

Chapter 2

ADAPTIVE NONPARAMETRIC TESTS FOR SINGLE SAMPLE LOCATION PROBLEM

Abstract

The present chapter considers a location test in a single sample setting without any assumption about the symmetry of the continuous distribution. Two adaptive test procedures are suggested - one is a probabilistic approach while the other is a deterministic approach. The deterministic approach is based on calculating a measure of symmetry and using it as a basis for choosing between the sign test and the Wilcoxon signed rank test. The probabilistic approach is also a combination of the sign test and the Wilcoxon signed rank test according to evidence of asymmetry provided by the p-value from the triples test for symmetry given in Randles *et al* (1980). A simulation study shows that the proposed deterministic approach using a simple measure of symmetry is as good as Lemmer (1993) in terms of power and attainment of the nominal size. The probabilistic approach has been shown to be superior to the other existing competitors.

2.1 Introduction

Let X_1, X_2, \dots, X_n denote a random sample from a continuous population with distribution function (d.f.) $F(x-\theta)$, where θ is the unknown population median. We consider the problem of testing

$$H_0 : \theta = \theta_0 \tag{2.1.1}$$

against some composite alternative H_1 . No assumption is made regarding the symmetry of the distribution.

Several distribution free tests are available in the literature for this single sample location problem. The sign (S^+) test is a valid test for H_0 irrespective of the skewness of the distribution. If no shape assumption is made on the underlying distribution, the sign test is uniformly most powerful size α for a one sided hypothesis about the median. For $H_0 : \theta = \theta_0$, the sign test only uses information in the sign of the random variables $X_1 - \theta_0, X_2 - \theta_0, \dots, X_n - \theta_0$. It does not take into account any metric information on how far the observation is from θ_0 . However, for a distribution that is symmetric about θ_0 , the vector of the absolute deviations of the observations from θ_0 is a sufficient statistic and hence it seems reasonable to make use of this information. Thus if we assume that the distribution is symmetric, then the Wilcoxon signed rank (W^+) test will be more efficient for testing the location than the S^+ test. This assumption seems to be realistic in many important situations including the paired data design. However, if the underlying distribution is asymmetric, the W^+ test is no longer distribution free and therefore may not maintain its nominal size. The danger in using W^+ test is that even a true null hypothesis may be rejected because of the skewness of the distribution.

Very often we have no idea regarding the skewness of the distribution. Some combinations of the S^+ and W^+ tests are proposed in the literature to obtain a reasonable power while maintaining the nominal significance level. Lemmer (1987) proposed the following procedure (L1): If the signed rank test has a highly significant value, reject H_0 , if it has an intermediate value, calculate the value of the sign test statistic and reject H_0 if this value is significant. For suitable critical values the test is shown to be robust and has good power properties compared to the sign test. However a disadvantage of the procedure is that it assumes that the user knows the direction of skewness of the distribution. Lemmer (1993) suggests two alternative procedures. Each adaptive procedure uses in its first stage some measure or test of symmetry to decide whether the W^+ test or the S^+ test should be used to test the hypothesis

about the median of the distribution. One of the procedures, denoted by L2, is based on calculating a measure of symmetry in its first stage and using it as a basis for choosing between the S^+ test and the W^+ test. The drawback of this procedure is the discontinuous nature of the test selection method. We may have a situation where a very small change in one observation value in the data actually results in a different choice of the test statistic. The other test (L3) is based on calculating the runs statistic of symmetry (McWilliams 1990). A disadvantage with the procedure is that the runs test may give highly significant values, not because the distribution is asymmetric but because it is symmetric about the true value of the median different from the one specified by the null hypothesis.

The purpose of the present chapter is to propose two adaptive test procedures for testing H_0 . The first adaptive procedure (AD1) has a probabilistic approach which uses the p-value from the triples test for symmetry given in Randles *et al.* (1980) and it is shown to be superior compared to the other existing test procedures. The second adaptive test has a deterministic approach like L2. It is observed that the highly recommended test for symmetry is not necessarily the best one to use as a preliminary test in such an adaptive procedure. The second proposed adaptive test (AD2) considering a simple measure of skewness, however, performs well in terms of power and attainment of nominal size. Baklizi (2005) used the p-value from the triples test to obtain modified Wilcoxon scores and developed an adaptive rank test. But our objective is to develop adaptive procedures for the location problem without the assumption of symmetry combining the sign and signed rank tests. The proposed adaptive test procedures are introduced in Section 2.2. In Section 2.3 we illustrate the proposed adaptive test procedures using an example. Section 2.4 presents a relative comparison of the proposed procedures to the existing competitors. Some asymptotic properties of the proposed adaptive procedures are discussed in Section 2.5. Finally Section 2.6 gives brief concluding remarks.

2.2 The Proposed Adaptive Tests

As in Section 2.1, let X_1, X_2, \dots, X_n denote a random sample from a continuous d.f. $F(x - \theta)$. We refer to $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ as the order statistics for the random sample X_1, X_2, \dots, X_n . In practice one seldom knows whether the distribution is symmetric. So here we propose adaptive procedures which allow the more powerful signed rank test to be used for fairly symmetric distributions, but otherwise prescribe the sign test.

2.2.1 The Probabilistic Approach

Before introducing the proposed adaptive rule we present a review of the triples test proposed by Randles *et al.* (1980). The null hypothesis for the triples test is that the underlying population is symmetric about θ against the alternative that it is asymmetric. Let

$$h(x_1, x_2, x_3) = \frac{1}{3} [\text{sign}(x_1 + x_2 - 2x_3) + \text{sign}(x_1 + x_3 - 2x_2) + \text{sign}(x_2 + x_3 - 2x_1)],$$

where $\text{sign}(x) = 1, 0, -1$ as $x >, =, < 0$. The triples test is then based on the U-statistic

$$\hat{\eta} = \frac{1}{\binom{n}{3}} \sum_{i < j < k} h(X_i, X_j, X_k). \quad (2.2.1)$$

This U-statistic estimates

$$E(\hat{\eta}) = \eta = P(X_1 + X_2 - 2X_3 > 0) - P(X_1 + X_2 - 2X_3 < 0),$$

with

$$\text{Var}(\hat{\eta}) = \sigma^2/n = \binom{n}{3}^{-1} \sum_{c=1}^3 \binom{3}{c} \binom{n-3}{3-c} \zeta_c,$$

where

$$\zeta_c = \text{Var}[f_c^*(X_1, \dots, X_c)]$$

and

$$f_c^*(X_1, \dots, X_c) = E[f^*(x_1, \dots, x_c, X_{c+1}, \dots, X_3)].$$

Note that if the underlying distribution is symmetric, $X_1 + X_2 - 2X_3$ has the same distribution as $-(X_1 + X_2 - 2X_3)$, and therefore $\eta = 0$. Hence we can use $\hat{\eta}$ as a statistic for testing

$$H_{0,S} : \eta = 0 \text{ versus } H_{1,S} : \eta \neq 0.$$

It follows that $\sqrt{n}(\hat{\eta} - \eta)/\sigma$ has a limiting standard normal distribution. Reject the null hypothesis of symmetry if $|T| > \tau_{\alpha/2}$, where $\tau_{\alpha/2}$ is the upper $\alpha/2$ th quantile of the standard normal distribution, and

$$T = \frac{\sqrt{n}\hat{\eta}}{\hat{\sigma}}, \quad \hat{\sigma}^2 = n \binom{n}{3}^{-1} \sum_{c=1}^3 \binom{3}{c} \binom{n-3}{3-c} \hat{\zeta}_c,$$

$$\hat{\zeta}_1 = \frac{1}{n} \sum_{i=1}^n (\hat{h}_1(X_i) - \hat{\eta})^2, \quad \hat{\zeta}_2 = \frac{1}{\binom{n}{2}} \sum_{j < k} (\hat{h}_2(X_j, X_k) - \hat{\eta})^2,$$

$$\hat{h}_1(X_i) = \frac{1}{\binom{n-1}{2}} \sum_{j < k, j \neq i, i \neq k} h(X_i, X_j, X_k),$$

$$\hat{h}_2(X_j, X_k) = \frac{1}{n-2} \sum_{i \neq j, i \neq k} h(X_i, X_j, X_k),$$

$$\hat{\zeta}_3 = \frac{1}{9} - \hat{\eta}^2.$$

We now introduce the proposed adaptive rule. Let the p-value of the triples test be denoted by

$$p_t = P_{H_{0,S}}(|T| \geq t).$$

The p-value can be considered as the amount of evidence against symmetry of the distribution present in the data. Whenever p_t is observed, perform a Bernoullian trial with probability of success p_t . If success occurs, use the W^+ test; otherwise, use the S^+ test. In other words our adaptive test rule is : Reject H_0 with probability p_t if $W^+ > w_\alpha^+$ and with probability $(1 - p_t)$ if $S^+ > s_\alpha^+$ or equivalently we may say that

accept H_0 with probability p_i if $W^+ \leq w_\alpha^+$ and with probability $(1 - p_i)$ if $S^+ \leq s_\alpha^+$, where w_α^+ and s_α^+ are the upper α -critical values for the W^+ and the S^+ tests, respectively. We require to make randomization to get exact size α for each of the tests.

2.2.2 The Deterministic Approach

Adaptive procedures using the concept of Hogg, referred here as the *deterministic approach*, are based on selector statistics which are functions of the sample observations with the purpose to select the type of underlying distribution. Here we introduce a simple measure of symmetry as a selector statistic. We may consider $(X_{(n)} - \tilde{X}) / (\tilde{X} - X_{(1)})$ as a measure of symmetry, where \tilde{X} denotes the median of the distribution. For a symmetric distribution the median is expected to be equidistant from both the extremes, while for a positively skewed distribution the median will be closer to the minimum and for a negatively skewed distribution it will be closer to the maximum. Now the quantity is further divided by $X_{(n)} - X_{(1)}$ to express it as a pure number. Thus the proposed measure of symmetry is

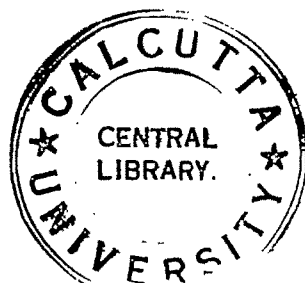
$$Q = \frac{X_{(n)} - 2\tilde{X} + X_{(1)}}{X_{(n)} - X_{(1)}} \quad (2.2.2)$$

The measure (2.2.2) has the limits -1 and 1.

The proposed adaptive test statistic is then given by

$$AD2 = S^+ I(|Q| > c) + W^+ I(|Q| \leq c), \quad (2.2.3)$$

where $I(x)$ is an indicator function assuming the value 1 or 0 according as x is true or false. Different values of c are examined and $c = 0.075$ is found to be the best choice in terms of robustness of the test.



2.3 An Example

In this section we would like to present a numerical example illustrating the application of the two proposed adaptive procedures. For this purpose we use the data given in Randles & Wolfe [1979,p 36] based on the study of the age distribution of a common mayfly species, *Stenacron interpunctatum*, among the various habitats in Big Darby Creek, Ohio. The data gave 10 measurements of head width (in micrometer divisions, 1 division = 0.0345 mm) of the mayflies in a particular habitat. The null hypothesis is that the median head width for mayflies from that particular habitat is 22 micrometer divisions against the alternative that it is greater than 22 *i.e.* the problem is to test

$$H_0 : \theta = 22 \quad vs \quad H_1 : \theta > 22.$$

To perform the AD1 test we need to compute the p-value for the triples test. We thus obtain the triples test statistic $T = 0.645$ and consequently the observed p-value is $p_t = 0.519$. We now perform a Bernoullian trial with probability of success 0.519. Suppose *failure* occurs. Then we must use the S^+ test. The observed value of the S^+ test statistic is 7 while $s_{0.05}^+ = 8$. Therefore at 5% level of significance we accept H_0 .

We also perform the AD2 test. For this we calculate $|Q| = 0.0556$. The observed $|Q|$ is less than $c = 0.075$ and we use the W^+ test. Observed $W^+ = 49$ while $w_{0.05}^+ = 44$ and therefore H_0 is rejected at the 5% level of significance. The difference in decision for the two adaptive procedures is because of the fact that the performance of both the adaptive tests depend on the criterion to measure the skewness of the distribution.

2.4 Relative Comparison of the Competing Tests

In this section we present the results of the simulation study for relative comparisons among the proposed adaptive procedures and the other existing competitors. The results are given only for the upper-tailed alternatives. For this study we consider

the following distributions:

- (I) Standard normal distribution: $N(0,1)$
- (II) Cauchy distribution centered at the origin and scale parameter 1: $C(0,1)$
- (III) Uniform distribution: $U(0,1)$
- (IV) Lognormal distribution: $LN(\mu, \sigma^2)$ with $\mu=3$, $\sigma^2=0.1, 0.2, 0.4, 0.6, 1.2, 1.5$.
- (V) Pareto distribution: The density function is

$$f(x) = \frac{ka^k}{x^{k+1}}, \quad x > a,$$

where $a(>0)$ is the location parameter and $k(>0)$ is the shape parameter. The median of the distribution is $a2^{1/k}$. Here we take median=20 and $k=0.5, 1, 2, 10, 20$.

The nominal significance level of the test is taken to be $\alpha=0.05$ and sample size $n=20$. The empirical size and power of the test are computed as the relative frequency with which a particular test rejects the null hypothesis H_0 . The results of the simulation study involving 5000 replications of the random sampling process are presented in Table 2.1. We investigate the power of the test at $\theta=\xi_{0.5}, \xi_{0.55}, \xi_{0.6}, \xi_{0.65}, \xi_{0.7}$, where ξ_q is the q th quantile of the distribution. The L1 and L3 tests are not included in the comparison since they are found to be less powerful than the L2 test.

For normal distribution we observe that the proposed adaptive test AD1 is only second to the W^+ test in terms of power comparison while the AD2 test and the L2 test are quite close. When the underlying distribution is Cauchy the S^+ test is better than the W^+ test and both the AD2 and L2 tests staying quite close to the S^+ test. The AD1 test however performs better than the W^+ test. With the uniform distribution the W^+ test is again the best with the AD1 test in the second position way ahead of the other competitors. Here again AD2 and L2 are quite close. We observe that if the underlying distribution is symmetric both the proposed adaptive procedures produce empirical size close to the nominal level of significance.

Table 2.1 : Empirical size and power of the tests

	q	S^+	W^+	AD1	AD2	L2
N(0,1)	0.50	0.050	0.047	0.051	0.051	0.054
	0.55	0.115	0.128	0.126	0.115	0.127
	0.60	0.224	0.284	0.251	0.240	0.235
	0.65	0.381	0.489	0.441	0.423	0.408
	0.70	0.568	0.719	0.638	0.618	0.601
C(0,1)	0.50	0.048	0.051	0.045	0.049	0.049
	0.55	0.111	0.115	0.102	0.111	0.113
	0.60	0.231	0.206	0.211	0.215	0.218
	0.65	0.377	0.330	0.350	0.373	0.381
	0.70	0.573	0.492	0.517	0.548	0.574
U(0,1)	0.50	0.048	0.050	0.049	0.056	0.052
	0.55	0.117	0.161	0.147	0.127	0.127
	0.60	0.218	0.387	0.317	0.277	0.272
	0.65	0.380	0.651	0.557	0.484	0.470
	0.70	0.575	0.851	0.736	0.655	0.658
LN(3,0.1)	0.50	0.052	0.090	0.058	0.055	0.057
	0.55	0.115	0.197	0.146	0.126	0.121
	0.60	0.226	0.376	0.281	0.244	0.240
	0.65	0.385	0.613	0.442	0.411	0.424
	0.70	0.567	0.793	0.640	0.591	0.592
LN(3,0.2)	0.50	0.049	0.107	0.062	0.054	0.059
	0.55	0.113	0.242	0.141	0.131	0.129
	0.60	0.229	0.435	0.270	0.242	0.249
	0.65	0.375	0.650	0.427	0.405	0.406
	0.70	0.567	0.828	0.623	0.579	0.595
LN(3,0.4)	0.50	0.047	0.134	0.065	0.055	0.057
	0.55	0.119	0.289	0.140	0.120	0.125
	0.60	0.212	0.482	0.250	0.237	0.233
	0.65	0.373	0.685	0.410	0.392	0.390
	0.70	0.560	0.860	0.600	0.565	0.575

Table 2.1 (*Continued*)

	q	S^+	W^+	AD1	AD2	L2
LN(3,0.6)	0.50	0.048	0.145	0.058	0.052	0.048
	0.55	0.116	0.313	0.141	0.111	0.118
	0.60	0.216	0.506	0.248	0.217	0.233
	0.65	0.376	0.719	0.408	0.382	0.387
	0.70	0.563	0.866	0.598	0.591	0.584
LN(3,1.2)	0.50	0.050	0.204	0.052	0.053	0.048
	0.55	0.113	0.360	0.121	0.115	0.113
	0.60	0.231	0.582	0.236	0.234	0.222
	0.65	0.374	0.769	0.383	0.378	0.375
	0.70	0.572	0.896	0.581	0.569	0.588
LN(3,1.5)	0.50	0.046	0.213	0.054	0.052	0.048
	0.55	0.114	0.392	0.120	0.115	0.112
	0.60	0.229	0.583	0.231	0.218	0.234
	0.65	0.379	0.774	0.379	0.398	0.368
	0.70	0.556	0.905	0.552	0.571	0.577
Pareto (k=0.5)	0.50	0.052	0.322	0.050	0.050	0.055
	0.55	0.117	0.513	0.127	0.124	0.117
	0.60	0.284	0.761	0.299	0.299	0.292
	0.65	0.625	0.946	0.623	0.615	0.632
	0.70	0.951	0.999	0.952	0.949	0.945
Pareto (k=1)	0.50	0.046	0.253	0.049	0.047	0.048
	0.55	0.126	0.460	0.129	0.119	0.117
	0.60	0.296	0.723	0.298	0.306	0.298
	0.65	0.611	0.945	0.618	0.630	0.629
	0.70	0.954	0.999	0.950	0.947	0.941
Pareto (k=2)	0.50	0.052	0.223	0.054	0.050	0.047
	0.55	0.122	0.412	0.134	0.123	0.116
	0.60	0.289	0.697	0.304	0.303	0.295
	0.65	0.631	0.941	0.615	0.613	0.621
	0.70	0.951	0.998	0.950	0.947	0.947

Table 2.1 (*Continued*)

	q	S^+	W^+	AD1	AD2	L2
Pareto	0.50	0.052	0.180	0.059	0.049	0.048
(k=10)	0.55	0.130	0.362	0.133	0.123	0.120
	0.60	0.286	0.664	0.315	0.299	0.290
	0.65	0.634	0.921	0.631	0.628	0.640
	0.70	0.951	0.999	0.948	0.966	0.950
Pareto	0.50	0.055	0.180	0.063	0.051	0.055
(k=20)	0.55	0.136	0.382	0.139	0.128	0.140
	0.60	0.286	0.648	0.305	0.282	0.300
	0.65	0.620	0.933	0.630	0.635	0.630
	0.70	0.946	0.999	0.953	0.948	0.950

The W^+ test tends to be anti-conservative for the upper-tail testing when the underlying distribution is lognormal. Thus in this situation we observe that the AD1 test, providing adequate control on the size, emerges as the best test among the competing procedures. The AD2 and L2 tests remain quite close as before and are ahead of the S^+ test. For the Pareto distribution again the W^+ test fails to maintain its nominal size. The proposed adaptive test AD1 has high power maintaining the nominal significance level reasonably. The AD2 and L2 tests also perform quite well and sometimes they are as good as the AD1 test. The S^+ test in this case although remains at the last place but very competitive.

Discussion: The simulation study performed here clearly indicates that the AD1 and AD2 tests are robust for nearly all cases. The AD1 test tends to be non-robust only for the lognormal distribution with σ^2 between 0.2 and 0.4 and for the Pareto distribution with $k = 20$. On the other hand, the W^+ test does not hold its nominal level very well when the underlying distribution is asymmetric. Thus it is not justifiable to include it in any power comparisons as the high powers might easily

be attributed to the inflated levels. Both the proposed adaptive procedures perform better than the S^+ test in most of the situations. The proposed test AD1 has the nearest power to the W^+ test which is the best test when the underlying distribution is symmetric. In case of small departure from symmetry the AD1 test is the best test among the competing tests. For moderate skewness the AD1 test is still the most powerful test but the size deviates a little from the nominal level of significance. Again if the distribution is highly skewed the S^+ , AD1, AD2 and L2 tests are seen to exhibit similar performance. So if one has no idea about the skewness of the distribution the proposed adaptive test procedures should be used and in such a case the AD1 test seems to be the most powerful among the existing competitors.

2.5 Some Asymptotic Properties

We now consider some asymptotic properties of the proposed adaptive test statistics. We know that, under either hypothesis, S^+ is the sum of independently and identically distributed (iid) Bernoulli variables with probability of success

$$p = P(X > \theta_0), \quad 0 < p < 1.$$

So we can apply the central limit theorem to assert that

$$\frac{S^+ - E(S^+)}{\sqrt{Var(S^+)}}$$

has an approximate standard normal distribution. It follows that

$$E(S^+) = n[1 - F(-\theta)],$$

$$Var(S^+) = n[1 - F(-\theta)]F(-\theta)$$

with

$$E_{H_0}(S^+) = \frac{n}{2}, \tag{2.5.1}$$

and

$$\text{Var}_{H_0}(S^+) = \frac{n}{4}. \quad (2.5.2)$$

Then, under H_0 ,

$$\frac{(S^+ - \frac{n}{2})}{\sqrt{\frac{n}{4}}} \quad (2.5.3)$$

has a limiting standard normal distribution.

Again, it can be shown that, W^+ is a linear combination of iid Bernoulli random variables with probability of success $1/2$ under H_0 , provided the underlying distribution is symmetric. Thus, if the underlying distribution is symmetric, we know that the null mean and variance of the W^+ statistic are

$$E_{H_0}(W^+) = \frac{n(n+1)}{4}, \quad (2.5.4)$$

and

$$\text{Var}_{H_0}(W^+) = \frac{n(n+1)(2n+1)}{24}. \quad (2.5.5)$$

Hence, applying Liapounov central limit theorem, it can be shown that, if the underlying distribution is symmetric, the standardized Wilcoxon signed rank statistic, viz.,

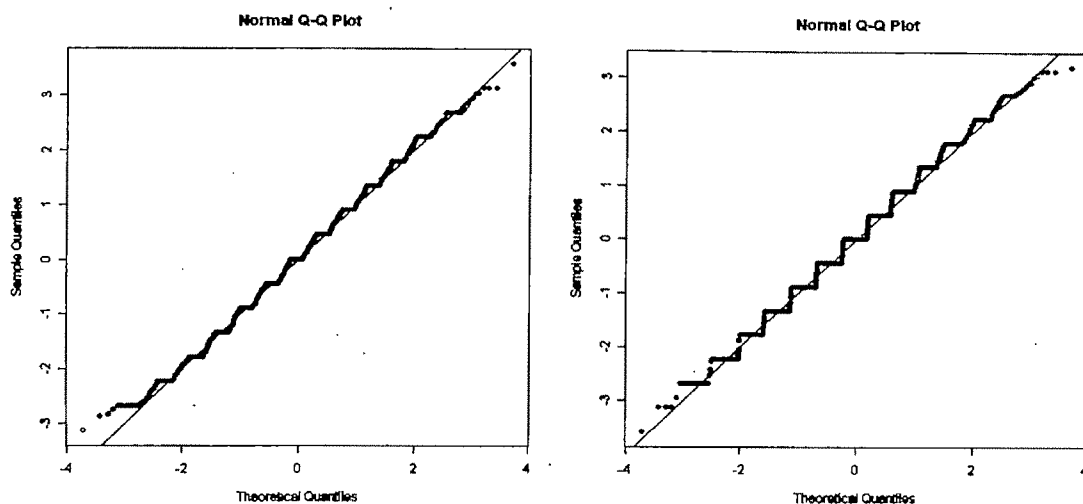
$$\frac{(W^+ - \frac{n(n+1)}{4})}{\sqrt{\frac{n(n+1)(2n+1)}{24}}} \quad (2.5.6)$$

is asymptotically standard normal under H_0 .

Also note that the magnitude of the p-value of the triples test, involved in the proposed AD1 test, is an indicator of the amount of asymmetry of the distribution present in the data. If the data do not present sufficient evidence of asymmetry, the p-value approaches 1. If the underlying distribution is asymmetric, the p-value tends to 0. Keeping the above results in mind we consider the following standardized form of AD1 to study its asymptotic behavior:

$$AD1^* = \frac{(W^+ - \frac{n(n+1)}{4})}{\sqrt{\frac{n(n+1)(2n+1)}{24}}} I(U^* \leq p_t) + \frac{(S^+ - \frac{n}{2})}{\sqrt{\frac{n}{4}}} I(U^* > p_t), \quad (2.5.7)$$

Figure 1: Normal Q-Q Plot for $N(0,1)$ and $LN(3,0.4)$ distribution



where U^* is uniformly distributed over $(0,1)$ and is independent of X_1, X_2, \dots, X_n .

At the first stage a normal Q-Q (Quantile-Quantile) plot (Figure 1) is used to check the asymptotic normality of $AD1^*$ under H_0 . The normal Q-Q plot is a very useful visual tool for assessing whether the distribution of a given variable follows a normal distribution. The Q-Q plot plots the empirical quantiles against the theoretical quantiles for normal distribution. When the distribution of the variable under examination has the same shape as the reference distribution, the normal distribution in this case, the Q-Q plot is linear. We have generated Q-Q plots using standard normal and lognormal distributions for different σ^2 and the normal Q-Q plot seems to be linear even for sample size as small as $n = 20$. We present here the normal Q-Q plots for $N(0,1)$ distribution and $LN(3,0.4)$ distribution. These normal Q-Q plots provide us with a fair indication about the asymptotic normality of $AD1^*$ under H_0 .

We now proceed to verify the asymptotic normality of $AD1^*$ under H_0 theoretic-

cally in the following result:

Result 2.1 *Suppose X_1, X_2, \dots, X_n is a random sample from a continuous population with d.f. $F(x-\theta)$. The statistic $AD1^*$ has asymptotically standard normal distribution under H_0 .*

Proof: Let $\Psi_n(\tau)$ denote the distribution function corresponding to $AD1^*$. Then we can write

$$\Psi_n(\tau) = P_{H_0}(AD1^* \leq \tau) = P_{H_0}(A_n \cap E_n) + P_{H_0}(B_n \cap E_n^c),$$

where where A_n, B_n and E_n denote, respectively, the events $\left[\frac{(W^+ - \frac{n(n+1)}{4})}{\sqrt{\frac{n(n+1)(2n+1)}{24}}} \leq \tau \right]$, $\left[\frac{(S^+ - \frac{n}{2})}{\sqrt{\frac{n}{4}}} \leq \tau \right]$ and $[U^* \leq p_t]$.

Note that

$$\lim_{n \rightarrow \infty} P(E_n) = 1 \text{ or } 0 \quad (2.5.8)$$

according as the underlying distribution symmetric or asymmetric i.e. $\eta = 0$ or otherwise. Also, note that, for $\eta = 0$,

$$\lim_{n \rightarrow \infty} P_{H_0}(A_n) = \Phi(\tau), \quad (2.5.9)$$

and, for any η ,

$$\lim_{n \rightarrow \infty} P_{H_0}(B_n) = \Phi(\tau), \quad (2.5.10)$$

where $\Phi(\tau)$ is the d.f. of a standard normal variable. We now consider the following two cases:

Case 1. When $\eta = 0$

$$P_{H_0}(B_n \cap E_n^c) \leq P_{H_0}(E_n^c) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus

$$\lim_{n \rightarrow \infty} P_{H_0}(B_n \cap E_n^c) = 0.$$

Moreover, combining (2.5.8), (2.5.9) and

$$P_{H_0}(A_n) + P_{H_0}(E_n) - 1 \leq P_{H_0}(A_n \cap E_n) \leq \min(P_{H_0}(A_n), P_{H_0}(E_n)),$$

we find that

$$\lim_{n \rightarrow \infty} P_{H_0}(A_n \cap E_n) = \Phi(\tau).$$

Hence

$$\lim_{n \rightarrow \infty} \Psi_n(\tau) = \Phi(\tau). \quad (2.5.11)$$

Case 2. When $\eta \neq 0$

$$P_{H_0}(A_n \cap E_n) \leq P_{H_0}(E_n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which gives

$$\lim_{n \rightarrow \infty} P_{H_0}(A_n \cap E_n) = 0.$$

Again, combining (2.5.8), (2.5.10) and

$$P_{H_0}(B_n) + P_{H_0}(E_n^c) - 1 \leq P_{H_0}(B_n \cap E_n^c) \leq \min(P_{H_0}(B_n), P_{H_0}(E_n^c))$$

we obtain

$$\lim_{n \rightarrow \infty} P_{H_0}(B_n \cap E_n^c) = \Phi(\tau).$$

Thus,

$$\lim_{n \rightarrow \infty} \Psi_n(\tau) = \Phi(\tau). \quad (2.5.12)$$

Finally, combining (2.5.11) and (2.5.12), we get

$$\lim_{n \rightarrow \infty} \Psi_n(\tau) = \Phi(\tau)$$

without any assumption regarding the symmetry of the underlying distribution. \square

Similarly we can establish the asymptotic normality of the standardized form of AD2 under H_0 . The asymptotic power property of the adaptive test procedures depend on the criteria and the statistic used. The power of the adaptive test converges to the power of the better component i.e., the power of our adaptive test, under a sequence of Pitman's local alternatives, converges to the power of the W^+ test if the underlying population is symmetric and to that of the S^+ test if the underlying population is asymmetric. This is shown in the following result.

Result 2.2 *Suppose $F(x)$ has the density $f(x)$ at all real x . Then the common asymptotic power of the adaptive tests under the sequence of local alternatives*

$$\theta_n = \theta_0 + \frac{b}{\sqrt{n}}, b > 0 \quad (2.5.13)$$

is

$$I(\eta = 0)\Phi(\sqrt{12}b \int f^2(y)dy - \tau_\alpha) + I(\eta \neq 0)\Phi(2bf(0) - \tau_\alpha).$$

Under the sequence of local alternatives (2.5.13) with $\eta = 0$, the asymptotic power (see Hettmansperger, 1984; pp 62-70) of the W^+ test is given by

$$\Phi(\sqrt{12}b \int f^2(y)dy - \tau_\alpha),$$

and, under (2.5.13), the S^+ test has asymptotic power

$$\Phi(2bf(0) - \tau_\alpha).$$

Using the same technique as in Result 2.1 the result follows immediately.

2.6 Concluding Remarks

In this chapter we have developed two adaptive procedures for testing the hypothesis about the median of the distribution without the assumption of symmetry. We must be aware of the dangers associated with the use of such adaptive procedures. Even though the final test is performed at the desired level of significance α , in the overall testing procedure the actual level may be quite different from the nominal level of significance. The tests which are best in their own rights are not necessarily the best one to use as the preliminary test in an adaptive procedure. So the robustness of adaptive procedures should be carefully examined. For example, we may consider the triples test which is a highly recommended test for testing whether a continuous univariate population is symmetric. We have used it in the first stage of our probabilistic approach and the overall test have been found to be robust enough. But, when we use the same test as the preliminary test in the deterministic approach, the robustness property is not maintained. So the choice of the preliminary test for such an adaptive procedure should be made with extra care. The deterministic approach (AD2) proposed in this chapter is based on a very simple measure of symmetry and it maintains the designated level α fairly accurately while displaying power as good as the L2 test. In the L2 test the measure of symmetry used in the first stage is based on \bar{U}_γ , \bar{M}_γ and \bar{L}_γ , which denote the average of the $\gamma \cdot n$ largest, middle and smallest order statistics, respectively (see Randles and Wolfe, 1979, p 389). If this measure falls in a specific interval, then the distribution is considered to be symmetric. Here γ and the interval are both chosen subjectively. In the AD2 test only the choice of the point c is subjective and the test performs satisfactorily unless there is outlier in the data.

From the results of the simulation study shown in Section 2.4 it is clear that the proposed adaptive test procedure (AD1) based on the probabilistic approach is

reasonably robust and has high power compared to the other competitors referred in this chapter. So when nothing is known about the skewness of the distribution we must use the proposed AD1 test. Few asymptotic properties of the proposed adaptive test procedure are discussed in Section 2.5. Under H_0 , the standardized form of the proposed test statistic is asymptotically normally distributed. The power of the adaptive test under a sequence of Pitman's local alternatives is shown to converge to the asymptotic power of the better component.

Each adaptive procedure, described in this chapter, uses in its first stage some measure or test of symmetry to decide whether the signed rank test or the sign test should be used to test the hypothesis about the median of the distribution. We can also construct an adaptive scheme that will first classify the underlying distribution as having light, medium or heavy tails. Then we may suggest suitable one-sample nonparametric tests for these classifications. In the next chapter, we extend our study to the two-sample set up and consider the hypothesis that two symmetric distributions of the continuous type and of the same shape have identical location. The only reason for inserting the symmetry assumption is the fact that we wish to demonstrate the use of a tailweight measure as the selector statistic to classify the underlying distribution.