Chapter 2

Reliability Properties of Residual Life Time and Inactivity Time of Series and Parallel System

2.1 Introduction

Let $X$ and $Y$ be two statistically independent random variables with an absolutely continuous distribution function $F(\cdot)$ and $G(\cdot)$, survival function $\bar{F}(\cdot) = 1 - F(\cdot)$ and $\bar{G}(\cdot) = 1 - G(\cdot)$ and probability density function $f(\cdot)$ and $g(\cdot)$ respectively. Suppose that

$$\{ x \in \mathbb{R} : f(x) > 0 \} = \{ x \in \mathbb{R} : g(x) > 0 \} = (0, \infty) = S \text{ (say)},$$

where $\mathbb{R} = (-\infty, \infty)$.

Let $X$ and $Y$ denote the lifetimes of two components, say $C_1$ and $C_2$. A series (parallel) system comprising of components $C_1$ and $C_2$ functions if and only if all (at least one) of its component function(s). Clearly, $\min(X,Y)$ and $\max(X,Y)$ are respectively the lifetime of series and parallel systems comprising
of components $C_1$ and $C_2$; here $\min(X, Y)$ (\max(X, Y)) denotes the minimum (maximum) of $X$ and $Y$ respectively. The residual life of $X$ with age/time $t \geq 0$ is given by

$$X_t = (X - t \mid X > t), \ t \geq 0,$$

and inactivity time of $X$ at time $t \geq 0$ is given by

$$X_{(t)} = (t - X \mid X \leq t), \ t \geq 0.$$

For a fixed $t \geq 0$, the survival functions of $X_t$ and $X_{(t)}$ are given by

$$S_{R,t}(x) = P(X_t > x) = \begin{cases} 1 & \text{if } x < 0, \\ \frac{F(x+t)}{F(t)} & \text{if } x \geq 0 \end{cases},$$

and

$$S_{I,t}(x) = P(X_{(t)} > x) = \begin{cases} 1 & \text{if } x < 0 \\ \frac{F(t-x)}{F(t)} & \text{if } 0 \leq x < t \\ 0 & \text{if } x \geq t \end{cases},$$

respectively. We denote $F_{R,t}(x) = 1 - S_{R,t}(x)$ and $F_{I,t}(x) = 1 - S_{I,t}(x)$ as the corresponding cumulative distribution functions.

For reliability engineers, the study of reliability properties of series and parallel systems is of great importance. Block ct. al. (1998), Chandra & Roy (2001), Pellesey & Petakos (2002), Li & Zhang (2003), Li & Lu (2003), Li & Zuo (2004) and Misra et al. (2008) studied reliability properties of residual life/inactivity time. The stochastic comparisons of residual life time and inactivity time in series and parallel systems is discussed by Li & Lu (2003) and Li & Zhang (2003). It may be noted that

- the residual life of series (parallel) system having components $X$ and $Y$ is $\left(\min(X, Y)\right)_t \left(\max(X, Y)\right)_t$;

- the inactivity time of series (parallel) system having components $X$ and $Y$ is $\left(\min(X, Y)\right)_{(t)} \left(\max(X, Y)\right)_{(t)}$. 

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• the lifetime of the series (parallel) system having residual lives $X_t$ and $Y_t$ is
  $\min(X_t, Y_t) \cdot \max(X_t, Y_t)$;

• the lifetime of series (parallel) system having inactivity times $X(t)$ and $Y(t)$ is
  $\min(X(t), Y(t)) \cdot \max(X(t), Y(t))$.

Let $\eta_f(x) = -f'(x)/f(x), \ x \in S$ and $\eta_g(x) = -g'(x)/g(x), \ x \in S$ denote the
eta functions of random variable $X$ and $Y$ respectively. Glaser (1980) demonstrated that the eta functions play a vital role in the study of the failure rates. We use the terms increasing and decreasing instead of non-decreasing and non-increasing, respectively. Next, we include below some definitions of stochastic orders which are standard in the literature [cf. Shaked & Shanthikumar (2007)].

**Definition 2.1.1:**

The random variable $X$ is said to be smaller than random variable $Y$ in the

(a) likelihood ratio (lr) ordering ($X \preceq_{lr} Y$) if $\frac{g(x)}{f(x)}$ increases in $x \in S$;

(b) reversed failure rate (rfr) ordering ($X \preceq_{rfr} Y$) if $\frac{G(x)}{F(x)}$ increases in $x \in S$;

(c) usual stochastic (st) ordering ($X \preceq_{st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$, for all $x \in \mathbb{R}$.

Now we present some notions of ageing (cf. Barlow and Proschan (1981)):

**Definition 2.1.2:**

The random variable $X$ is said to have

(d) increasing failure rate (ifr) if the failure rate function $\frac{f(x)}{F(x)}$ is increasing in $x \in S$;

(e) decreasing failure rate (dfr) if the failure rate function $\frac{f(x)}{F(x)}$ is decreasing in $x \in S$.
(i) decreasing reversed failure rate (DRFR) if the reversed failure rate function 
\[ \frac{f(x)}{r(x)} \] is decreasing in \( x \in S \).

Li & Zhang (2003) proved that if \( X \) and \( Y \) are independent and identically distributed, then for all \( t \geq 0 \), \( (\max(X, Y))_t \leq_{st} \max(X_t, Y_t) \). Similar results for inactivity time have also been proved. Li & Lu (2003) strengthen the results of Li & Zhang (2003) and proved that if \( X \) and \( Y \) are independent and identically distributed, then for all \( t \geq 0 \),

(i) \( (\max(X, Y))_t \leq_{tr} \max(X_t, Y_t) \);

(ii) \( (\max(X, Y))_t \leq_{fr} \max(X_t, Y_t) \); and

(iii) \( \min(X_0, Y_0) \leq_{fr} (\min(X, Y))_t \).

Li & Lu (2003) also proved that, if \( X \) and \( Y \) are independent (not necessarily identical distributed), then for all \( t \geq 0 \),

(i) \( (\max(X, Y))_t \leq_{fr} \max(X_t, Y_t) \);

(ii) \( (\max(X, Y))_t \leq_{fr} \max(X_t, Y_t) \); and

(iii) \( \min(X_0, Y_0) \leq_{fr} (\min(X, Y))_t \).

In section 2.2.2 of the chapter, we obtain some new results on stochastic comparisons of residual life time and inactivity time in series and parallel systems. Assuming that \( X \) and \( Y \) are independent, but not necessarily identical distributed and letting \( X \leq_{fr} Y \), \( \eta_f < 0 \) and \( \eta_g > 0 \), (or \( Y \leq_{fr} X \), \( \eta_f > 0 \) and \( \eta_g < 0 \)), we proved that the parallel system of used components, i.e., \( \max(X_t, Y_t) \), is better than the used parallel system, i.e., \( (\max(X, Y))_t \), in the sense of likelihood ratio order. Further, assuming \( X \) and \( Y \) are independent, but not necessarily identical distributed and letting \( X \leq_{tr} Y \), (or \( Y \leq_{tr} X \)), we proved that for any \( t \geq 0 \),

\[ (\max(X, Y))_t \leq_{tr} \max(X_t, Y_t) \];

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and
\[ \min(X_{(t)}, Y_{(t)}) \preceq_{lr} (\min(X, Y))_{(t)}. \]

In section 2.3, we prove various ageing properties of used/inactive parallel/series systems and the parallel/series system of used/inactive components. Finally, some examples are provided to support the obtained results of sections 2.2 and 2.3 in section 2.4 by taking Weibull and Gompertz distributions into consideration.

\section{2.2 Stochastic Comparison}

Li & Lu (2003) proved that if \( X \) and \( Y \) are independent and identically distributed then for any \( t \geq 0, (\max(X, Y))_{(t)} \preceq_{lr} \max(X_{(t)}, Y_{(t)}). \) They also proved that if \( X \) and \( Y \) are independent, but not necessarily identically distributed, then for any \( t \geq 0, (\max(X, Y))_{(t)} \leq_{fr} \max(X_{(t)}, Y_{(t)}). \) In the following theorem, we find the sufficient conditions for \((\max(X, Y))_{(t)} \preceq_{lr} \max(X_{(t)}, Y_{(t)}\) to hold when \( X \) and \( Y \) are independent, but not necessarily identically distributed.

**Theorem 2.2.1:**

If \( X \preceq_{lr} Y \) or \( Y \preceq_{lr} X \) then for any \( t \geq 0, \ (\max(X, Y))_{(t)} \preceq_{lr} \max(X_{(t)}, Y_{(t)}). \)

**Proof:**

Let \( t \geq 0 \) be fixed. Let \( H_{1,t}(x) \) and \( h_{1,t}(x) \) denote respectively the cumulative distribution function and probability density function of random variable \((\max(X, Y))_{(t)}\). Then for \( 0 \leq x \leq t, \)

\[
H_{1,t}(x) = P[(\max(X, Y))_{(t)} \leq x] = \frac{F(t)G(t) - F(t-x)G(t-x)}{F(t)G(t)},
\tag{2.2.1}
\]

and

\[
h_{1,t}(x) = \frac{F(t-x)g(t-x) + f(t-x)G(t-x)}{F(t)G(t)}.
\tag{2.2.2}
\]

\hspace{3cm} 25
Let $M_{1,t}(x)$ and $m_{1,t}(x)$ denote the cumulative distribution function and probability density function of random variable $\max(X(t), Y(t))$. For $0 \leq x \leq t$, we have

$$M_{1,t}(x) = P\left((\max(X(t), Y(t)) \leq x\right) \leq t,$$

\[
M_{1,t}(x) = \left(\frac{F(t) - F(t-x)}{F(t)}\right)\left(\frac{G(t) - G(t-x)}{G(t)}\right), \quad (2.2.3)
\]

and

$$m_{1,t}(x) = \frac{(F(t) - F(t-x))g(t-x) + (G(t) - G(t-x))f(t-x)}{F(t)G(t)}. \quad (2.2.4)$$

For $0 \leq x < t$, we consider

$$R_{1,t}(x) = \frac{m_{1,t}(x)}{h_{1,t}(x)}$$

\[
= \frac{(F(t) - F(t-x))g(t-x) + (G(t) - G(t-x))f(t-x)}{F(t-x)g(t-x) + f(t-x)G(t-x)}
\]

\[
= -1 + \frac{F(t)g(t-x) + G(t)f(t-x)}{F(t-x)g(t-x) + f(t-x)G(t-x)}.
\]

For $0 \leq x < t$, it is easy to verify that

\[
R'_{1,t}(x) = \frac{f(t-x)g(t-x)}{[F(t-x)g(t-x) + f(t-x)G(t-x)]^2} \left[2F(t)g(t-x) + 2G(t)f(t-x)
\right.
\]

\[
+ \sum_{j=1}^{\infty} \left[(\eta(t-x) - \eta_j(t-x)) [G(t)F(t-x) - G(t-x)F(t)]\right]. \quad (2.2.5)
\]

We will prove the assertion for the case $X \leq_{ir} Y$. Similarly, the assertion follows for the case $Y \leq_{ir} X$. It may be noted that

\[
X \leq_{ir} Y \Leftrightarrow \ln\left(\frac{g(t)}{f(t)}\right) \text{ is increasing in } t \in (0, \infty) \Leftrightarrow \eta_f(t) \geq \eta_g(t), \ \forall t > 0.
\]

\[
(2.2.6)
\]

Also,

\[
X \leq_{ir} Y \Rightarrow X \leq_{rf} Y \Leftrightarrow F(u)G(v) \geq F(v)G(u), \ \forall 0 \leq u \leq v < \infty.
\]

\[
(2.2.7)
\]

Using (2.2.6) and (2.2.7) in (2.2.5), we conclude that $R'_{1,t}(x) \geq 0, \ \forall 0 \leq x < t$, i.e., $(\max(X, Y))_{(t)} \leq_{ir} \max(X(t), Y(t))$. 

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The following corollary is an immediate consequence of Theorem 2.2.1.

**Corollary 2.2.1:**
If \( X \equiv_{st} Y \), then \( \max(X, Y)_{(t)} \leq_{lr} \max(X_{(t)}, Y_{(t)}) \).

**Remark 2.2.1:**
The result stated in Corollary 2.2.1 is by Li and Lu (2003).

Li & Lu (2003) proved that if \( X \) and \( Y \) are independent and identically distributed, then for any \( t \geq 0 \), \( \min(X, Y)_{(t)} \geq_{lr} \min(X_{(t)}, Y_{(t)}) \). They also proved that if \( X \) and \( Y \) are independent, but not necessarily identically distributed, then for any \( t \geq 0 \), \( \min(X, Y)_{(t)} \geq_{fr} \min(X_{(t)}, Y_{(t)}) \). In the following theorem, we find the sufficient conditions for \( \min(X, Y)_{(t)} \geq_{lr} \min(X_{(t)}, Y_{(t)}) \) to hold when \( X \) and \( Y \) are independent, but not necessarily identically distributed.

**Theorem 2.2.2:**
If \( X \leq_{lr} Y \) or \( Y \leq_{lr} X \), then for any \( t \geq 0 \), \( \min(X, Y)_{(t)} \geq_{lr} \min(X_{(t)}, Y_{(t)}) \).

**Proof:**
We fix \( t \geq 0 \). Let \( H_{2,t}(x) \) and \( h_{2,t}(x) \) denote respectively the cumulative distribution function and probability density function of random variable \( \min(X, Y)_{(t)} \).

For \( 0 \leq x \leq t \), we have

\[
H_{2,t}(x) = P \left( \min(X, Y)_{(t)} \leq x \right) = \frac{(1 - F(t - x))(1 - G(t - x)) - (1 - F(t))(1 - G(t))}{1 - (1 - F(t))(1 - G(t))}, \quad (2.2.8)
\]

and

\[
h_{2,t}(x) = \frac{(1 - F(t - x))g(t - x) + (1 - G(t - x))f(t - x)}{1 - (1 - F(t))(1 - G(t))}. \quad (2.2.9)
\]

For \( 0 \leq x \leq t \), let \( M_{2,t}(x) \) and \( m_{2,t}(x) \) denote respectively the cumulative distribution function and probability density function of random variable \( \min(X_{(t)}, Y_{(t)}) \).
Then, for \(0 \leq x \leq t\),
\[
M_{2,t}(x) = P(\min(X(t), Y(t)) \leq x) = 1 - \frac{F(t-x)G(t-x)}{F(t)G(t)}. \tag{2.2.10}
\]
and
\[
m_{2,t}(x) = \frac{f(t-x)G(t-x) + F(t-x)g(t-x)}{F(t)G(t)}. \tag{2.2.11}
\]
For \(0 \leq x < t\), we consider
\[
R_{2,t}(x) = \frac{h_{2,t}(x)}{m_{2,t}(x)} = \frac{F(t)G(t)}{1 - (1 - F(t))(1 - G(t))} \left( \frac{(1 - F(t-x))g(t-x) + (1 - G(t-x))f(t-x)}{F(t-x)g(t-x) + f(t-x)G(t-x)} \right)
= A(t)Z_t(x),
\]
where
\[
A(t) = \frac{F(t)G(t)}{1 - (1 - F(t))(1 - G(t))},
\]
and, for \(0 \leq x < t\),
\[
Z_t(x) = -1 + \frac{g(t-x) + f(t-x)}{F(t-x)g(t-x) + f(t-x)G(t-x)}.
\]
Clearly, for \(0 \leq x < t\),
\[
Z_t'(x) = \frac{1}{[F(t-x)g(t-x) + G(t-x)F(t-x)]^2} \left[ - [F(t-x)g(t-x) + f(t-x)G(t-x)] \\
\left[ g'(t-x) + f'(t-x) \right] + [g(t-x) + f(t-x)][F(t-x)g'(t-x) + f(t-x)g(t-x) \\
+ f'(t-x)G(t-x) + g(t-x)f(t-x)] \right]
= \frac{1}{[F(t-x)g(t-x) + G(t-x)f(t-x)]^2} \left[ 2g^2(t-x)f(t-x) + 2f^2(t-x)g(t-x) \\
+ [F(t-x)\{g(t-x)f'(t-x) + f(t-x)g'(t-x)\}] + [G(t-x)\{g(t-x)f'(t-x) \\
- f(t-x)g'(t-x)\}] \right]
= \frac{f(t-x)g(t-x)}{[F(t-x)g(t-x) + G(t-x)f(t-x)]^2} \left[ 2g(t-x) + 2f(t-x) \\
+ [\{\eta_f(t-x) - \eta_g(t-x)\}\{F(t-x) - G(t-x)\}] \right]. \tag{2.2.12}
\]
We will prove the assertion for the case $X \leq_{tr} Y$. Similarly, the assertion follows for the case $Y \leq_{tr} X$. It may be noted that, as in the proof of the Theorem 2.2.1,

\[ X \leq_{tr} Y \iff \eta_f(t) \geq \eta_f(t), \quad \forall t > 0. \tag{2.2.13} \]

Also,

\[ X \leq_{tr} Y \Rightarrow X \leq_{st} Y \iff F(u) \geq G(u), \quad \forall 0 \leq u < \infty. \tag{2.2.14} \]

Using (2.2.13) and (2.2.14) in (2.2.12), we conclude that $Z_i^t(x) \geq 0$, $\forall 0 \leq x < t$, i.e., $(\min(X,Y))_{(t)} \geq_{tr} \min(X(t), Y(t))$.

The following corollary is an immediate consequence of Theorem 2.2.2.

**Corollary 2.2.2:**

If $X =_{st} Y$, then $(\min(X,Y))_{(t)} \geq_{tr} \min(X(t), Y(t))$.

**Remark 2.2.2:**

The result stated in Corollary 2.2.2 is in by Li and Lu (2003).

Li & Lu (2003) proved that if $X$ and $Y$ are independent and identically distributed, then for any $t \geq 0$, $(\max(X,Y))_{(t)} \leq_{tr} \max(X(t), Y(t))$. They also proved that if $X$ and $Y$ are independent, but not necessarily identically distributed, then for any $t \geq 0$, $(\max(X,Y))_{(t)} \leq_{fr} \max(X(t), Y(t))$. In the following theorem, we find the sufficient conditions for $(\max(X,Y))_{(t)} \leq_{tr} \max(X(t), Y(t))$ to hold when $X$ and $Y$ are independent, but not necessarily identically distributed.

**Theorem 2.2.3:**

If $X \prec_{fr} Y$, $\eta_f < 0$ and $\eta_f > 0$ or $Y \prec_{fr} X$, $\eta_f > 0$ and $\eta_f < 0$, then for any $t \geq 0$, $(\max(X,Y))_{(t)} \leq_{tr} \max(X(t), Y(t))$.

**Proof:**

Let $t \geq 0$ be fixed. Let $H_{3,t}(x)$ and $h_{3,t}(x)$ denote respectively the cumulative distribution function and probability density function of random variable
\((\max(X,Y))_t\). Then for \(x \geq 0\),

\[
H_{3,t}(x) = P[(\max(X,Y))_t \leq x] = \frac{F(t+x)G(t+x) - F(t)G(t)}{1 - F(t)G(t)},
\]

(2.2.15)

and

\[
h_{3,t}(x) = \frac{f(t+x)G(t+x) + g(t+x)F(t+x)}{1 - F(t)G(t)}.
\]

(2.2.16)

Let \(M_{3,t}(x)\) and \(m_{3,t}(x)\) denote the cumulative distribution function and probability density function of random variable \(\max(X_t,Y_t)\).

For \(x \geq 0\), we have

\[
M_{3,t}(x) = P(\max(X_t,Y_t) \leq x) = \left( \frac{F(t+x) - F(t)}{1 - F(t)} \right) \left( \frac{G(t+x) - G(t)}{1 - G(t)} \right),
\]

(2.2.17)

and

\[
m_{3,t}(x) = \frac{(G(t+x) - G(t)) f(t+x) + (F(t+x) - F(t)) g(t+x)}{(1 - F(t))(1 - G(t))}.
\]

(2.2.18)

For \(x \geq 0\), we consider

\[
R_{3,t}(x) = \frac{m_{3,t}(x)}{h_{3,t}(x)} = \left( \frac{1 - F(t)G(t)}{(1 - F(t))(1 - G(t))} \right) \left( \frac{(G(t+x) - G(t)) f(t+x) + (F(t+x) - F(t)) g(t+x)}{(1 - F(t))(1 - G(t))} \right)
\]

\[
= B(t) U_t(x),
\]

where

\[
B(t) = \frac{1 - F(t)G(t)}{(1 - F(t))(1 - G(t))},
\]

and

\[
U_t(x) = 1 - \frac{f(t+x)G(t)+g(t+x)F(t)}{f(t+x)G(t+x) + g(t+x)F(t+x)}.
\]

For \(x > 0\), it may be easily verified that

\[
U_t'(x) = \frac{f(t+x)g(t+x)}{[f(t+x)G(t+x) + g(t+x)F(t+x)]^2} \left[ 2(G(t) f(t+x) + g(t+x) F(t)) \right.
\]

\[
+ (G(t+x)F(t) - G(t)F(t+x)) (\eta_b(t+x) - \eta_f(t+x)) \left( \eta_b(t+x) - \eta_f(t+x) \right)
\]

(2.2.19)
We will prove the assertion for the case $X \leq_{rf} Y$, $\eta_f \leq 0$ and $\eta_g \geq 0$. Similarly, the assertion follows for the case $Y \leq_{rf} X$, $\eta_f \geq 0$ and $\eta_g \leq 0$. It may be noted that

$$X \leq_{rf} Y \iff F(u)G(v) \geq F(v)G(u), \quad \forall 0 \leq u \leq v < \infty. \quad (2.2.20)$$

Using (2.2.20), $\eta_f \leq 0 \& \eta_g \geq 0$ in (2.2.19), we conclude that $U'_t(x) \geq 0$, $\forall x \geq 0$, i.e., $(\max(X,Y))_t \geq_{fr} \max(X_t,Y_t)$.

### 2.3 Ageing Properties

In this section, we discuss the various ageing properties of the residual life time and inactivity time in series and parallel systems. The following property proves that if the random variables $X$ and $Y$ have DRFR, then this property is preserved by the random variable $\max(X_t,Y_t)$.

**Property 2.3.1:**

Suppose that the random variables $X$ and $Y$ have DRFR. Then for any $t \geq 0$, the random variable $\max(X_t,Y_t)$ has DRFR.

**Proof:**

Fix $t > 0$. Let $\lambda_t(x)$ and $\mu_t(x)$ denote respectively the reversed failure rates of $X_t$ and $Y_t$ and let $M_{r,t}(x)$ denote the cumulative distribution function of $\max(X_t,Y_t)$. Let $F_{R,t}(x)$ and $G_{R,t}(x)$ denote respectively the cumulative distribution functions of $X_t$ and $Y_t$. Then for $x \geq 0$,

$$\lambda_t(x) = \frac{f(x+t)}{F(x+t) - F(t)}, \quad \mu_t(x) = \frac{g(x+t)}{G(x+t) - G(t)}, \quad \text{and} \quad M_{r,t}(x) = F_{R,t}(x)G_{R,t}(x).$$

$X$ has DRFR implies that $F(x)f'(x) \leq f^2(x), \forall x > 0$, which in turn implies that $\lambda_t(x) \leq 0, \forall x > 0$ (i.e., $X_t$ has DRFR or equivalently $\ln(F_{R,t}(x))$ is concave in $x \in (0, \infty)$). Similarly, $Y$ has DRFR implies that $Y_t$ has DRFR (i.e., $\ln(G_{R,t}(x))$...
is concave in $x \in (0, \infty)$). Thus if $X$ and $Y$ have DRFR then

$$\ln (M_{x,t}(x)) = \ln (F'_{R,t}(x)G_{R,t}(x)) = \ln (F'_{R,t}(x)) + \ln (G_{R,t}(x))$$

is concave in $x \in (0, \infty)$, i.e., $(X_t, Y_t)$ has DRFR.

In the following property, we prove that if the random variables $X$ and $Y$ have DRFR, then the random variable $(\max(X, Y))_{(t)}$ has IFR.

**Property 2.3.2:**

Suppose that the random variables $X$ and $Y$ have DRFR. Then for any $t \geq 0$, the random variable $(\max(X, Y))_{(t)}$ has IFR.

**Proof:**

Fix $t \geq 0$. It is obvious that if the random variables $X$ and $Y$ have DRFR, then $\max(X, Y)$ also has DRFR. Also it is easy to verify that if a non-negative random variable $Z$ has DRFR, then for any $s \geq 0$, the random variable $Z_{(s)} = (s - Z|Z \leq s)$ has IFR. Thus under the hypothesis of the theorem, $\max(X, Y)$ has DRFR, which in turn implies that $(\max(X, Y))_{(t)}$ has IFR.

In the following property, we prove that if the random variables $X$ and $Y$ have IFR, then the random variable $\max(X(t), Y(t))$ has DRFR.

**Property 2.3.3:**

Suppose that the random variables $X$ and $Y$ have IFR. Then for any $t \geq 0$, the random variable $\max(X(t), Y(t))$ has DRFR.

**Proof:**

Fix $t \geq 0$. It is obvious that if the random variables $X$ and $Y$ have IFR, then random variables $X(t)$ and $Y(t)$ have DRFR. This in turn implies that $\max(X(t), Y(t))$ has DRFR.

In the following property, we prove that if the random variables $X$ and $Y$ have DRFR, then the random variable $\min(X(t), Y(t))$ has IFR.
Property 2.3.4:
Suppose that the random variables $X$ and $Y$ have DRFR. Then for any $t \geq 0$, the random variable $\min(X_{(t)}, Y_{(t)})$ has IFR.

Proof:
Fix $t \geq 0$. It is obvious that if random variables $X$ and $Y$ have DRFR, then random variables $X_{(t)}$ and $Y_{(t)}$ have IFR. This in turn implies that $(\min(X, Y))_{(t)}$ has IFR.

The following property proves that if the random variables $X$ and $Y$ have IFR, then this property is preserved by the random variable $\min(X, Y)$.

Property 2.3.5:
Suppose that the random variables $X$ and $Y$ have IFR. Then for any $t \geq 0$, the random variable $\min(X_{t}, Y_{t})$ has IFR.

Proof:
Fix $t \geq 0$. It is obvious that if random variables $X$ and $Y$ have IFR, then random variables $X_{t}$ and $Y_{t}$ have IFR. This in turn implies that $\min(X_{t}, Y_{t})$ has IFR.

The following property proves that if the random variables $X$ and $Y$ have IFR, then this property is preserved by the random variable $(\min(X, Y))_{t}$.

Property 2.3.6:
Suppose that the random variables $X$ and $Y$ have IFR. Then, for any $t \geq 0$, the random variable $(\min(X, Y))_{t}$ has IFR.

Proof:
Fix $t \geq 0$. It is obvious that if random variables $X$ and $Y$ have IFR, then $\min(X, Y)$ also has IFR. Also it is easy to verify that if a non-negative random variable $Z$ has IFR, then for any $s \geq 0$, the random variable $Z_{s} = (Z - s|Z > s)$ has IFR. Thus under the hypothesis of the theorem, $\min(X, Y)$ has IFR, which in turn implies that $(\min(X, Y))_{t}$ has IFR.
2.4 Examples

Weibull and Gompertz distribution are important life distributions which are used in reliability modelling. In this section, we provide some examples to support the theory developed in Sections 2.2 and 2.3. For the detailed study of these distributions, Marshall and Olkin (2007) may be referred.

**Weibull distribution**

Consider that the random variable $X$ has Weibull distribution with parameters $(\alpha, \lambda)$ and with survival function

$$\bar{F}(x) = e^{-(\lambda x)^{\alpha}}, \ x > 0, \ \lambda > 0, \ \alpha > 0.$$  

The corresponding probability density function is given by

$$f(x) = \alpha \lambda^\alpha x^{\alpha-1} e^{-(\lambda x)^{\alpha}}, \ x > 0, \ \lambda > 0, \ \alpha > 0,$$

and

$$f'(x) = \alpha \lambda^\alpha x^{\alpha-2} e^{-(\lambda x)^{\alpha}} ((\alpha - 1) - \alpha x \lambda^\alpha).$$

Clearly, $f'(x) \leq 0$, if $\alpha \leq 1$. Similarly, let $Y$ follows Weibull $(\beta, \mu)$. It may be noted that $f' \leq 0$ and $g' \leq 0 \iff F$ and $G$ are concave $\Rightarrow \ln F$ and $\ln G$ are concave $\iff X$ and $Y$ have DRFR. Hence, if $X$ and $Y$ follows Weibull($\alpha, \lambda$), $\alpha \leq 1$, and Weibull($\beta, \mu$), $\beta \leq 1$, then the sufficient conditions of Property 2.3.1, 2.3.2 and 2.3.4 are satisfied.

It is well known that the random variable $X$, which follows Weibull($\alpha, \lambda$), has IFR if $\alpha \geq 1$ (Barlow and Proschan (1981)). Therefore, if $X$ and $Y$ follows Weibull($\alpha, \lambda$) $\alpha \geq 1$ and Weibull $(\beta, \mu)$, $\beta \geq 1$, then the sufficient conditions of Property 2.3.3, 2.3.5 and 2.3.6 are satisfied.

In order to observe when $X \leq_{tr} Y$ ($Y \leq_{tr} X$), we consider

$$\frac{g(x)}{f(x)} = \frac{\beta \mu^\beta}{\alpha \lambda^\alpha} \psi(x),$$

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where \( \psi(x) = x^{\beta - \alpha} e^{(\lambda x)^\alpha} e^{-\mu x^\beta} \).

Further,

\[
\psi'(x) = x^{\beta - \alpha - 1} e^{(\lambda x)^\alpha} e^{-\mu x^\beta} \left((\beta - \alpha) + \alpha \lambda^\alpha x^\alpha - \beta \mu^\beta x^\beta\right) \\
= x^{\beta - \alpha - 1} e^{(\lambda x)^\alpha} e^{-\mu x^\beta} \left(\alpha(\lambda^\alpha x^\alpha - 1) + \beta(1 - \mu^\beta x^\beta)\right).
\]

It may be noted that if we take \( \alpha = \beta, \lambda \geq \mu (\alpha = \beta, \lambda \leq \mu) \), then \( \psi'(x) \geq 0 (\psi'(x) \leq 0) \), i.e., \( X \leq_{lr} Y (Y \leq_{lr} X) \).

Hence, if \( X \) and \( Y \) follows Weibull(\( \alpha, \lambda \)) and Weibull(\( \beta, \mu \)) respectively such that \( \alpha = \beta \), it is clear from the above arguments (cf. Theorems 2.2.1 and 2.2.2) that,

\[
(max(X, Y))_{(t)} \leq_{lr} max(X_{(t)}, Y_{(t)}),
\]

and

\[
(min(X, Y))_{(t)} \geq_{lr} min(X_{(t)}, Y_{(t)}).
\]

**Gompertz distribution**

If we consider the random variable \( X \) which has Gompertz distribution with scale parameter \( \lambda \) and frailty parameter \( \xi \), i.e., \( X \) follows Gompertz(\( \lambda, \xi \)). Then, the random variable \( X \) has survival function

\[
\bar{F}(x) = e^{-\xi(e^{\lambda x} - 1)}, \ x > 0, \ \lambda > 0, \ \xi > 0,
\]

its probability density function is

\[
f(x) = \lambda \xi e^{\lambda x - \xi(e^{\lambda x} - 1)}, \ x \geq 0, \ \lambda \geq 0, \ \xi \geq 0,
\]

and

\[
f'(x) = \lambda^2 \xi e^{\lambda x - \xi(e^{\lambda x} - 1)}(1 - \xi e^{\lambda x}). \tag{2.4.1}
\]

On applying the Maclaurin’s series to \( e^{\lambda x} \) in expression (2.4.1), we have

\[
f'(x) = \lambda^2 \xi e^{\lambda x - \xi(e^{\lambda x} - 1)} \left(1 - \xi \left(1 + \lambda x + \frac{(\lambda x)^2}{2!} + \ldots\right)\right).
\]
If we choose $\xi > 1$, then $f'(x) \leq 0$. Similarly, let $Y$ follows Gompertz($\mu, \eta$). It may be noted that

$$f' \leq 0 \text{ and } g' \leq 0 \iff F \text{ and } G \text{ are concave } \iff \ln F \text{ and } \ln G \text{ are concave } \iff X \text{ and } Y \text{ have DRFR.}$$

Clearly, if $X$ and $Y$ follows Gompertz($\lambda, \xi$), $\xi > 1$, and Gompertz($\mu, \eta$), $\eta > 1$, respectively, then the sufficient conditions of Property 2.3.1, 2.3.2 and 2.3.4 are satisfied.

In order to observe when $X \leq_{ir} Y \ (Y \leq_{ir} X)$, we consider for the case when $\lambda = \mu$,

$$\frac{g(x)}{f(x)} = \frac{\eta e^{-\eta (e^{\lambda x} - 1)}}{\xi e^{-\xi (e^{\lambda x} - 1)}} = \frac{\eta}{\xi} e^{(\lambda x - 1)(\xi - \eta)} ,$$

which is increasing (decreasing) in $x$ if $\xi \geq \eta \ (\xi \leq \eta)$. Therefore, if we take $\lambda = \mu$, $\xi \geq \eta \ (\lambda = \mu, \xi \leq \eta)$, then $X \leq_{ir} Y \ (Y \leq_{ir} X)$.

Hence, if $X$ and $Y$ follows Gompertz($\lambda, \xi$) and Gompertz($\mu, \eta$) respectively such that $\lambda = \mu$, it is clear from the above arguments (cf. Theorems 2.2.1 and 2.2.2) that,

$$(\max(X, Y))_{(x)} \leq_{ir} \max(X_{(x)}, Y_{(x)}),$$

and

$$(\min(X, Y))_{(x)} \geq_{ir} \min(X_{(x)}, Y_{(x)}).$$

### 2.5 Conclusions

The stochastic comparison of residual life and inactivity time of series and parallel systems had been studied in the literature when the random variables are independent and identically distributed. In this chapter, such results are extended
when the condition of identically distribution is omitted. By assuming that \( X \) and \( Y \) are independent, but not necessarily identically distributed and letting \( X \leq_{lr} Y, \ \eta_f \leq 0 \) and \( \eta_g \geq 0, \) (or \( Y \leq_{lr} X, \ \eta_f \geq 0 \) and \( \eta_g \leq 0 \)) we proved that the parallel system of used components, i.e., \( \max(X_t, Y_t) \), is better than the used parallel system, i.e., \( \max(X, Y) \), in the sense of likelihood ratio order. Also, by assuming \( X \) and \( Y \) are independent, but not necessarily identically distributed and letting \( X \leq_{lr} Y, \) (or \( Y \leq_{lr} X \)), we proved that, for any \( t \geq 0, \)

\[
(\max(X, Y))(t) \leq_{lr} \max(X(t), Y(t));
\]

and

\[
\min(X(t), Y(t)) \leq_{lr} (\min(X, Y))(t).
\]

Also, we proved various ageing properties of used/inactive parallel/series systems and the parallel/series system of used/inactive components. The obtained results are supported by well known distributions, such as weibull and gompertz distributions.