Chapter - 6

Semi Smooth Graceful Labeling and Its Applications
6.1 **Introduction**: 

In 1966 A. Rosa [33] defined $\alpha$-valuation as a $\beta$-valuation (graceful labeling) with an additional property. A graph which admits $\alpha$-valuation is necessarily bipartite. A natural generalization of graceful graph is the notion of $k$-graceful graph. It is obvious that 1-graceful is graceful and a graph which admits $\alpha$-valuation is always $k$-graceful graph, $\forall k \in \mathbb{N}$. Ng [31] has identified some graphs that are $k$-graceful, $\forall k \in \mathbb{N}$, but do not have $\alpha$-valuation.

6.2 **Definition and Known Results**: 

**Definition 6.2.1**: A bipartite graceful graph $G$ with graceful labeling $f$ is said to be *smooth graceful graph* if it admits an injective function $g : V(G) \longrightarrow \{0, 1, \ldots, \lfloor \frac{q-1}{2} \rfloor, \lfloor \frac{q+2}{2} \rfloor + l, \ldots, q + l\}$ such that its induced edge labeling map $g^* : E(G) \longrightarrow \{1 + l, 2 + l, \ldots, q + l\}$ defined as $g^*(e) = |g(u) - g(v)|$, for every edge $e = (u, v) \in E(G)$, for any $l \in \mathbb{N}$ is a bijection.

**Definition 6.2.2**: A *semi smooth graceful graph* $G$, we mean it is a bipartite graph with $|E(G)| = q$ and the property that for all non-negative integer $l$, there is an integer $t$ ($1 \leq t \leq q$) and an injective function $g : V(G) \longrightarrow \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\}$ such that the induced edge labeling function $g^* : E(G) \longrightarrow \{1 + l, 2 + l, \ldots, q + l\}$ defined as $g^*(e) = |g(u) - g(v)|$ is a bijection for every edge $e = (u, v) \in E(G)$.

Smooth graceful graph will help to produce new disconnected as well as connected graceful graphs. Kaneria and Jariya [17,18] define smooth graceful labeling. Every smooth graceful graph is also a semi smooth graceful graph. They proved cycle $C_n$ ($n \equiv 0 \pmod{4}$), path $P_n$, grid graph $P_n \times P_m$ and complete bipartite graph $K_{2,n}$ are smooth graceful graphs. Kaneria, Viradia and Makadia [23] proved that there is a path union of a semi smooth graceful graph, star of a semi smooth graceful graph and cycle graph of a semi smooth graceful graph are graceful. They also proved step grid graph $St_n$, Cycle graph $C(t \cdot H)$ and $C^m(t \cdot C_n)$ are smooth graceful graphs, where $t \equiv 0 \pmod{4}$, $n \equiv 0 \pmod{4}$, $m \in \mathbb{N}$ and $H$ is a semi smooth graceful graph.
6.3 Semi Smooth Graceful Labeling of Some Graphs:

**Theorem**—6.3.1: $B_{m,n}$ is a semi smooth graceful graph.

**Proof:** We know that $B_{m,n}$, bistar graph obtained by joining apex vertices of $K_{1,m}$ and $K_{1,n}$ by a new edge. Let $v_0, v_1, \ldots, v_m, u_0, u_1, \ldots, u_n$ be vertices of $B_{m,n}$. Obviously $E(B_{m,n}) = \{(u_0, v_0), (v_0, v_1), \ldots, (v_0, v_m), (u_0, u_1), \ldots, (u_0, u_n)\}$ i.e. $|V(B_{m,n})| = m+n+2$ and $|E(B_{m,n})| = m + n + 1$. We shall define a vertex labeling function $f : V(B_{m,n}) \rightarrow \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\}$, where $q = |E(B_{m,n})| = m+n+1$, $t = m+1$ and $l$ is the arbitrary non-negative integer as follows,

$$f(v_0) = q = m+n+1+l; \ f(u_0) = m;$$
$$f(v_i) = i - 1, \quad \forall \ i = 1, 2, \ldots, m;$$
$$f(u_j) = q - j + l, \quad \forall \ j = 1, 2, \ldots, n.$$  

Above labeling pattern give rise a semi smooth graceful labeling to the graph $B_{m,n}$, as it will produce edge labels $(v_0, v_1) = q+l, (v_0, v_2) = q-1+l, \ldots, (v_0, v_m) = q-m+1+l = n + 2 + l, (u_0, v_0) = n + 1 + l, (u_0, u_1) = n + l, (u_0, u_2) = n - 1 + l, \ldots, (u_0, u_n) = 1 + l$. Thus, $f$ and its induced function $f^*$ both are bijective maps. Therefore, $B_{m,n}$ is a semi smooth graceful graph.

**Illustration**—6.3.2: $B_{3,5}$ and its semi smooth graceful labeling shown in figure—6.1. Here $q = |E(B_{3,5})| = 9.$
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Theorem—6.3.3 : \( S'(B_{m,n}) \) is a semi smooth graceful graph.

Proof : Let \( G = S'(B_{m,n}) \), splitting graph of \( B_{m,n} \). Let \( V(G) = \{v_0, v_1, \ldots, v_m, u_0, u_1, \ldots, u_n, v_0', v_1', \ldots, v_m', u_0', u_1', \ldots, u_n'\} \). It is obvious that \( E(S'(B_{m,n})) = E(B_{m,n}) \cup \{(v_0, v_i') \ (i = 1, 2, \ldots, m), (v_0', v_i)(i = 1, 2, \ldots, m), (v_0', u_0), (u_0', v_0), (u_0', u_j)(j = 1, 2, \ldots, n), (u_0, u_j') \ (j = 1, 2, \ldots, n) \}. \) i.e. \( |V(S'(B_{m,n}))| = 2|V(B_{m,n})| = 2(m + n + 2) \) and \( |E(S'(B_{m,n}))| = 3|E(B_{m,n})| = 3(m + n + 1) \). We shall define a vertex labeling function \( f : V(G) \to \{0, 1, \ldots, t - 1, t + l, t + l + 1, \ldots, q + l\} \), where \( q = |E(G)| = 3(m + n + 1), \ t = 2m + 2 \) and \( l \) is the arbitrary non-negative integer as follows,

\[
\begin{align*}
  f(v_0) &= q + l, \\
  f(u_0) &= 2m, \\
  f(v_i) &= m + (i - 1), & \forall \ i = 1, 2, \ldots, m; \\
  f(v'_i) &= m - i, & \forall \ i = 1, 2, \ldots, m; \\
  f(u_j) &= 2m + 2j + l, & \forall \ j = 1, 2, \ldots, n; \\
  f(u'_j) &= 2(m + n) + j + l, & \forall \ j = 1, 2, \ldots, n.
\end{align*}
\]

Above defined labeling pattern give rise a semi smooth graceful labeling to the graph \( G \).

Illustration—6.3.4 : \( S'(B_{4,5}) \) and its semi smooth graceful labeling shown in figure—6.2. Here \( q = |E(S' B_{3,5})| = 30 \).

\[\text{figure—6.2} \quad S'(B_{4,5}) \text{ and its semi smooth graceful labeling.}\]

Theorem—6.3.5 : \( S'(P_n) \) is a semi smooth graceful graph.

Proof : Let \( P_n \) be the path on consecutive \( v_1, v_2, \ldots, v_n \) vertices. For the graph \( S'(P_n) \) added vertices corresponding to \( v_1, v_2, \ldots, v_n \) are \( v_1', v_2', \ldots, v_n' \). It is obvious that \( |V(S(P_n))| = 2n \) and \( |E(S'(P_n))| = 3n - 3 \).
We shall define a labeling function $f : V(S'(P_n)) \rightarrow \{0, 1, \ldots, t - l, t + l + 1, \ldots, q + l\}$, where $q = |E(S'(P_n))| = 3n - 3$, $t = 3\lfloor \frac{n}{2} \rfloor - 1$ and $l$ is the arbitrary non-negative integer as follows,

$$f(v_i) = q - \frac{3}{2}(i - 1) + l, \quad \text{when } i \text{ is odd},$$

$$= \left(\frac{3i}{2}\right) - 2, \quad \text{when } i \text{ is even, } \forall i = 1, 2, \ldots, n;$$

$$f(v'_i) = f(v_i) - 1, \quad \forall i = 1, 2, \ldots, n.$$

Above defined labeling pattern give rise a semi smooth graceful labeling to the graph $S'(P_n)$ and so $S'(P_n)$ is a semi smooth graceful graph.

**Illustration—6.3.6 :** $S'(P_6)$ and its semi smooth graceful labeling shown in figure—6.3.

Here $q = |E(S'(P_7))| = 15$.

![figure—6.3](image-url)  

$S'(P_6)$ and its semi smooth graceful labeling.

**Theorem—6.3.7 :** Let $G$ be a semi smooth graceful graph. A graph obtained by joining $G$ and $B_{m,n}^2$ with an arbitrary path $P_r$ is graceful.

**Proof :** Let $K$ be the graph obtained by joining a semi smooth graph $G$, $B_{m,n}^2$ with an arbitrary path $P_r$. Let $q_1 = |E(G)|$, $q_2 = r - 1$, $q_3 = 2(m + n) + 1$. Let $G$ be a semi smooth graceful graph with semi smooth vertex labeling function $f : V(G) \rightarrow \{0, 1, \ldots, t - 1, t + l, t + l + 1, \ldots, q_1 + l\}$, whose induced edge labeling function is absolute difference of end vertices for each edge and $t \in \{1, 2, \ldots, q_1\}$, $l$ be an arbitrary non-negative integer.

Since $G$ is a bipartite graph, we shall have following partitions of $V(G)$.

$$V_1 = \{u \in V(G)/f(u) < t \}$$

$$V_2 = \{u \in V(G)/f(u) \geq t\}$$
Let \( w_0 \in V(G) \) be such that \( f(w_0) = t - 1 \), such vertex would lies in \( V_1 \), otherwise \( 1 + l \) edge label can not be produce in \( G \). In fact \( 1 + l \) edge label can be produce by the vertices of \( G \) whose vertex labels are \( t - 1 \), \( t + l \) and they are adjacent in \( G \). Let \( w_0 = w_1, w_2, \ldots, w_r \) be the vertices of \( P_r \). Let \( u_i(i = 0, 1, \ldots, m), v_j(j = 0, 1, \ldots, n) \) be vertices of \( B_{m,n}^2 \) with \( w_r = v_0 \). Here \( |E(K)| = q_1 + q_2 + q_3 = q_1 + r + 2(m + n) \). Now we define a vertex labeling function \( g : V(K) \to \{0, 1, \ldots, q_1 + q_2 + q_3\} \) as follows.

\[
g(w) = f(w), \quad \text{when } w \in V_1; \\
g(w) = f(w) - l + r + 2(m + n), \quad \text{when } w \in V_2; \\
g(w_i) = t + r + 2(m + n) - \frac{i}{2}, \quad \text{when } i \text{ is even}, \\
g(w_i) = t + \frac{i-3}{2}, \quad \text{when } i \text{ is odd}, \forall i = 1, 2, \ldots, r;
\]

**Case-I** : \( r \) is odd.

\[
g(v_0) = g(w_r), \quad g(u_0) = g(w_r) + q_3; \\
g(v_j) = g(w_r) + j, \quad \forall j = 1, 2, \ldots, n; \\
g(u_i) = g(w_r) + n + i, \quad \forall i = 1, 2, \ldots, m;
\]

**Case-II** : \( r \) is even.

\[
g(v_0) = g(w_{r-1}) + q_3, \quad g(u_0) = g(w_{r-1}) + 1; \\
g(v_j) = g(w_{r-1}) + j + 1, \quad \forall j = 1, 2, \ldots, n; \\
g(u_i) = g(w_{r-1}) + n + i + 1, \quad \forall i = 1, 2, \ldots, m.
\]

Above labeling function give rise a graceful labeling to the graph \( K \) and so, it is graceful.

**Illustration-6.3.8** : A graph obtained by joining \( B_{3,4} \) and \( B_{3,3}^2 \) with path \( P_6 \) and its graceful labeling is shown in figure-6.4

![Graph](image-url)
6.4 Semi Smooth Graceful Labeling and $\alpha$-Valuation:

**Theorem—6.4.1:** A smooth graceful labeling $g$ on a graph $G$ is also $\alpha$-labeling for the graph $G$.

**Proof:** Let $G$ be a smooth graceful graph with a smooth graceful labeling $g : V(G) \rightarrow \{0, 1, \ldots, \lfloor \frac{q-1}{2} \rfloor, \lfloor \frac{q+1}{2} \rfloor + l, \lfloor \frac{q+3}{2} \rfloor + l, \ldots, q + l \}$ and its induced edge labeling function $g^* : E(G) \rightarrow \{1 + l, 2 + l, \ldots, q + l \}$ defined as $g^*(e) = |g(u) - g(v)|$ is a bijection, for every edge $e = (u, v) \in E(G)$, where $l$ is an arbitrary non-negative integer.

By taking $l = 0$, $g$ becomes a graceful labeling for the graph $G$ and we have $k = \lfloor \frac{q-1}{2} \rfloor$ ($0 \leq k \leq q - 1$) such that for every $e = (x, y) \in E(G)$, we must have either $f(x) \in \{0, 1, \ldots, k\}$ and $f(y) \in \{k + 1, k + 2, \ldots, q\}$ or $f(x) \in \{k + 1, k + 2, \ldots, q\}$ and $f(y) \in \{0, 1, \ldots, k\}$. i.e. for every edge $e = (x, y) \in E(G)$ either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$.

Thus, $g$ becomes an $\alpha$-labeling for the graph $G$.

**Theorem—6.4.2:** A graph $G$ which admits an $\alpha$-labeling is semi smooth graceful graph.

**Proof:** Let $G$ be a graph and $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ be an $\alpha$-labeling for $G$. i.e. $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ is an injective map and its induced edge labeling function $f^* : E(G) \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e) = |f(u) - f(v)|$ is a bijection, for every edge $e = (u, v) \in E(G)$. Moreover $\exists$ an integer $k$ ($0 \leq k \leq q - 1$) such that for every $e = (x, y) \in E(G)$, either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. Obviously $G$ is a bipartite graph with vertex set partition $V_1 = \{v \in V/ f(v) \leq k\}$ and $V_2 = \{v \in V/ f(v) > k\}$. Let $l$ be an arbitrary non-negative integer and take $t = k + 1$.

We shall define $g : V(G) \rightarrow \{0, 1, \ldots, t - 1, t + l, t + l + 1, \ldots, q + l\}$ and its induced edge labeling function $g^* : E(G) \rightarrow \{1 + l, 2 + l, \ldots, q + l\}$ with as $g^*(e) = |g(u) - g(v)|$, \( \forall e = (u, v) \in E(G) \) as follows.

\[
g(w) = f(w), \quad \text{when } w \in V_1
\]
\[
g(w) = f(w) + l, \quad \text{when } w \in V_2.
\]

Now for each $e = (u, v) \in E(G)$, we see that
\[ g^*(e) = g^*((u, v)) = |g(u) - g(v)| = |f(u) - f(v)| + l = f^*(e) + l. \]

Therefore \( g^*(E) = \{1 + l, 2 + l, \ldots, q + l\} \), as \( f^*(E) = \{1, 2, \ldots, q\} \). Moreover \( g^* \) is a bijection as \( f^* \). Thus, \( g \) is a semi smooth graceful labeling and so, \( G \) is a semi smooth graceful graph.

**Theorem—6.4.3:** For a semi smooth graceful graph \( H \), the path union \( P(m \cdot H) \) is smooth graceful graph, where \( m \) is even.

**Proof:** Let \( H \) be a semi smooth graceful graph and \( G = P(m \cdot H) \), where \( m \) is even. It is obvious that \( P(m \cdot H) \) is a bipartite as \( H \) is bipartite. Let \( f : V(H) \rightarrow \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\} \) be a semi smooth graceful labeling for \( H \), where \( q = |E(H)|, 1 \leq t \leq q \) and \( l \) be an arbitrary non-negative integer. Let \( p = |V(H)| \) and \( V(H) = \{v_1, v_2, \ldots, v_p\} \).

Let \( v_{i,j} (1 \leq j \leq p) \) be vertices for \( i^{th} \) copy \( H^{(i)} \) of \( P(m \cdot H), \forall i = 1, 2, \ldots, m \) with \( v_{1,j} = v_j (1 \leq j \leq p) \). Obviously \( P = |V(G)| = mp \) and \( Q = |E(G)| = m(q+1) - 1 \).

In [23] Kaneria, Viradia and Makadai proved that \( P(m \cdot H) \) is graceful graph with vertex labeling function \( g : V(G) \rightarrow \{0, 1, \ldots, Q\} \) defined by

\[
\begin{align*}
g(v_{1,j}) &= f(v_j), & \text{when } f(v_j) < t \\
g(v_{2,j}) &= g(v_{1,j}) + (Q - q) - l, & \text{when } f(v_j) \geq t, \forall j = 1, 2, \ldots, p; \\
g(v_{i,j}) &= g(v_{i-2,j}) + (q + 1), & \text{when } g(v_{i-2,j}) < \frac{Q}{2}, \\
g(v_{i,j}) &= g(v_{i-2,j}) - (q + 1), & \text{when } g(v_{i-2,j}) > \frac{Q}{2}; \\
g(v_{i,j}) &= g(v_{i-2,j}) - (q + 1), & \text{when } g(v_{i-2,j}) > \frac{Q}{2}, \forall j = 1, 2, \ldots, p, \forall i = 3, 4, \ldots, m. 
\end{align*}
\]

Where \( l \) be an arbitrary non-negative integer.

Since \( G \) is a graceful graph with graceful vertex labeling function \( g, \exists e = (v_{1,s}, v_{1,r}) \in E(H^{(1)}) \subset E(G) \) such that \( g^*(e) = Q, g(v_{1,s}) = Q \) and \( g(v_{1,r}) = 0 \). Obviously \( g(v_{m,s}) = q + (\frac{m}{2} - 1)(q + 1) = \frac{m}{2}(q + 1) - 1 \) and \( g(v_{m,r}) = Q - q - (\frac{m}{2} - 1)(q + 1) = \frac{m}{2}(q + 1). \)
Therefore, $g^*(v_{m,s}, v_{m,r}) = 1$. Take $T = \frac{n}{2}(q + 1)$. From above labeling pattern $g$ on $P(m \cdot H)$, we can see that for each $e = (x, y) \in E(P(m \cdot H))$ either $g(x) < \frac{Q}{2}$ and $g(y) \geq \frac{Q}{2}$, i.e. either $g(x) \leq T - 1$ and $g(y) \geq T$ or $g(x) \geq T$ and $g(y) \leq T - 1$.

Thus, for any arbitrary non-negative integer $L$, we get $T = \lfloor \frac{Q + 1}{2} \rfloor$ and a vertex labeling function $h: V(G) \rightarrow \{0, 1, \ldots, T - 1, T + 1, T + L + 1, \ldots, Q + L\}$ defined by

$$
\begin{align*}
    h(v_{1,j}) &= g(v_{1,j}), \quad \text{when } g(v_{1,j}) < T \\
                 &= g(v_{1,j}) + L, \quad \text{when } g(v_{1,j}) \geq T,
\end{align*}
$$

$$
\begin{align*}
    h(v_{i,j}) &= g(v_{i,j}), \quad \text{when } g(v_{i,j}) < T \\
                 &= g(v_{i,j}) + L, \quad \text{when } g(v_{i,j}) \geq T,
\end{align*}
$$

∀ $j = 1, 2, \ldots, p$; ∃ $i = 2, 3, \ldots, m$.

So, $h$ becomes a smooth graceful vertex labeling function for $G$. Therefore, $G = P(m \cdot H)$ is a smooth graceful graph, where $m$ is even.

**Corollary−6.4.4 :** For a graph $H$ which admits an $\alpha$-labeling, the path union $P(m \cdot H)$ also admits an $\alpha$-labeling, where $m$ is even.

**Proof:** Let $H$ be a graph which admits an $\alpha$-labeling say $f$ on $V(H)$. Then $f$ is also semi smooth vertex labeling function for $H$ by small change discussed in Theorem−6.4.2.

So, by Theorem−6.4.3 $P(m \cdot H)$ is smooth graceful graph ($m$ is even). Therefore, by Theorem−6.4.1 $P(m \cdot H)$ admits an $\alpha$-labeling.

**Illustration−6.4.5 :** $\alpha$-labeling function for $St_5$, smooth graceful labeling function for same graph and $\alpha$-labeling for the path union $P(4 \cdot St_5)$ are shown in figure−6.5 to 6.7 respectively.
\textbf{Illustration—6.4.6}: $\alpha$-labeling function for $K_{1,6}$, semi smooth graceful labeling function for same graph and $\alpha$-labeling for the path union $P(6 \cdot K_{1,6})$ are shown in figure—6.8 and 6.9 respectively.
6.5 Concluding Remarks:

In this chapter semi smooth graceful labeling is discussed in detail and survey of some existing results is carried out. We developed smooth graceful labeling and semi smooth graceful labeling for some graceful graph. It is generalized concept of graceful labeling. We used smooth graceful labeling technique to produce $\alpha$-labeling which is also generalization of graceful labeling. Seven new results are obtained. Investigations carried out here are novel and important. Labeling pattern is given in vary elegant way and it is demonstrated by means of sufficient examples.
References


[34] M A Seoul and M Z Youssef, Harmonious labeling of helms and related graphs, unpublished.


