Chapter - 4

Mean Labeling and Related Results
4.1 Introduction:

This chapter is targeted to explore the concepts of mean labeling. The concept of mean labeling was introduced by Somasundaram and Ponraj [36] in 2003.

Somasundaram and Ponraj [32−39] have proved that $P_n$, $C_n$, $K_{2,n}$, $C_m \cup P_n$, $P_m \times P_n$, $P_m \times C_n$ are mean graphs for any $m, n \in \mathbb{N}$, $K_n$ and $K_{1,n}$ are mean graphs if and only if $n \leq 3$, Bistar $B_{m,n}$ is a mean graph if and only if $m < n + 2$, barycentric subdivision of the star $K_{1,n}$ is a mean graph if and only if $n < 4$, the wheel $W_n$ is not a mean graph.

Jeyanthi, Ramya and Thangavelu [16] proved that $n$ copies of $K_{1,4}$ is super mean graph, the graphs obtained by identifying an endpoint of $P_m (m \geq 2)$ with each vertex of $C_n$ admits super mean labeling, the graph obtained by identifying an endpoint of two copies of $P_m (m \geq 2)$ with each vertex of $C_n$ admits super mean labeling, the graphs obtained by identifying an endpoint of three copies of $P_m (m \geq 2)$ admits super mean labeling, the graphs obtained by identifying an endpoint of four copies of $P_m (m \geq 2)$ admits super mean labeling.

4.2 Mean Labeling for Step Grid Graph:

Definition 4.2.1: A function $f$ is called mean labeling of a graph $G = (V, E)$ if $f : V \rightarrow \{0, 1, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e) = \lceil \frac{f(u)+f(v)}{2} \rceil$ is bijective for every edge $e = (u, v) \in E$. A graph $G$ is called mean graph if it admits a mean labeling.

Theorem−4.2.2: A step grid graph $St_n$ is a mean graph.

Proof: Let $G = St_n$ be a step grid graph of size $n$. Where mention each vertices of $n^{th}$ column like $u_{1,j}(1 \leq j \leq n)$, $(n-1)^{th}$ column like $u_{2,j}(1 \leq j \leq n)$, $(n-2)^{th}$ column like $u_{3,j}(1 \leq j \leq n-1)$, $(n-3)^{th}$ column like $u_{4,j}(1 \leq j \leq n-2)$ and similarly the first column like $u_{n,j}(1 \leq j \leq 2)$. 
We see that number of vertices in $G$ is $|V(G)| = p = \frac{1}{2}(n^2 + 3n - 2)$ and the number of edges in $G$ is $|E(G)| = q = n^2 + n - 2$.

We define a labeling function $f : V(G) \rightarrow \{0, 1, \ldots, q\}$ as follows.

$$f(u_1,n) = 0;$$

$$f(u_1,n-j) = f(u_1,n-j+1) + j, \quad \text{if } j \text{ is odd, } j \neq n,$$

$$= f(u_1,n-j+1) + j + 1, \quad \text{if } j \text{ is even, } j \neq n;$$

**Case I:** $n$ is even.

$$f(u_{2,1}) = f(u_{1,1}) + (n+1);$$

$$f(u_{i,1}) = f(u_{i-1,1}) + (n-i+2), \quad \text{if } i \text{ is odd, } i \neq 1,$$

$$= f(u_{i-1,1}) + (n-i+3), \quad \text{if } i \text{ is even, } i \neq 2;$$

$$f(u_{i,j}) = f(u_{i-1,j-1}) + 2; \quad \forall \ i = 2, 3, \ldots, n, \ \forall \ j = 2, 3, \ldots, n + 2 - i.$$

**Case II:** $n$ is odd.

$$f(u_{2,1}) = f(u_{1,1}) + n;$$

$$f(u_{i,1}) = f(u_{i-1,1}) + (n-i+3), \quad \text{if } i \text{ is odd, } i \neq 1,$$

$$= f(u_{i-1,1}) + (n-i+2), \quad \text{if } i \text{ is even, } i \neq 2;$$

$$f(u_{i,j}) = f(u_{i-1,j-1}) + 2, \quad \forall \ i = 2, 3, \ldots, n, \ \forall \ j = 2, 3, \ldots, n + 2 - i.$$

Above labeling pattern give rise a mean labeling to the graph $G$ and so, it is a mean graph.

**Illustration 4.2.3:** $St_6$ and its mean labeling shown in figure 4.1.
**Theorem**–4.2.4 : Path union of finite copies of step grid graph $St_n$ is a mean graph.

**Proof :** Let $G$ be a path union of $r$ copies of step grid graph $St_n$ ($r \in N$). Let $f$ be the function for mean labeling of $St_n$ as we mentioned in *Theorem*–2.1.

In graph $G$, We see that the vertices $|V(G)| = P = \frac{r}{2}(n^2 + 3n - 2)$ and the edges $|E(G)| = Q = r(n^2 + n - 1) - 1$. Let $u_{k,i,j}(1 \leq i, j \leq n)$ be vertices of $k^{th}$ copy of $St_n$, $\forall k = 1, 2, \ldots, r$. Where the vertices of $k^{th}$ copy of $St_n$ is $p = \frac{1}{2}(n^2 + 3n - 2)$ and edges of $k^{th}$ copy of $St_n$ is $q = n^2 + n - 2$.

Join the vertices $u_{k,1,n}$ to $u_{k+1,n,1}$, $\forall k = 1, 2, \ldots, r - 1$ by an edge to form the path union of $r$ copies of step grid graph.

We define a labeling function $g : V(G) \rightarrow \{0, 1, \ldots, Q\}$ as follows.

$$g(u_{1,i,j}) = f(u_{i,j}) + (Q - q), \quad \forall i, j = 1, 2, \ldots, n;$$

$$g(u_{k,i,j}) = f(u_{k-1,i,j}) - (q + 1), \quad \forall i, j = 1, 2, \ldots, n, \forall k = 2, 3, \ldots, r.$$

Above labeling pattern give rise a mean labeling to the given graph $G$. So, path union of finite copies of the step grid graph is a mean graph.

**Illustration**–4.2.5 : Path union of 4 copies of $St_5$ and its mean labeling shown in figure–4.2.

![Figure 4.2](image-url)
**Theorem—4.2.6**: Cycle of step grid graph \( C(r \cdot St_n) \) is a mean graph, where \( r \equiv 0 \pmod{2} \).

**Proof**: Let \( G = C(r \cdot St_n) \) be a cycle of step grid graph \( St_n \) with \( r \) copies. Let \( f \) be the function for mean labeling of \( St_n \) as we mentioned in Theorem—2.1.

In graph \( G \), we see that the vertices \( |V(G)| = P = \frac{r}{2}(n^2 + 3n - 2) \) and the edges \( |E(G)| = Q = r(n^2 + n - 1) \). Let \( u_{k,i,j}(1 \leq i, j \leq n) \) be vertices of \( k^{th} \) copy of \( St_n \), \( \forall k = 1, 2, \ldots, r \). Where the vertices of \( k^{th} \) copy of \( St_n \) is \( p = \frac{1}{2}(n^2 + 3n - 2) \) and edges of \( k^{th} \) copy of \( St_n \) is \( q = n^2 + n - 2 \).

Join the vertices \( u_{k,1,n} \) to \( u_{k+1,n,1} \), \( \forall k = 1, 2, \ldots, r - 1 \) and \( u_{r,1,n} \) with \( u_{1,n,1} \) by an edge to form the cycle of \( r \) copies of step grid graph.

We define a labeling function \( g : V(G) \rightarrow \{0, 1, \ldots, Q\} \) as follows.

\[
\begin{align*}
g(u_{1,i,j}) &= f(u_{i,j}) + (Q - q), \quad \forall i, j = 1, 2, \ldots, n; \\
g(u_{k,i,j}) &= g(u_{k-1,i,j}) - (q + 1), \quad \forall i, j = 1, 2, \ldots, n, \forall k = 2, 3, \ldots, \frac{r}{2}; \\
g(u_{\frac{r}{2}+1,n,1}) &= g(u_{\frac{r}{2},n,1}) - (q + 1); \\
g(u_{\frac{r}{2}+1,i,j}) &= g(u_{\frac{r}{2},i,j}) - (q + 2), \quad \forall i, j = 1, 2, \ldots, n, \text{ except } (n,1); \\
g(u_{\frac{r}{2}+2,n,1}) &= g(u_{\frac{r}{2}+1,n,1}) - (q + 2); \\
g(u_{\frac{r}{2}+2,i,j}) &= g(u_{\frac{r}{2}+1,i,j}) - (q + 1), \quad \forall i, j = 1, 2, \ldots, n, \text{ except } (n,1); \\
g(u_{k,i,j}) &= g(u_{k-1,i,j}) - (q + 1), \quad \forall i, j = 1, 2, \ldots, n, \forall k = \frac{r}{2} + 3, \frac{r}{2} + 4, \ldots, r.
\end{align*}
\]

Above labeling pattern give rise a mean labeling to the given cycle of step grid graph.

**Illustration—4.2.7**: \( C(6 \cdot St_4) \) and its mean labeling shown in figure—4.3.
Theorem 4.2.8: \( C(t \cdot K_{2,m}) \) is a mean graph, where \( m \in N \) and \( t \equiv 0 \pmod{2} \).

Proof: Let \( G = C(t \cdot K_{2,m}) \), where \( m \in N \). We see that the number of vertices in \( G \) is \( |V(G)| = P = t(m + 2) \) and the number of edges in \( G \) is \( |E(G)| = Q = 2tm + t \). Let \( u_{i,1} \), \( u_{i,2} \), \( v_{i,j} \) (\( 1 \leq j \leq m \)) be vertices of \( i^{th} \) copy of \( K_{2,m}^1 \) in \( G \), \( \forall i = 1, 2, \ldots, t \). Now join the \( u_{i,2} \) with \( u_{i+1,1} \), \( \forall i = 1, 2, \ldots, t - 1 \) and \( u_{t,2} \) with \( u_{1,1} \) by an edge.

We know that the labeling function \( f : V(K_{2,m}^{(1)}) \rightarrow \{0, 1, \ldots, q = 2m\} \) defined by,

\[
\begin{align*}
  f(u_{1,1}) &= q, \\
  f(u_{1,2}) &= 0, \\
  f(v_{1,j}) &= q - (2j - 1), \quad \forall j = 1, 2, \ldots, m,
\end{align*}
\]

is a mean labeling to the graph \( K_{2,m} \).

Now we define a labeling function \( g : V(G) \rightarrow \{0, 1, \ldots, Q\} \) as follows,

\[
\begin{align*}
  g(u_{1,j}) &= f(u_{1,j}) + (Q - q), \quad \forall j = 1, 2; \\
  g(v_{1,j}) &= f(v_{1,j}) + (Q - q), \quad \forall j = 1, 2, \ldots, m; \\
  g(u_{i,j}) &= g(u_{i-1,j}) - (q + 1), \quad \forall j = 1, 2, \forall i = 2, 3, \ldots, \frac{t}{2}; \\
  g(v_{i,j}) &= g(v_{i-1,j}) - (q + 1), \quad \forall j = 1, 2, \ldots, m, \forall i = 2, 3, \ldots, \frac{t}{2}; \\
  g(u_{\frac{i}{2}+1,1}) &= g(u_{\frac{i}{2},1}) - (q + 1), \\
  g(u_{\frac{i}{2}+1,2}) &= g(u_{\frac{i}{2},2}) - (q + 2); \\
  g(v_{\frac{i}{2}+1,j}) &= g(v_{\frac{i}{2},j}) - (q + 2), \quad \forall j = 1, 2, \ldots, m; \\
  g(u_{\frac{i}{2}+2,1}) &= g(u_{\frac{i}{2}+1,1}) - (q + 2), \\
  g(u_{\frac{i}{2}+2,2}) &= g(u_{\frac{i}{2}+1,2}) - (q + 1); \\
  g(v_{\frac{i}{2}+2,j}) &= g(v_{\frac{i}{2}+1,j}) - (q + 1), \quad \forall j = 1, 2, \ldots, m; \\
  g(u_{i,j}) &= g(u_{i-1,j}) - (q + 1), \quad \forall j = 1, 2 \text{ and } \forall i = \frac{t}{2} + 3, \frac{t}{2} + 4, \ldots, t; \\
  g(v_{i,j}) &= g(v_{i-1,j}) - (q + 1), \quad \forall j = 1, 2, \ldots, m, \forall i = \frac{t}{2} + 3, \frac{t}{2} + 4, \ldots, t.
\end{align*}
\]

Above labeling pattern give rise a mean labeling to the graph \( G \) obtained by taking cycle of \( K_{2,m} \) and so, \( G \) is a mean graph.

Illustration 4.2.9: \( C(4 \cdot K_{2,5}) \) and its mean labeling shown in figure 4.4.
4.3 Union as well as Path Union of Mean Graphs

Theorem 4.3.1: \( \cup_{i=1}^{t} (P_{n_i} \times P_{m_i}) \), union of finite number of grid graphs is a mean graph.

Proof: Let \( G = \cup_{i=1}^{t} (P_{n_i} \times P_{m_i}) \). Let \( u_{i,j,k} \ (1 \leq j \leq n_i, 1 \leq k \leq m_i) \) be vertices of \( i^{th} \) copy of grid \( P_{n_i} \times P_{m_i} \) in \( G \), \( \forall \ i = 1, 2, \ldots, t \).

We know that each grid \( (P_{n_i} \times P_{m_i}) \) (where \( n_i \leq m_i \)) is a mean graph on \( p_i = n_i m_i \) vertices and \( q_i = 2m_i n_i - (m_i + n_i) \) edges with following vertex labeling functions

\[
\begin{align*}
    f_i(u_{i,j,k}) &= f_i(u_{i,j-1,k+1}) - 2, & \forall \ k = m_i - 1, m_i - 2, \ldots, 1, \ \forall \ j = 2, 3, \ldots, n_i; \\
    f_i(u_{i,1,k}) &= k^2 - 1, & \text{when } 1 \leq k \leq n_i - 1; \\
    f_i(u_{1,j,k}) &= q_i - (n_i - j)^2, & \forall \ j = 1, 2, \ldots, n_i; \\
    f_i(u_{1,1,k}) &= f_i(u_{1,1,n_i-1}) + (2n_i - 1)(k - n_i + 1), & \text{when } n_i \leq k \leq m_i; \\
    f_i(u_{i,j,m_i}) &= q_i - (n_i - j)^2, & \forall \ j = 1, 2, \ldots, n_i; \\
    f_i(u_{i,j,m_i}) &= q_i - (n_i - j)^2, & \forall \ j = 1, 2, \ldots, n_i;
\end{align*}
\]

In graph \( G \), we see that the number of vertices \( |V(G)| = P = \sum_{i=1}^{t} m_i n_i \) and the edges \( |E(G)| = Q = \sum_{i=1}^{t} q_i \). We define a labeling function \( g : V(G) \longrightarrow \{0, 1, \ldots, Q\} \) as follows.

\[
\begin{align*}
    g(u_{1,2,1}) &= f_1(u_{1,2,1}) + (Q - q_1 + 1); \\
    g(u_{1,2,k}) &= f_1(u_{1,2,k}) + (Q - q_1), & \forall \ k = 2, 3, \ldots, m_1; \\
    g(u_{1,j,k}) &= f_1(u_{1,j,k}) + (Q - q_1), & \forall \ j = 1, 3, \ldots, n_1, \forall \ k = 1, 2, \ldots, m_1;
\end{align*}
\]
\[ g(u_{i,n,m}) = f_i(u_{i,n,m}) + \sum_{t=i+1}^t q_t + 1, \quad \forall \ i = 2, 3, \ldots , t - 1; \]
\[ g(u_{i,2,1}) = f_i(u_{i,2,1}) + 1 + \sum_{t=i+1}^t q_t, \quad \forall \ i = 2, 3, \ldots , t - 1; \]

**Case I :** \( n_i \neq 2 \)
\[ g(u_{i,n_i,k}) = f_i(u_{i,n_i,k}) + \sum_{t=i+1}^t q_t, \quad \forall \ k = 1, 2, \ldots , m_i - 1, \quad \forall \ i = 2, 3, \ldots , t - 1; \]

**Case II :** \( n_i = 2 \)
\[ g(u_{i,2,k}) = f_i(u_{i,2,k}) + \sum_{t=i+1}^t q_t, \quad \forall \ k = 2, 3, \ldots , m_i - 1, \quad \forall \ i = 2, 3, \ldots , t - 1; \]
\[ g(u_{i,j,k}) = f_i(u_{i,j,k}) + \sum_{t=i+1}^t q_t, \quad \forall \ j = 1, 3, 4, \ldots , n_i - 1, \quad \forall \ k = 1, 2, \ldots , m_i; \]
\[ g(u_{t,n_t,m_t}) = f_t(u_{t,n_t,m_t}) + 1; \]
\[ g(u_{t,n_t,k}) = f_t(u_{t,n_t,k}), \quad \forall \ k = 1, 2, \ldots , m_t - 1; \]
\[ g(u_{t,j,k}) = f_t(u_{t,j,k}), \quad \forall \ j = 1, 2, \ldots , m_t - 1, \quad \forall \ k = 1, 2, \ldots , m_t. \]

Above labeling pattern give rise a mean labeling to the graph \( G \) and so, it is a mean graph.

**Illustration**–4.3.2 : \( (P_2 \times P_3) \cup (P_3 \times P_4) \cup (P_2 \times P_4) \cup (P_3 \times P_6) \) and its mean labeling shown in figure–4.5.

Here \( Q = \sum_{l=1}^4 q_l = 7 + 17 + 10 + 27 = 61, \sum_{l=3}^4 q_l = 37, \sum_{l=4}^4 q_l = 27 \) and \( Q - q_1 = 54. \)
Theorem—4.3.3 : Let \( G_i(1 \leq i \leq t) \) be \( t \) connected mean graphs on \( |V(G_i)| = p_i \) vertices and \( |E(G_i)| = q_i \) edges, \( \forall i = 1, 2, \ldots, t \). Then the path union of \( G_i(1 \leq i \leq t) \) by paths of arbitrary length is also a mean graph.

Proof : Let \( G = \langle G_1, P_{n_1}, G_2, P_{n_2}, \ldots, G_{t-1}, P_{n_{t-1}}, G_t \rangle \) be the path union of graphs \( G_i \) \((1 \leq i \leq t)\) by path of arbitrary length. Let \( V(G_i) = \{v_{i,j}/j = 1, 2, \ldots, p_i\} \), \( E(G_i) \) be the edges set for \( G_i \), where \( |E(G_i)| = q_i, \forall i = 1, 2, \ldots, t \). Let \( V(P_{n_i}) = \{w_{i,k}/k = 1, 2, \ldots, n_i\} \), \( E(P_{n_i}) = \{(w_{i,k}, w_{i,k+1})/k = 1, 2, \ldots, n_i - 1\}, \forall i = 1, 2, \ldots, t \).

Since a mean graph \( G_i(1 \leq i \leq t) \) will always have vertices with labels \( q_i, q_{i-1} \) and 0, w.l.o.g. we may assume that vertex label of \( v_{i,1} \) is \( q_i \) and vertex label of \( v_{i,p_i} \) is 0, \( \forall i = 1, 2, \ldots, t \). Now take \( v_{i,p_i} = w_{i,0} \) and \( w_{i,n_i} = v_{i+1,1} \) for each \( i = 1, 2, \ldots, t - 1 \) to form the graph \( G = \langle G_1, P_{n_1}, G_2, P_{n_2}, \ldots, G_{t-1}, P_{n_{t-1}}, G_t \rangle \) arbitrary path union of graphs.

Suppose \( G_i(1 \leq i \leq t) \) be mean graphs with following vertex labeling function \( f : V(G_i) \rightarrow \{0, 1, \ldots, q_i\} \) defined by, \( f_i(v_{i,1}) = q_i \) and \( f_i(v_{i,p_i}) = 0, \forall i = 1, 2, \ldots, t \) as we mentioned earlier. It is obvious that \( P_{n_i}(1 \leq i \leq t - 1) \) is mean graph with vertex labeling function \( g_i : V(P_{n_i}) \rightarrow \{0, 1, \ldots, n_i\} \) defined by,

\[
g_i(w_{i,k}) = n_i - k, \quad \forall k = 1, 2, \ldots, n_i, \forall i = 1, 2, \ldots, t - 1.
\]

In graph \( G \), we see that the number of vertices \( P = |V(G)| = \sum_{i=1}^t p_i + \sum_{i=1}^{t-1} (n_i - 2) \) and the edges \( Q = |E(G)| = \sum_{i=1}^t q_i + \sum_{i=1}^{t-1} (n_i - 1) \). We define a labeling function \( g : V(G) \rightarrow \{0, 1, \ldots, Q\} \) as follows.

\[
g(v_{1,j}) = f_1(v_{1,j}) + (Q - q_1), \quad \forall j = 1, 2, \ldots, p_1;

g(w_{1,k}) = g_1(w_{1,k}) + (Q - q_1 - n_1), \quad \forall k = 2, 3, \ldots, n_1 - 1;

g(v_{i,j}) = f_i(v_{i,j}) + (Q - \sum_{l=1}^{i-1} q_l - \sum_{l=1}^{i-1} n_l), \quad \forall j = 1, 2, \ldots, p_i, \forall i = 2, 3, \ldots, t - 1;

g(w_{i,k}) = g_i(w_{i,k}) + (Q - \sum_{l=1}^{i} (q_l + n_l)), \quad \forall k = 2, 3, \ldots, n_i - 1, \forall i = 2, 3, \ldots, t - 1;

g(v_{t,i}) = f_t(v_{t,i}), \quad \forall i = 1, 2, \ldots, p_t.
\]

Above labeling pattern give rise a mean labeling to the graph \( G \) and so, path union of \( G_i(1 \leq i \leq t) \) by paths of arbitrary length is a mean graph.

Illustration—4.3.4 : \( \langle P_3 \times P_4, P_3, C_8, P_6, B_{2,1}^2, P_2, K_{2,4} \rangle \) and its mean labeling shown in figure—4.6.

Here \( Q = 17 + 3 + 8 + 6 + 9 + 2 + 8 = 53 \).
Corollary 4.3.5: Let $G_i (1 \leq i \leq t)$ be connected mean graphs on $p_i$ vertices and $q_i$ edges $(1 \leq i \leq t)$. Then $P_n(G_1, G_2, \ldots G_t)$ is a mean graph.

Corollary 4.3.6: Let $G_i (1 \leq i \leq t)$ be connected mean graphs. Then $P(G_1, G_2, \ldots G_t)$ and $\langle G_1, G_2, \ldots G_t \rangle$ are mean graph.

4.4 Some Cycle of Graphs with Mean Labeling:

Theorem 4.4.1: $C(r \cdot B_{n,n}^2)$ is a mean graph, where $r \equiv 0 \pmod{2}$.

Proof: Let $G = C(r \cdot B_{n,n}^2)$ be a cycle of the square of bistar $B_{n,n}^2$ with $r$ copies. Let $f$ be the mean labeling function for $B_{n,n}^2$ defined as

\[
\begin{align*}
  f(u) &= 0; \\
  f(v) &= 4n + 1; \\
  f(v_j) &= 2j, \quad \forall \ j = 1, 2, \ldots, n; \\
  f(u_j) &= 2n + 2j, \quad \forall \ j = 1, 2, \ldots, n.
\end{align*}
\]

In graph $G$ we see that the vertices $|V(G)| = P = r(2n + 2)$ and the edges $|E(G)| = Q = r(4n + 2)$. Let $u_{i,j}$ $(0 \leq j \leq n)$ be vertices of $i^{th}$ copy of $B_{n,n}^2$, $\forall \ i = 1, 2, \ldots, r$, where the vertices of $i^{th}$ copy of $B_{n,n}^2$ is $p = 2n + 2$ and edges of $i^{th}$ copy of $B_{n,n}^2$ is $q = 4n + 1$. 

Join the vertices $v_{i,0}$ to $u_{i+1,0}, \forall i = 1, 2, \ldots, r - 1$, and $v_{r,1}$ to $v_{1,0}$ by an edge to form the cycle of $r$ copies of the square of bistar.

We define a labeling function $g : V(G) \rightarrow \{0, 1, \ldots, Q\}$ as follows,

\begin{align*}
g(v_{1,j}) &= f(v_j) + (Q - q), \quad \forall j = 0, 1, 2, \ldots, n; \\
g(u_{1,j}) &= f(u_j) + (Q - q), \quad \forall j = 0, 1, 2, \ldots, n; \\
g(u_{i,j}) &= g(u_{i-1,j}) - (q + 1), \quad \forall j = 0, 1, 2, \ldots, n, \forall i = 2, 3, \ldots, r; \\
g(v_{i,j}) &= g(v_{i-1,j}) - (q + 1), \quad \forall j = 0, 1, 2, \ldots, n, \forall i = 2, 3, \ldots, r; \\
g(u_{r+1,j}) &= g(u_{r,j}) - (q + 2), \quad \forall j = 0, 1, 2, \ldots, n; \\
g(v_{r+1,j}) &= g(v_{r,j}) - (q + 2), \quad \forall j = 0, 1, 2, \ldots, n; \\
g(u_{i,j}) &= g(u_{i-1,j}) - (q + 1), \quad \forall j = 0, 1, 2, \ldots, n, \forall i = \frac{r}{2} + 2, \frac{r}{2} + 3, \ldots, r; \\
g(v_{i,j}) &= g(v_{i-1,j}) - (q + 1), \quad \forall j = 0, 1, 2, \ldots, n, \forall i = \frac{r}{2} + 2, \frac{r}{2} + 3, \ldots, r.
\end{align*}

Above labeling pattern give rise a mean labeling to the given cycle of $B^2_{n,n}$. Hence, $C(r \cdot B^2_{n,n})$ is a mean graph.

**Illustration**—4.4.2 : $C(4 \cdot B^3_{3,3})$ and its mean labeling shown in figure—4.7.
**Theorem—4.4.3:** \( C(r \cdot M(C_n)) \) is a mean graph, where \( r \equiv 0 \pmod{2} \).

**Proof:** Let \( G = C(r \cdot M(C_n)) \) be a cycle of middle graph \( M(C_n) \) with \( r \) copies. Let \( f \) be the mean labeling function for \( M(C_n) \) defined as

**Case—I:** \( n \equiv 1 \pmod{2} \)

\[
\begin{align*}
    f(v_j) &= 3(j-1), & \forall j &= 1, 2, \ldots, \frac{n-1}{2}, \\
    f(v'_j) &= 3j - 1, & \forall j &= 1, 2, \ldots, \frac{n-1}{2}; \\
    f(v_j) &= \frac{3n-1}{2}, & \text{for } j &= \frac{n+1}{2}, \\
    f(v'_j) &= \frac{3n+5}{2}, & \text{for } j &= \frac{n+1}{2}; \\
    f(v_j) &= 3j, & \forall j &= \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n, \\
    f(v'_j) &= 3j - 1, & \forall j &= \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n.
\end{align*}
\]

**Case—II:** \( n \equiv 0 \pmod{2} \)

\[
\begin{align*}
    f(v_j) &= 3(j-1), & \forall j &= 1, 2, \ldots, \frac{n}{2}, \\
    f(v'_j) &= 3j - 1, & \forall j &= 1, 2, \ldots, \frac{n}{2}; \\
    f(v_n) &= 3n; & f(v'_n) &= 3n - 2; & f(v'_{\frac{3n}{2}+1}) &= \frac{3n}{2}; \\
    f(v_j) &= 3j + 2, & \forall j &= \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1; \\
    f(v'_j) &= 3j - 5, & \forall j &= \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n - 1.
\end{align*}
\]

In graph \( G \) we see that the vertices \( |V(G)| = P = 2rn \) and the edges \( |E(G)| = Q = 3rn + r \). Let \( v_{i,j} \) \( (1 \leq j \leq n) \), \( v'_{i,j} \) \( (1 \leq j \leq n) \) be vertices of \( i^{th} \) copy of \( M(C_n) \), \( \forall i = 1, 2, \ldots, r \), where the vertices of \( i^{th} \) copy of \( M(C_n) \) is \( p = 2n \) and edges of \( i^{th} \) copy of \( M(C_n) \) is \( q = 3n \).

Join the vertices \( v_{i,1} \) to \( v_{i+1,n} \), \( \forall i = 1, 2, \ldots, r - 1 \) and \( v'_{r,n-1} \) to \( v_{1,n} \) by an edge to form the cycle of \( r \) copies of middle graph.

We define a labeling function \( g : V(G) \rightarrow \{0, 1, \ldots, Q\} \) as follows,

\[
\begin{align*}
    g(v_{i,j}) &= f(v_j) + (Q - q), & \forall j &= 1, 2, \ldots, n, \\
    g(v'_{i,j}) &= f(v'_j) + (Q - q), & \forall j &= 1, 2, \ldots, n; \\
    g(v_{i,j}) &= g(v_{i-1,j}) - (q + 1), & \forall j &= 1, 2, \ldots, n, \forall i = 2, 3, \ldots, \frac{n}{2}, \\
    g(v'_{i,j}) &= g(v'_{i-1,j}) - (q + 1), & \forall j &= 1, 2, \ldots, n, \forall i = 2, 3, \ldots, \frac{n}{2}; \\
    g(v_{\frac{3n}{2}+1,j}) &= g(v'_{\frac{3n}{2}+1,j}) - (q + 2), & \forall j &= 1, 2, \ldots, n, \\
    g(v'_{\frac{3n}{2}+1,j}) &= g(v'_{\frac{3n}{2}+1,j}) - (q + 2), & \forall j &= 1, 2, \ldots, n;
\end{align*}
\]
Above labeling pattern give rise a mean labeling to the given cycle of middle graph. Hence, \( C(r \cdot M(C_n)) \) is a mean graph.

**Theorem 4.4.4:** \( C(r \cdot Df_n) \) is a mean graph for \( n \) even, where \( r \equiv 0 (\text{mod } 2) \).

**Proof:** Let \( G = C(r \cdot Df_n) \) be a cycle of double fan \( Df_n \) with \( r \) copies. Let \( f \) be the mean labeling function for \( Df_n \) defined as

\[
\begin{align*}
f(u) &= 0; \\
f(v) &= 3n - 1; \\
f(v_j) &= 3j - 1, & j \equiv 1 (\text{mod } 2), \\
f(v_j) &= 3j - 2, & \text{otherwise}
\end{align*}
\]

In graph \( G \) we see that the vertices \(|V(G)| = P = r(n + 2)\) and the edges \(|E(G)| = Q = 3rn\). Let \( u_i, v_{i,j} \) (\( 0 \leq j \leq n \)) be vertices of \( i^{th} \) copy of \( Df_n \), \( \forall \ i = 1, 2, \ldots, r \), where the vertices of \( i^{th} \) copy of \( Df_n \) is \( p = n + 2 \) and edges of \( i^{th} \) copy of \( Df_n \) is \( q = 3n - 1 \).

Join the vertices \( u_i \) to \( v_{i+1,0} \), \( \forall \ i = 1, 2, \ldots, r - 1 \) and \( v_{r,1} \) to \( v_{1,0} \) by an edge to form the cycle of \( r \) copies of double fan.

We define a labeling function \( g : V(G) \to \{0, 1, \ldots, Q\} \) as follows,

\[
\begin{align*}
g(u_1) &= f(u) + (Q - q), \\
g(v_{1,0}) &= f(v) + (Q - q), \\
g(v_{i,j}) &= f(v_j) + (Q - q), & \forall \ j = 1, 2, \ldots, n; \\
g(u_i) &= g(u_{i-1}) - (q + 1), & \forall \ i = 2, 3, \ldots, \frac{r}{2}, \\
g(v_{i,j}) &= g(v_{i-1,j}) - (q + 1), & \forall \ j = 0, 1, 2, \ldots, n, \forall \ i = 2, 3, \ldots, \frac{r}{2}; \\
g(u_{\frac{r}{2}+1}) &= g(u_{\frac{r}{2}}) - (q + 2), \\
g(v_{\frac{r}{2}+1,j}) &= g(v_{\frac{r}{2},j}) - (q + 2), & \forall \ j = 0, 1, 2, \ldots, n; \\
g(u_i) &= g(u_{i-1}) - (q + 1), & \forall \ i = \frac{r}{2} + 2, \frac{r}{2} + 3, \ldots, r, \\
g(v_{i,j}) &= g(v_{i-1,j}) - (q + 1), & \forall \ j = 0, 1, 2, \ldots, n, \forall \ i = \frac{r}{2} + 2, \frac{r}{2} + 3, \ldots, r.
\end{align*}
\]

Above labeling pattern give rise a mean labeling to the given cycle of double fan. Hence, \( C(r \cdot Df_n) \) is a mean graph when \( r \) and \( n \) both are even.
4.5 Concluding Remarks:

Here we have discussed mean labeling for step grid graph $St_n$, path union of finite copies of the step grid graph $St_n$, cycle graph of the step grid graph $St_n$, cycle graph of the complete bipartite graph $K_{2,m} \cup \bigcup_{i=1}^{t}(P_{n_i} \times P_{m_i})$ union of finite number of grid graphs, arbitrary path union of mean graphs, cycle graph of the square of bistar $B_{n,n}^2$, the cycle graph of middle graph of $C_n$, cycle graph of double fan graph $Df_n$. This work contribute some new results to the family of mean labeling. After 2003 similar results and similar labelings for various graphs are introduced by so many researchers like a new labeling parameter called mean number of a graph, mean cordial labeling and mean cordial graph, geometric mean, pair sum labeling etc. in 2007 by Sundaram, Ponraj and Somasundaram [40]. Penultimate chapter−5 is aimed to discuss graceful labeling and one point union of graphs with their graceful labeling.