Chapter - 2

Fundamental Concepts and Definitions
2.1 **INTRODUCTION** :

This chapter is intended to introduce notations, definitions and terminology which are needed for the subsequent chapters.

2.2 **BASIC DEFINITIONS** :

**Definition 2.2.1**: (Graph)

A graph \( G = (V(G), E(G)) \) consists of two finite sets: \( V(G) \), the vertex set of the graph which is a nonempty set of element called vertices, and \( E(G) \), the edge set of the graph which is a possibly empty set of element called edges, such that each edge \( e \) in \( E(G) \) is assigned an unordered pair of vertices \( (u, v) \), called the end vertices of \( e \).

**Definition 2.2.2**: (Order of a Graph)

The number of vertices in \( G \) is called the order of a graph \( G \). It is denoted by \( |V(G)| \).

**Definition 2.2.3**: (Size of a Graph)

The number of edges in \( G \) is called the size of a graph \( G \). It is denoted by \( |E(G)| \).

**Definition 2.2.4**: (Loop)

An edge of a graph which joins a vertex to itself is called a loop. A loop at the vertex \( v_i \) is an edge \( e = (v_i, v_i) \).

**Definition 2.2.5**: (Multiple edges)

If two vertices of a graph are joined by more than one edge then these edge are called multiple edges.

**Definition 2.2.6**: (Simple graph)

A graph which has neither loops nor multiple edges is called a simple graph.
Definition 2.2.7: (Adjacent vertices)
If two vertices of a graph are joined by an edge then these vertices are called adjacent vertices.

Definition 2.2.8: (Degree of vertices)
The number of edges incident on vertex $v$ of any graph $G$ is called degree of $v$. It is denoted by $deg(v)$ or $d(v)$.

Definition 2.2.9: (Incident edges)
Two edges that have an end vertex in common are called incident edges.

Definition 2.2.10: (Endpoint/Pendent vertex)
A vertex of a graph of degree 1 is called endpoint or pendent vertex. An edge of the graph $G$ which is incident with a pendent vertex is called a pendent edge.

Definition 2.2.11: (Connected and Disconnected graph)
A graph is said to be connected if there is a path between every pair of vertices of $G$. A graph which is not connected is called a disconnected graph.

Illustration—2.1.12: Let us consider the following graph $G$. 

![Graph Image](figure-2.1)
In graph $G$ shown in figure-2.1

- Order of graph $G$ is 8, Size of graph $G$ is 8.
- $e_6$ forms a loop at $v_5$, $e_3$ and $e_4$ are multiple edges.
- $v_1$ and $v_2$ are adjacent vertices by $e_1$ edge.
- $d(v_5) = 3$, $d(v_3) = 2$, $d(v_6) = 1$.
- $e_1$ and $e_2$ are incident edges at $v_2$ vertex.
- $v_1$, $v_6$ and $v_8$ are endpoints.
- $G$ is disconnected graph.

**Definition 2.2.13 : (Walk)**

A walk is defined as a finite alternating sequence of vertices and edges of the form $v_0e_1v_1e_2\ldots e_nv_n$ which start and end a vertex and each edge in the sequence is incident on the vertex immediately preceding and succeeding it in the sequence.

- The number of edges in a walk is called the length of the walk.
- Beginning and ending vertices are equal then it is called a closed walk.

**Definition 2.2.14 : (Path)**

A walk in which no vertex is repeated is called a path. A path with $n$ vertices is denoted by $P_n$.

**Definition 2.2.15 : (Trail)**

A walk in which no edge is repeated is called a trail.

**Definition 2.2.16 : (Cycle)**

A closed path in which no vertex is repeated except the terminal vertex is called a cycle. A cycle with $n$ vertices is denoted by $C_n$. 
Definition 2.2.17: (Unicyclic graph)

A graph $G$ with exactly one cycle is called a unicyclic graph.

Illustration—2.1.18: consider the following graph $G$ shown in figure—2.2.

For this graph we note the followings:

- $W = v_2e_2v_3e_3v_4e_9v_5e_4v_3e_5v_6e_6v_2$ is a closed walk.
- $P_4 = v_2e_2v_3e_3v_4e_9v_5$ is a path of length 3.
- $C_5 = v_2e_2v_3e_3v_4e_9v_5e_8v_1e_1v_2$ is a cycle of length 5.

Definition 2.2.19: (Euler graph)

Let $G = (V, E)$ be a graph. A closed trail in $G$ is called an Euler line if it contains all the edges of the graph $G$. A graph $G$ is called an Euler graph if it admits an Euler line.

Definition 2.2.20: (Tree)

A graph $G$ is called a acyclic graph if it contains no cycle. A graph $T$ is called a tree if it contains a acyclic graph.

Definition 2.2.21: (Caterpillar)

A caterpillar is a tree with the property that the removal of its pendant vertices leaves a path. This path is known as spine of the caterpillar.
Definition 2.2.22 : (Complete graph)
A complete graph is a simple graph such that every pair of vertices is joined by an edge. A complete graph on $n$ vertices is denoted by $K_n$.

Definition 2.2.23 : (Bipartite graph)
A graph $G$ is said to be bipartite if the vertices can be partitioned into two disjoint subsets $V_1$ and $V_2$ such that for every edge $e_i = (v_i, v_j) \in E(G)$, $v_i \in V_1$ and $v_j \in V_2$.

Definition 2.2.24 : (Complete bipartite graph)
A simple bipartite graph, whose two vertices are adjacent if and only if they are in different partite sets is called a complete bipartite graph. If partite sets $V_1$ and $V_2$ are having $m$ and $n$ vertices respectively then the related complete bipartite graph is denoted by $K_{m,n}$ and $V_1$ is called $m$-vertices part and $V_2$ is called $n$-vertices part of $K_{m,n}$.

Definition 2.2.25 : (Regular graph)
A regular graph is a graph if degree of each vertices are same.

Definition 2.2.26 : ($k$-regular graph)
A regular graph with vertices of degree $k$ is called a $k$-regular graph.

Definition 2.2.27 : (Chord)
A chord of a cycle $C_n$ is an edge joining two non adjacent vertices of $C_n$ and such cycle $C_n$ is known as cycle with one chord.

Definition 2.2.28 : (Twin chords)
Two chords of a cycle $C_n(n \geq 5)$ are said to be twin chords if they form a triangle with an edge of the cycle $C_n$.

Definition 2.2.29 : (Star graph)
A complete bipartite graph $K_{1,n}$ is known as star graph.
Definition 2.2.30:  (Banana tree)

A banana tree is a tree which is obtained from a family of stars by joining one end vertex of each star to a new vertex.

Definition 2.2.31:  (Super subdivision)

Let $G = (V, E)$ be a graph. A graph $H$ is called a super subdivision of $G$ if $H$ is obtained from $G$ by replacing every edge $e_i$ of $G$ by a complete bipartite graph $K_{2, m_i}$ for some $m_i, 1 \leq i \leq q$ in such a way that the ends of each $e_i$ are merged with the two vertices part of $K_{2, m_i}$ after removing the edge $e_i$ from the graph $G$.

Definition 2.2.32:  ($t$-super subdivision)

Let $G = (V, E)$ be a graph with $p$ vertices and $q$ edges. A graph $H$ is said to be a $t$-super subdivision of $G$ if $H$ is obtained from $G$ by replacing every edge $e$ of $G$ by a complete bipartite graph $K_{2, t}$ for some $t \in N$.

Definition 2.2.33:  (Cartesian product)

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ two graphs. Then cartesian product of $G_1$ and $G_2$ which is denoted by $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $u = (u_1, u_2), v = (v_1, v_2)$ such that $u$ and $v$ are adjacent in $G_1 \times G_2$ whenever $(u_1 = v_1$ and $u_2$ adjacent to $v_2)$ or $(u_2 = v_2$ and $u_1$ adjacent to $v_1)$.

Definition 2.2.34:  (Tensor product)

The tensor product $G \times H$ of graphs $G$ and $H$ is a graph such that the vertex set of $G \times H$ is the cartesian product $V(G) \times V(H)$ and any two vertices $(u, u')$ and $(v, v')$ are adjacent in $G \times H$ if and only if $u'$ is adjacent with $v'$ and $u$ is adjacent with $v$.

Definition 2.2.35:  (Grid graph)

The cartesian product of two paths $P_n, P_m$ is known as grid graph, which is denoted by $P_n \times P_m$. 
Definition 2.2.36: (Wheel)

The wheel graph $W_n$ is defined to be the join $K_1 + C_n$. The vertex corresponding to $K_1$ is called the apex vertex and vertices corresponding to cycle are called rim vertices while the edges corresponding to cycle are called rim edges. We continue to recognize apex of wheel as the apex of respective graphs obtained from wheel.

Definition 2.2.37: (Gear graph)

The gear graph $G_n$ is obtained from the wheel by subdividing each of its rim edges.

Definition 2.2.38: (Helms)

The helm $H_n$ is the graph obtained from a wheel $W_n$ by attaching a pendant edge to each rim vertex.

Definition 2.2.39: (Flower graph)

The flower graph $Fl_n$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to the apex of the helm.

Definition 2.2.40: (Web graph)

A Web graph is the graph obtained by joining all the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of the outer cycle.

Definition 2.2.41: (Petersen graph)

The Petersen graph is an undirected graph with 10 vertices and 15 edges, as shown in figure 2.3.

Definition 2.2.42: (Double fan)

The double fan $Df_n$ is obtained by $P_n + 2K_1$. 
Definition 2.2.43 : (Middle graph)

The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident on it.

Definition 2.2.44 : (Bistar)

Bistar $B_{n,n}$ is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$ by an edge.

Definition 2.2.45 : (The square of bistar)

$B^2_{n,n}$ is the square of $B_{n,n}$ obtained by joining each pair of two vertices of $B_{n,n}$ at distance 2 by a new edge.

Definition 2.2.46 : (Cycle of graph)

Let $G$ be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$, $n \geq 2$ be $n$ copies of $G$. Let $v \in V(G)$. The graph obtained by joining vertex $v$ of $G^{(i)}$ with same vertex of $G^{(i+1)}$ by an edge, $\forall i = 1, 2, \ldots, n-1$ and the same vertex $v$ of $G^{(n)}$ with the vertex $v$ of $G^{(1)}$ by an edge is called cycle of graph. It is denoted by $C(n \cdot G)$. If we replace $G$ by $C(n \cdot G)$, such graph becomes $C(n \cdot C(n \cdot G))$, we denote it by $C^2(n \cdot G)$. In general for any $t \geq 2$, $C^t(n \cdot G) = C(n \cdot C^{t-1}(n \cdot G))$. It is obvious that, $C(n \cdot K_1) = C_n$.

Definition 2.2.47 : (Mean graph)

A function $f$ is called mean labeling of a graph $G = (V, E)$ if $f : V \rightarrow \{0, 1, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e) = \lceil \frac{f(u)+f(v)}{2} \rceil$ is bijective for every edge $e = (u, v) \in E$. A graph $G$ is called mean graph if it admits a mean labeling.
**Definition 2.2.48 : (Star of G)**

Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. Let $G^{(0)}, G^{(1)}, \ldots, G^{(p)}$ be $p+1$ copies of $G$. Join each vertex $v_i$ of $G^{(0)}$ with the corresponding vertex $v_i$ of $G^{(i)}$, $\forall i = 1, 2, \ldots, p$. Such graph is known as star of $G$ and it is denoted by $G^*$. We call $G^{(0)}$ as central copy of $G^*$. It is obvious that $K^*_1 = K_2$ and $K^*_2 = P_6$.

**Definition 2.2.49 : (Path union)**

Let $G$ be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$, $n \geq 2$ be $n$ copies of $G$. The graph obtained by joining vertex $v$ of $G^{(i)}$ with same vertex of $G^{(i+1)}$ by an edge, $\forall i = 1, 2, \ldots, n-1$ is called the path union of graph $G$, It is denoted by $P(n \cdot G)$. If $G = K_1$ then $P(n \cdot K_1) = P_n$.

**Definition 2.2.50 : (Path union of graphs)**

Let $G_1, G_2, \ldots, G_t$ be connected graphs. Consider $P_{n_1}, P_{n_2}, \ldots, P_{n_{t-1}}$ paths on vertices $n_1, n_2, \ldots, n_{t-1}$ respectively. Then path union of graphs by path of arbitrary length is denoted by $(G_1, P_{n_1}, G_2, P_{n_2}, \ldots, G_{t-1}, P_{n_{t-1}}, G_t)$ and such graph obtained by joining two graphs $G_i, G_{i+1}$ by a path $P_n (1 \leq i \leq t-1)$.

If we replace each paths $P_{n_1}, P_{n_2}, \ldots, P_{n_{t-1}}$ by a path $P_n$, i.e. $P_{n_1} = P_n = P_{n_2} = \ldots = P_{n_{t-1}}$, such path union of graphs $G_1, G_2, \ldots, G_t$ we shall denote it by $P_n(G_1, G_2, \ldots, G_t)$.

**Definition 2.2.51 : (Join sum of graphs)**

Consider $t$ copies of a graph $G_0$. Then graph $G = \langle G_0^{(1)}; G_0^{(2)}; \ldots; G_0^{(t)} \rangle$ obtained by joining two copies of the graph $G_0^{(i)}$ and $G_0^{(i+1)}$ by a vertex $1 \leq i \leq t-1$ is called join sum of graphs.

**Definition 2.2.52 : (Step grid graphs)**

Take $P_n, P_{n-1}, \ldots, P_2$ paths on $n, n, n-1, n-2, \ldots, 3, 2$ vertices and arrange them vertically. A graph obtained by joining horizontal vertices by edges of given successive paths is known as a step grid graph of size $n$, where $n \geq 3$. It is denoted by $St_n$. It is obvious that $|V(St_n)| = \frac{1}{2}(n^2 + 3n - 2)$ and $|E(St_n)| = n^2 + n - 2$.

Above definition introduced by Kaneria and Makadia [19].
Definition 2.2.53: (One point union)

A graph $G$ obtained by replacing each edge of $K_{1,t}$ by a path $P_n$ on $n+1$ vertices is called one point union for $t$ copies of path $P_n$. We denote such graph $G$ by $P^n_t$.

Definition 2.2.54: (Barycentric subdivision)

If every edge of a graph $G$ is subdivided by a new vertex then the resulting graph is called barycentric subdivision of the graph $G$. In other word 1-super subdivision of $G$ is known as barycentric subdivision of the graph $G$.

Definition 2.2.55: (Open star of graphs)

A graph obtained by replacing each vertex of $K_{1,n}$ except the apex vertex by the connected graphs $G_1, G_2, \ldots, G_n$ is known as open star of graphs. We denote such graph by $S(G_1, G_2, \ldots, G_n)$.

If we replace each vertices of $K_{1,n}$ except the apex vertex by a connected graph $G$. i.e. $G_1 = G, G_2 = G, \ldots, G_n = G$, such open star of graph is denoted by $S(n \cdot G)$.

Definition 2.2.56: (One point union for path of graphs)

A graph $G$ obtained by replacing each vertices of $P^n_t$ except the central vertex by the connected graphs $G_1, G_2, \ldots, G_{tn}$ is known as one point union for path of graphs. We shall denote such graph $G$ by $P^n_t(G_1, G_2, \ldots, G_{tn})$, where $P^n_t$ is the one point union of $t$ copies of path $P_n$.

If we replace each vertices of $P^n_t$ except the central vertex by a connected graph $H$, i.e. $G_1 = G_2 = \ldots = G_{tn} = H$, such one point union of path graphs, we shall denote it by $P^n_t(tn \cdot H)$.

Definition 2.2.57: (Index of cordiality)

The index of cordiality for $G$ is $n$ if union of $n$ copies of $G$ is cordial, but union of less than $n$ copies of $G$ do not have cordial labeling.
Definition 2.2.58: (Ternary vertex labeling and 3-equitable labeling)

Let \( G = (V, E) \) be a graph. A mapping \( f : V \rightarrow \{0, 1, 2\} \) is called ternary vertex labeling of \( G \) and \( f(v) \) is called label of the vertex \( v \) of \( G \) under \( f \).

For an edge \( e = uv \) the induced edge labeling \( f^*: E \rightarrow \{0, 1, 2\} \) is given by \( f^*(e) = |f(u) - f(v)| \). Let \( v_f(0), v_f(1), v_f(2) \) be the number of vertices of \( G \) having labels 0, 1 and 2 respectively under \( f \) and let \( e_f(0), e_f(1), e_f(2) \) be the number of edges having labels 0, 1 and 2 respectively under \( f^* \).

A ternary vertex labeling of a graph \( G \) is called 3-equitable labeling if \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| \leq 1 \), \( 0 \leq i, j \leq 2 \). A graph \( G \) is called 3-equitable graph if it admits 3-equitable labeling.

Definition 2.2.59: (t copies of stars)

Consider \( t \) copies of stars namely \( K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t} \). Then the \( G = \langle K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t} \rangle \) is the graph obtained by joining apex vertices of \( K_{1,n_i} \) and \( K_{1,n_{i+1}} \) to a vertex \( u_i \), where \( 1 \leq i \leq t - 1 \).

Definition 2.2.60: (Splitting graph)

Let \( G = (V, E) \) be a graph. Then the splitting graph of \( G \) is denoted by \( S'(G) \), which obtained by adding to each \( v \in V \), by a new vertex \( v' \) such that \( v' \) is adjacent to those vertex of \( G \), which are adjacent to \( v \) in \( G \). i.e. If \( V = \{v_1, v_2, \ldots, v_n\} \), then take \( V' = \{v'_1, v'_2, \ldots, v'_n\} \) and \( S'(G) = (V \cup V', E \cup \{(u, v'), (u', v) / (u, v) \in E\}) \). Observe that \( |V(S'(G))| = 2|V(G)| \) and \( |E(S'(G))| = 3|E(G)| \).

Definition 2.2.61: (\( \alpha \)-labeling)

A function \( f \) is called \( \alpha \)-labeling of a graph \( G = (V, E) \) if \( f \) is a graceful labeling for \( G \) and there exist an integer \( k \) (\( 0 \leq k \leq q - 1 \)) such that for every \( e = (x, y) \in E(G) \), either \( f(x) \leq k < f(y) \) or \( f(y) \leq k < f(x) \). A graph \( G \) with an \( \alpha \)-labeling is necessarily bipartite graph.
Definition 2.2.62: (k-graceful labeling)

A function $f$ is called $k$-graceful labeling of a graph $G = (V, E)$ if $f : V(G) → \{0, 1, \ldots, k + q − 1\}$ is injective and the induced function $f^* : E(G) → \{k, k + 1, k + 2, \ldots, k + q − 1\}$ defined as $f^*(e) = |f(u) − f(v)|$ is bijective for every edge $e = (u, v) ∈ E(G)$. A graph $G$ is called $k$-graceful graph if it admits a $k$-graceful labeling.

Definition 2.2.63: (Harmonious graph)

A function $f$ is called harmonious labeling of a graph $G$ if $f : V(G) → \{0, 1, 2, \ldots, q − 1\}$ is injective and the induced function $f^* : E(G) → \{0, 1, 2, \ldots, q − 1\}$ defined as $f^*(e = (u, v) = (f(u) + f(v)) \pmod{q}$ is bijective, $∀ e = (u, v) ∈ E(G)$. A graph which admits harmonious labeling is called harmonious graph.

Definition 2.2.64: (Super graph)

A subgraph of a graph $G$ is a graph $H$ such that $V(H) ⊆ V(G)$ and $E(H) ⊆ E(G)$. In such a case, $G$ is known as a super graph of $H$.

Definition 2.2.65: (Super mean graph)

Let $G$ be a graph with $p = |V(G)|$ and $q = |E(G)|$. Let $f : V(G) → \{1, 2, 3, \ldots, p + q\}$ be an injection. For each edge $e = (u, v)$, let $f^*(e) = \frac{f(u) + f(v)}{2}$ if $f(u) + f(v)$ is even and $f^*(e) = \frac{f(u) + f(v) + 1}{2}$ if $f(u) + f(v)$ is odd. Then $f$ is called a super mean labeling, if $f(V) ∪ \{f^*(e) : e ∈ E(G)\} = \{1, 2, 3, \ldots, p + q\}$. A graph that admits a super mean labeling is called a super mean graph.

Definition 2.2.66: (Mean cordial graph)

Let $G$ be a simple graph. A function $f : V(G) → \{0, 1, 2\}$ and its induced edge labeling function $f^* : E(G) → \{0, 1\}$ defined by $f^*(e = (u, v)) = \lfloor \frac{f(u) + f(v)}{2} \rfloor$, for each $e = (u, v) ∈ E(G)$ is called a mean cordial labeling if $|e_f(0) − e_f(1)| ≤ 1$ and $|v_f(i) − v_f(j)| ≤ 1, ∀ i, j ∈ \{0, 1, 2\}$, where $e_f(0) = \text{the number of edges with 0 labels}$, $e_f(1) = \text{the number of edges with 1 label}$, $v_f(i) = \text{the number of vertices with label } i$, $∀ i = 0, 1, 2$. A graph $G$ is said to be mean cordial graph if it admits a mean cordial labeling.
Definition 2.2.67: (Geometric mean graph)

Let $G$ be a graph with $p = |V(G)|$ and $q = |E(G)|$. A vertex labeling function $f : V(G) \rightarrow \{1, 2, \ldots, q+1\}$ is injective and its induced function $f^* : E(G) \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = (u, v)) = \lfloor \sqrt{f(u)f(v)} \rfloor$ or $\lceil \sqrt{f(u)f(v)} \rceil$ is bijective, for every edge $e = (u, v) \in E(G)$, then $f$ is called a geometric mean labeling for $G$. A graph $G$ is called a geometric mean graph if it admits a geometric mean labeling.

Definition 2.2.68: (Pair sum labeling)

Let $G$ be a graph with $p = |V(G)|$ and $q = |E(G)|$. An injective map $f : V(G) \rightarrow \{\pm 1, \pm 2, \ldots, \pm p\}$ is called a pair sum labeling if the induced edge function $f^* : E(G) \rightarrow \mathbb{Z} - \{0\}$ defined by $f^*(u, v) = f(u) + f(v)$ is one-one and $f^*(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{q/2}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{q-1}/2\} \cup \{k_{q+1}/2\}$ according as $q$ is even or odd. A graph is called a pair sum graph if it admits a pair sum labeling.

2.3 Concluding Remarks:

This chapter provides a brief account of basic concepts, definitions and notations which are prerequisites for the advancement of the remaining chapters. Other standard terminology and notations we refer to Harary [14], West [45], Gross and Yellen [13], Clark and Holton [7]. The next chapter—3 is focused on cordial labeling.