
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} G_{n-k}^{(\alpha)}(ax; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(a-1; \lambda) G_{k-i}^{(\alpha-1)}(ay; \lambda)
\]

\[= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} G_{n-k}^{(\alpha)}(ax; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(b-1; \lambda) G_{k-i}^{(\alpha-1)}(by; \lambda). \tag{2.3.1}
\]

**Remark 2.2.6.** On setting \( y = 0 \) and \( \alpha = 1 \) in Theorem (2.2.3), we have

**Corollary 2.2.6.** Let for all integers \( a > 0, b > 0 \) and \( n \geq 0 \) with \( \lambda \in \mathbb{C} \), we have

\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} (\lambda)^{i+j} a^{n-k} E_k(bx + \frac{b}{a} i + j; \lambda) E_{n-k}(\lambda)
\]

\[= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} (\lambda)^{i+j} b^{n-k} E_k(ax + \frac{a}{b} i + j; \lambda) E_{n-k}(\lambda). \tag{2.2.16}
\]

### 2.3 Symmetric identities for the Apostol-Genocchi polynomials

In this part of the chapter, we give general symmetry identities for the generalized Apostol-Genocchi polynomials \( G_n^{(\alpha)}(x; \lambda) \) by utilizing generating functions (2.1.12) and (2.1.15).

**Theorem 2.3.1.** Let for all integers \( a > 0, b > 0 \) and \( n \geq 0 \) with \( \alpha \geq 1, \lambda \in \mathbb{C} \), the following relation holds true:

\[
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} G_{n-k}^{(\alpha)}(bx; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(a-1; \lambda) G_{k-i}^{(\alpha-1)}(ay; \lambda)
\]

\[= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} G_{n-k}^{(\alpha)}(ax; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(b-1; \lambda) G_{k-i}^{(\alpha-1)}(by; \lambda). \tag{2.3.1}
\]
Proof. Let

\[ A(t) = \frac{(2t)^{2\alpha-1}e^{ab(x+y)t}(1 - \lambda(-e^{abt}))}{(\lambda e^{at} + 1)^{\alpha}(\lambda e^{bt} + 1)^{\alpha}}, \]  

(2.3.2)

\[ = \frac{1}{a^\alpha b^{\alpha-1}} \left( \frac{2at}{\lambda e^{at} + 1} \right)^\alpha e^{abzt} \left( \frac{1 - \lambda(-e^{at})^b}{\lambda e^{bt} + 1} \right)^\alpha \left( \frac{2bt}{\lambda e^{bt} + 1} \right)^z e^{abyt}, \]

\[ = \frac{1}{a^\alpha b^{\alpha-1}} \left( \sum_{n=0}^{\infty} G_n^{(a)}(bx; \lambda) \frac{(at)^n}{n!} \right) \left( \sum_{n=0}^{\infty} M_n(a - 1; \lambda) \frac{(bt)^n}{n!} \right) \left( \sum_{n=0}^{\infty} G_n^{(a-1)}(ay; \lambda) \frac{(bt)^n}{n!} \right), \]

\[ A(t) = \frac{1}{a^\alpha b^{\alpha}} \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{b+1} G_{n-k}^{(a)}(bx; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(a - 1; \lambda) G_{k-i}^{(a-1)}(ay; \lambda) \right) \frac{t^n}{n!}. \]

(2.3.3)

Following similar plan, we have

\[ A(t) = \frac{1}{a^\alpha b^{\alpha}} \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} G_{n-k}^{(a)}(ax; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(b - 1; \lambda) G_{k-i}^{(a-1)}(by; \lambda) \right) \frac{t^n}{n!}. \]

(2.3.4)

Finally when the coefficients of \( \frac{t^n}{n!} \) in the last two equations (2.3.3) and (2.3.4) are compared, we get the expected result.

Remark 2.3.1. On setting \( \lambda = 1 \) in Theorem 2.3.1, the result precipitates to a similar known result of Yang (2008, (9)).

Corollary 2.3.1. Let for all integers \( a > 0, b > 0 \) and \( n \geq 0, \alpha \geq 1 \) with \( \lambda \in \mathbb{C} \), then

\[ \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} G_{n-k}^{(a)}(bx; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(a - 1; \lambda) G_{k-i}^{(a-1)}(ay) \]

\[ = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} G_{n-k}^{(a)}(ax; \lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(b - 1; \lambda) G_{k-i}^{(a-1)}(by). \]

(2.3.5)

Remark 2.3.2. On setting \( y = 0 \) and \( \alpha = 1 \) in Theorem (2.3.1), we achieve the relation

Corollary 2.3.2. Let for all integers \( a > 0, b > 0 \) and \( n \geq 0 \) with \( \lambda \in \mathbb{C} \), we have

\[ \sum_{i=0}^{n} \binom{n}{i} a^{i-1} b^{n-i} G_i(bx; \lambda) M_{n-i}(a-1; \lambda) = \sum_{i=0}^{n} \binom{n}{i} b^{i-1} a^{n-i} G_i(ax; \lambda) M_{n-i}(b-1; \lambda). \]

(2.3.6)
Remark 2.3.3. On setting $x = 0$ in (2.3.6), we get the relation

$$\sum_{i=0}^{n} \binom{n}{i} a^{i-1} b^{n-i} G_i(\lambda) M_{n-i}(a-1; \lambda) = \sum_{i=0}^{n} \binom{n}{i} b^{i-1} a^{n-i} G_i(\lambda) M_{n-i}(b-1; \lambda).$$

(2.3.7)

Remark 2.3.4. In (2.3.7) if $\lambda = 1$, then a result similar to the result of Tuenter (2001) is obtained.

$$\sum_{i=0}^{n} \binom{n}{i} a^{i-1} b^{n-i} G_i M_{n-i}(a-1) = \sum_{i=0}^{n} \binom{n}{i} b^{i-1} a^{n-i} G_i M_{n-i}(b-1).$$

(2.3.8)

Theorem 2.3.2. For each pair of positive integers $a, b$ and all integers $n \geq 0$ with $\alpha \geq 1, \lambda \in \mathbb{C}$, the below identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{k} b^{n-k} G_k^{(\alpha)}(ax + b; \lambda) G_{n-k}^{(\alpha)}(ay + \frac{a}{b} j; \lambda) = \sum_{k=0}^{n} \binom{n}{k} b^{k} a^{n-k} G_k^{(\alpha)}(bx + \frac{a}{b} i; \lambda) G_{n-k}^{(\alpha)}(by + \frac{a}{b} j; \lambda).$$

(2.3.9)

Proof. Let $A(t) = \frac{(2at)^{\alpha}(2bt)^{\alpha}}{(\lambda e^{at+1})^{\alpha+1}(\lambda e^{bt+1})^{\alpha+1}}$, one can see that the expression for $A(t)$ is symmetric in $a$ and $b$ and so we can expand $A(t)$ in two series to prove the theorem.

$$A(t) = \left(\frac{2at}{\lambda e^{at+1}}\right)^{\alpha} e^{abt} \left(\frac{\lambda e^{bt+1}}{\lambda e^{at+1}}\right)^{\alpha} \left(\frac{2bt}{\lambda e^{bt+1}}\right)^{\alpha} e^{abt} \left(\frac{\lambda e^{at+1}}{\lambda e^{bt+1}}\right)^{\alpha} e^{abt},$$

$$= \left(\frac{2at}{\lambda e^{at+1}}\right)^{\alpha} e^{abt} \sum_{i=0}^{a-1} \lambda^i e^{bti} \left(\frac{2bt}{\lambda e^{bt+1}}\right)^{\alpha} e^{abt} \sum_{j=0}^{b-1} \lambda^j e^{atj},$$

$$= \sum_{i=0}^{a-1} \lambda^i \left(\frac{2at}{\lambda e^{at+1}}\right)^{\alpha} e^{(bx + \frac{a}{b} i)at} \sum_{j=0}^{b-1} \lambda^j \left(\frac{2bt}{\lambda e^{bt+1}}\right)^{\alpha} e^{(ax + \frac{a}{b} j)bt},$$

$$= \left(\sum_{i=0}^{a-1} \lambda^i \sum_{k=0}^{\infty} G_k^{(\alpha)}(bx + \frac{b}{a} i; \lambda) (at)^k/k!\right) \left(\sum_{j=0}^{b-1} \lambda^j \sum_{n=0}^{\infty} G_n^{(\alpha)}(ay + \frac{a}{b} j; \lambda) (bt)^n/n!\right).$$

$$A(t) = \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \lambda^i \lambda^j G_k^{(\alpha)}(bx + \frac{b}{a} i; \lambda) G_n^{(\alpha)}(ay + \frac{a}{b} j; \lambda) \frac{t^{n+k}}{n!k!}.$$
Readjusting $n$ by $n - k$ in above equation, we get

$$A(t) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^{k} b^{n-k} G_{k}^{(\alpha)} (bx + \frac{b}{a} i; \lambda) G_{n-k}^{(\alpha)} (ay + \frac{a}{b} j; \lambda) \right) \frac{t^{n}}{n!}. \quad (2.3.10)$$

On the other side

$$A(t) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} a^{n-k} b^{k} G_{k}^{(\alpha)} (ax + \frac{a}{b} i; \lambda) G_{n-k}^{(\alpha)} (by + \frac{b}{a} j; \lambda) \right) \frac{t^{n}}{n!}. \quad (2.3.11)$$

Finally when the coefficients of $\frac{t^{n}}{n!}$ in the last two expressions for $A(t)$ are compared, we get the desired result.

**Remark 2.3.5.** On setting $\lambda = 1$ in Theorem (2.3.2), the following result is produced as a special case.

**Corollary 2.3.4.** Let $a > 0, b > 0$ and $n \geq 0$ with $\alpha \geq 1$, then we have

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^{k} b^{n-k} G_{k}^{(\alpha)} (bx + \frac{b}{a} i; \lambda) G_{n-k}^{(\alpha)} (ay + \frac{a}{b} j)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} b^{k} a^{n-k} G_{k}^{(\alpha)} (ax + \frac{a}{b} i; \lambda) G_{n-k}^{(\alpha)} (by + \frac{b}{a} j). \quad (2.3.12)$$

**Remark 2.3.6.** On setting $y = 0, \alpha = 1$ in Theorem (2.3.2), we obtain

**Corollary 2.3.5.** For all integers $a > 0, b > 0$ and $n \geq 0$ with $\lambda \in \mathbb{C}$, we have

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (\lambda)^{i+j} a^{k} b^{n-k} G_{k}^{(\alpha)} (bx + \frac{b}{a} i; \lambda) G_{n-k}^{(\alpha)} (\frac{a}{b} j; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\lambda)^{i+j} b^{k} a^{n-k} G_{k}^{(\alpha)} (ax + \frac{a}{b} i; \lambda) G_{n-k}^{(\alpha)} (\frac{b}{a} j; \lambda). \quad (2.3.13)$$

**Theorem 2.3.3.** Let for each pair of positive integers $a$ and $b$ and all integers $n \geq 0$ with $\alpha \geq 1, \lambda \in \mathbb{C}$, the identity below holds true:

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (\lambda)^{i+j} a^{k} b^{n-k} G_{k}^{(\alpha)} (bx + \frac{b}{a} i + j; \lambda) G_{n-k}^{(\alpha)} (ay; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\lambda)^{i+j} b^{k} a^{n-k} G_{k}^{(\alpha)} (ax + \frac{a}{b} i + j; \lambda) G_{n-k}^{(\alpha)} (by; \lambda). \quad (2.3.14)$$
Proof. The proof is alike to Theorem (2.3.2), but a slight change in the order of summation of the series is needed.

\[
A(t) = \frac{(2at)^\alpha (2bt)^\alpha e^{ab(x+y)t}(\lambda a e^{abt} + 1)(\lambda b e^{abt} + 1)}{(\lambda e^{at} + 1)^{\alpha+1}(\lambda e^{bt} + 1)^{\alpha+1}}.
\]

\[
A(t) = \left(\frac{2at}{\lambda e^{at} + 1}\right)^\alpha e^{abxt} \left(\frac{\lambda a e^{abt} + 1}{\lambda e^{bt} + 1}\right)^\alpha e^{abyt} \left(\frac{\lambda b e^{abt} + 1}{\lambda e^{at} + 1}\right),
\]

\[
= \left(\frac{2at}{\lambda e^{at} + 1}\right)^\alpha \sum_{j=0}^{b-1} \lambda^i e^{btj} \left(\frac{2bt}{\lambda e^{bt} + 1}\right)^\alpha e^{abyt} \sum_{j=0}^{b-1} \lambda^j e^{ajt},
\]

\[
= \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \lambda^{i+j} \left(\frac{2at}{\lambda e^{at} + 1}\right)^\alpha e^{(bx + \frac{b}{a}i + j)t} \left(\frac{2bt}{\lambda e^{bt} + 1}\right)^\alpha e^{abyt},
\]

\[
= \left(\sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \lambda^{i+j} \sum_{k=0}^{\infty} G^{(a)}_k (bx + \frac{b}{a}i + j; \lambda) \frac{(at)^k}{k!}\right) \left(\sum_{n=0}^{\infty} G^{(a)}_n (ay; \lambda) \frac{(bt)^n}{n!}\right),
\]

\[
A(t) = \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \lambda^{i+j} G^{(a)}_k (bx + \frac{b}{a}i + j; \lambda) a^k b^n G^{(a)}_n (ay; \lambda) \frac{t^{n+k}}{n!k!}.
\]

Readjusting \(n\) by \(n - k\) in above equation, we get

\[
A(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^k b^{n-k} G^{(a)}_k (bx + \frac{b}{a}i + j; \lambda) G^{(a)}_n (ay; \lambda) \right) \frac{t^n}{n!}.
\]

(2.3.15)

On the other side, we have

\[
A(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^{n-k} b^k G^{(a)}_k (ax + \frac{a}{b}i + j; \lambda) G^{(a)}_{n-k} (by; \lambda) \right) \frac{t^n}{n!}.
\]

(2.3.16)

Comparision of the coefficients of \(\frac{t^n}{n!}\) in the above two mentioned expressions for \(A(t)\) we come to the desired result.

Remark 2.3.7. On setting \(\lambda = 1\) in Theorem (2.3.3), we deduce the result given below

Corollary 2.3.6. Let \(a > 0, b > 0\) and \(n \geq 0, \alpha \geq 1\), then we have

\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^k b^{n-k} G^{(a)}_k (bx + \frac{b}{a}i + j) G^{(a)}_{n-k} (ay).
\]
\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} b^k a^{n-k} G_k^{(\alpha)}(ax + \frac{a}{b}i + j) G_{n-k}^{(\alpha)}(by).
\] (2.3.17)

**Remark 2.3.8.** Setting \( y = 0, \alpha = 1 \) in Theorem (2.3.3), we have

**Corollary 2.3.7.** For all integers \( a > 0, b > 0 \) and \( n \geq 0, \lambda \in \mathbb{C}, \) then

\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (\lambda)^i j^k a^{n-k} b^k G_k(a x + \frac{a}{b}i + j; \lambda) G_{n-k}(\lambda)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\lambda)^i j^k a^{n-k} b^k G_k(a x + \frac{a}{b}i + j; \lambda) G_{n-k}(\lambda).
\] (2.3.18)

**Concluding Remarks:** In this chapter, several properties of the generalized Apostol Euler and Apostol Genocchi polynomials in the form of symmetric identities are established by applying the generating functions (2.1.10), (2.1.12) and (2.1.15). These identities of the generalized polynomials extended comparatively to generalized Euler and Genocchi polynomials.