Chapter 1

INTRODUCTION

1.1 Background

In the last few decades the search for robust statistical procedures has become a principal area of inquest in statistical research. An enormous effort has been devoted to robust procedures because some statisticians are now concerned that the basic assumptions about underlying models might very well dominate the analysis of the data in many cases. Many classical parametric statistical procedures usually assume the underlying distribution to be normal. For example, the paired t-test requires that the distribution of the differences be approximately normal, while the unpaired t-test requires an assumption of normality to hold separately for both sets of observations. The acceptance of the normality assumption without much inquiry is definitely causing concern among a group of statisticians. According to Bancroft, "the property of normality of distribution of errors can itself be considered as a proper question of uncertainty in the model specification and hence just another problem involving inference for incompletely specified models".

Although statisticians have long been concerned about the implications of incorrect model assumptions, it has not been until the advent of extensive computer simulation that the seriousness of the erroneous assumptions has been clearly recognised. It is therefore desirable to devise some procedures that are free of this assumption concerning distribution. Motivated by this, statisticians tried to develop statistical procedures that are independent of the distribution of the random variable on which observations have been obtained. Nonparametric methods provide an alternative series of statistical techniques that do not make numerous or strict assumptions about the population from which the data have been sampled. There is a wide range of methods that can be used in different circumstance, *e.g.* the sign test is used as an alternative to the single sample or paired t-test, while a nonparametric alternative to the unpaired t-test is given by the Wilcoxon-Mann-Whitney test. More often than not, the nonparametric procedures are only slightly less efficient than their normal theory competitors when the underlying populations are normal, and they can be reasonably more efficient than these competitors when the underlying populations are not normal.

Much of the field of nonparametric statistics is based on rank statistics, particularly in the area of hypothesis testing. However the choice of a suitable rank test depends on the underlying distribution which may be symmetric or asymmetric and which may have short, medium or long tails. In practice we know either nothing about the underlying distribution or only have a partial little information. Thus it seems sensible to use procedures good for a broad class of possible underlying models, but which are not necessarily best for any of them. Such procedures are frequently characterized as being robust. One approach to achieving robustness is to use the data to estimate their distribution or equivalently scores associated with the rank statistic. Another approach, where we assume that the distribution of the given data belongs to a reasonably broad family of distributions, is to choose a subfamily which seems to have a reasonably close relation to the data with respect to some well defined characteristics. Then we select a rank test which is reasonably good for the main inference problem. As all these test procedures are *adapted* to the given data, they are called *adaptive nonparametric tests.* Throughout the present dissertation we have our development on the second method of adaptation where the final decision is reached in two stages.

For clarity of exposition we commence our discussion on adaptive tests with the single-sample location problem. We next consider the two-sample location problem under symmetry and then advance to the generalized Behrens-Fisher problem. Subsequently we reflect on the two-sample scale problem under symmetry. Finally we conclude the present dissertation with an adaptive test for the two-sample location problem based on a progressively censored scheme.

1.2 Brief Review of the Literature

There is a variety of works on adaptive testing over the last few decades. The advantage of adaptive method is that, for tests of significance, the adaptive approach is extremely effective when a wide class of distributions is considered. But we must also be aware of the possible dangers associated with two-staged adaptive rules. Even though the final test is conducted at the desired level α, the true level of significance in the overall testing procedure may be quite different from α. This is because in the second stage we are actually required to control the conditional probability of type I error, given that the data have determined the use of a particular transformation. This is usually quite difficult to do. Hájek (1962) developed a method to estimate the density, or rather the score function, from the data at hand and hence construct the rank test generated by this estimate of the score function. Such adaptive tests are generally asymptotically optimum for a broad class of densities, but their use involves tedious computations and the convergence rates for the asymptotic results may be quite slow. The first practical adaptive procedure for testing the null hypothesis that the center of a symmetric distribution equals a specified value was proposed by Randles and Hogg (1973). In the same paper they also proposed a two-sample adaptive procedure for testing the null hypothesis that two symmetric distributions have identical locations. A few years later Hogg, Fisher and Randles (1975) proposed a two-sample adaptive test that did not require the assumption of symmetric distributions. This two-sample test used measures of asymmetry and tailweight to select one of several rank tests. The supremacy of these adaptive procedures over the usual parametric procedures based on the sample means and the usual nonadaptive procedures based on ranks was established by extensive simulation studies.

Policello and Hettmansperger (1976) proposed an adaptive rank test for the single sample location problem. This test is not distribution-free but maintains its nominal levels reasonably well. An adaptive rank test, considered by Jones (1979), is distribution-free for the same problem. Further adaptive tests for the single sample location problem are proposed by Lemmer (1993) and Baklizi (2005). Some adaptive tests for location alternatives in the two-sample and c-sample problem can be found, among others, in Büning (1994, 1996, 1999), Ruberg (1986), Hüsler (1987), O'Gorman (1996, 1997a), Beier and Büning (1997), Büning and Kössler (1998), Büning and Rietz (2003), Kössler (2005) and Kössler and Kumar (2008). The proposed test by Ruberg (1986) for the two-sample problem, unlike Hogg's method, is a continuously adaptive rank based test that changes only slightly when small changes are made to the data. Unfortunately the test is rather complex and is found to have relatively low power in certain situations. For the one-way layout a continuously adaptive rank based test is considered by O'Gorman (1997b). This test has relatively good power compared to the F test and the Kruskal-Wallis test. Adaptive tests for the two-sample scale problem are proposed by Kössler (1994), Hall and Padmanabhan (1997) and Büning (2003) while Büning and Thadewald (2000) and Neuhäuser (2001) work on the location and scale problem. An adaptive distribution-free test for the multisample scale problem is studied by Büning and Rietz (2008). Büning (2002) introduced an adaptive test for the general two-sample problem based on tests of Kolmogorov-Smirnov and Cramer-von Mises type while John and Priebe (2007) developed another data-adaptive solution of the two-sample problem. Pecková and Fleming (2003) develop an adaptive test for the difference in survival distributions when the shape of the hazard ratio is unknown. Hill *et al.* (1988) and Sun (1997) develop adaptive nonparametric procedures for the problems of testing for ordered alternatives and multiple comparisons in the one-way analysis of variance (ANOVA) model. An adaptive distribution-free test for the regression problem is suggested by Hogg and Randles (1975). A more general approach to test the parameters in linear models is proposed by O'Gorman (2002). In this adaptive procedure the tests of significance are formulated as tests for a subset of regression coefficients in a linear model. One multivariate adaptive test procedure is proposed for the single sample location problem by Peters (1991). A selector statistic constructed from univariate Mahalanobis distances is used to choose the appropriate sign or signed rank procedure yielding a large sample test which performs well over a broad class of distributions.

A brief review of some adaptive robust procedures along with some suggestions for future applications and theory is given by Randles and Hogg (1973), Hogg (1976, 1982) and Hogg and Lenth (1984). For a wide discussion on adaptive statistical methods one can also go through the book by O'Gorman (2004).

1.3 Motivation of the Present Work

Most of the adaptive tests available in the literature, discussed in the previous section, is based on the concept of Hogg (1967). Hereafter, in the present dissertation, any such method will be referred as the *deterministic approach.* Suppose that each of k tests based on the statistics $T_1, T_2, ..., T_k$ is distribution-free over the class of distribution functions under consideration. Let S be some selector statistic used to chose a test from these k tests. Further, let U_S denote the set of all values of S with the following decomposition:

$$U_S = D_1 \cup D_2 \cup ... \cup D_K, \quad D_i \cap D_j = \emptyset \text{ for } i \neq j,$$

so that $S \in D_j$ corresponds to the decision to use the test based on T_j. In this way the observed data select the model that seems appropriate, and then we make the statistical inference for the situation under consideration.

A disadvantage with these usual deterministic adaptive procedures is the discontinuous nature of the test selection method. The test selection is likely to be affected if the value of the selector statistic is near the boundary between two partitioning sets. Here a small change in the data may move the observed value of the selector statistic over the boundary. Moreover the boundaries are defined empirically. Although their partitioning may give good results, more objective and stochastic considerations may lead to a different, even better, adaptive procedures. The choice of the test statistic depends not only on the selector statistic, but also on the sample size. In the present work we shall discuss some adaptive nonparametric test procedures that get rid of this drawback in the deterministic approach. The proposed adaptive procedures consist in calculating some classification probabilities, based on the p-values of pretests, to decide an appropriate test for the problem of interest. These so called *stochastic* or *probabilistic* adaptive procedures are shown to be effective and yet computationally simple enough to appeal to the practicing statistician.

The theory of hypothesis testing is heavily dependent on the pre-specified value of the level of significance. In the traditional method the decision function is determined in such a manner that the probability of Type II error is minimum subject to the conditions imposed by the chosen level. This method avoids the problem of interrelationship between the Type I and Type II error probabilities. In many cases the significance level is chosen arbitrarily. Moreover, in nonparametric statistics, usually we do not have adequate information about alternative distributions so that the Type II error considerations are not clear enough. So instead of prescribing a fixed size α test for the preliminary testing problem we consider the *observed level of significance*, which is herein called the p-value. The p-value is the smallest level of significance at which an experimenter would reject the null hypothesis on the basis of the observed data. The so called p-value gives an idea of how strongly the data contradict the hypothesis.

The p-value is defined as the probability, under null hypothesis, of obtaining a value of the test statistic equal to or more extreme than the observed value. For one-sided tests the p-value is a well defined quantity. Two define a p-value for two-sided tests we consider the following two quantities. An upper one-tailed p-value is the probability that the corresponding test statistic is greater than or equal to its observed value. Similarly, a lower one-tailed p-value is the probability that the corresponding test statistic is less than or equal to its observed value. For two-tailed test a definition of p-value can be given by the minimum of these two upper and lower p-values. Another definition which seems to be logical is given by the sum of the probability of a value equal to or more extreme than that observed in one tail and some probability from the opposite tail. However, this can give rise to various p-values for the same set of observations depending upon the choice of the probability added from the other tail. A reasonable solution is to consider a two-tailed p-value as twice the minimum of the upper and lower p-values when the null distribution is symmetric.

Under the alternative hypothesis we generally expect the p-values to be small. If the p-value is less than the desired level of significance, then we reject the null hypothesis. Since the p-value is a measure of evidence against the null hypothesis, the magnitude of the p-value is imperative to the interpretations and conclusions inferred from the observed data. The p-value is itself a random variable whose distribution, for the null hypothesis, is asymptotically uniform over $(0,1)$ if the test statistics obey some regularity assumptions. Moreover, for the alternative hypothesis, the p-value goes to zero with probability one as the sample size becomes large. Bahadur (1960) argues that the rate at which the p-value approaches zero is an indication of the asymptotic efficiency of the test and in this connection he studied limiting value of the *level actually attained*.

Note that in order to study the asymptotic properties of the proposed adaptive procedures we need to look into the asymptotic behavior of the p-values of the preliminary tests. We know that, in general, convergence in distribution does not imply convergence of the corresponding expected value. However, for every uniformly bounded continuous function g, convergence in distribution of the sequence of random variables $\{Y_n\}$ to the random variable Y does imply convergence of $E[g(Y_n)]$ to $E[g(Y)]$. Thus, as the sample size increases, here the expected p-value approaches $\frac{1}{2}$ and 0 under the null and the alternative hypotheses, respectively. Hereinafter, whenever we discuss about the limiting p-value we indeed refer to the limit of the expected p-value. P-values, with the additional information they provide, are more desirable than fixed levels in the construction of adaptive test procedures.

1.4 Summary of the Work Done

In this section we summarize the works done in the dissertation as follows:

Chapter 2: **Adaptive Nonparametric Tests for Single Sample Location Problem:**

Let X_1, X_2,...,X_n denote a random sample from a continuous population with distribution function (d.f.) F(x-θ), where θ is the unknown population median. We consider the problem of testing

$$H_0 : \theta = \theta_0$$

against some composite alternative H_1. No assumption is made regarding the symmetry of the distribution.

Several distribution free tests are available in the literature for this single sample location problem. The sign (S^+) test is a valid test for H_0 irrespective of the skewness of the distribution . If we assume that the distribution is symmetric, then the Wilcoxon signed rank (W^+) test will be more efficient for testing the location than the S^+ test. If the underlying distribution is asymmetric, the W^+ test is no longer distribution free and therefore may not maintain its nominal size. Our objective is to develop adaptive procedures for the location problem without the assumption of symmetry combining the sign and signed rank tests.

The null hypothesis for the preliminary test is that the underlying population is symmetric about θ against the alternative that it is asymmetric. Let

$$h(x_1, x_2, x_3) = \frac{1}{3}[sign(x_1 + x_2 - 2x_3) + sign(x_1 + x_3 - 2x_2) + sign(x_2 + x_3 - 2x_1)],$$

where $sign(x) = 1, 0, -1$ as $x >, =, < 0$. The triples test proposed by Randles *et al.* (1980) for testing symmetry versus asymmetry is then based on the U-statistic

$$\hat{\eta} = \frac{1}{\binom{n}{3}} \sum_{i<j<k} h(X_i, X_j, X_k).$$

Reject the null hypothesis of symmetry if $|T|>\tau_{\alpha/2}$, where

$$T = \frac{\sqrt{n}\hat{\eta}}{\hat{\sigma}},$$

with $\hat{\sigma}$ as a consistent estimate of the variance of the U-statistic $\hat{\eta}$.

We now introduce the proposed adaptive rules.

Probabilistic Rule: Let p_t denote the p-value corresponding to an observed $\hat{\eta}$. Our adaptive test rule AD1 is : Reject H_0 with probability p_t if $W^+>w_\alpha^+$ and with probability (1-p_t) if $S^+>s_\alpha^+$, where w_α^+ and s_α^+ are the upper α-critical values for the W^+ and the S^+ tests, respectively. We may require to make randomization to get exact size α for each of the tests.

Deterministic Rule: We refer to $X_{(1)}\leq X_{(2)}\leq .....\leq X_{(n)}$ as the order statistics for the random sample X_1, X_2,...,X_n. We may consider ($X_{(n)}$-$\widetilde{X}$)-($\widetilde{X}$-$X_{(1)}$) as a measure of symmetry, where $\widetilde{X}$ denotes the median of the observed distribution. The quantity is further divided by $X_{(n)}$-$X_{(1)}$ to express it as a pure number. Thus the proposed measure of symmetry is

$$Q = \frac{X_{(n)} - 2\widetilde{X} + X_{(1)}}{X_{(n)} - X_{(1)}}.$$

The measure has the limits -1 and 1.

The proposed adaptive test statistic is then given by

$$AD2 = S^{+}I(\mid Q \mid > c) + W^{+}I(\mid Q \mid \leq c),$$

where I(x) is an indicator function assuming the value 1 or 0 according as x is true or false. Different values of c are examined and c = 0.075 is found to be the best choice in terms of robustness of the test.

***Chapter 3*: Adaptive Nonparametric Tests for Two-Sample Location Problem Under Symmetry:**

Let X_1,X_2,...,X_{n_1} and Y_1,Y_2,...,Y_{n_2} be random samples from populations with continuous distribution functions (d.f.'s) F(x) and F(x-θ), $-\infty < \theta < \infty$, respectively. Suppose $F(x) + F(-x) = 1$ for all x. Then the objective is to test

$$H_0 : \theta = 0$$

against some composite alternative.

Under H_0, X_1,X_2,...,X_{n_1},Y_1,Y_2,...,Y_{n_2} represent a single random sample of size N=n_1+n_2 from a continuous symmetric d.f. F(x). A two-sample simple linear rank statistic is of the form

$$A_T = \sum_{j=1}^{n_2} a_T(R_j),$$

where R_j denotes the rank of Y_j among all N observations and $a_T(1), a_T(2),..., a_T(N)$ denote scores which satisfy nondecreasing and nonconstant conditions, i.e.,

$$a_T(1) \leq a_T(2) \leq ... \leq a_T(N), a_T(1) \neq a_T(N).$$

For any such choices of scores A_T provides a nonparametric test for H_0.

First, we consider the Wilcoxon-Mann-Whitney test (W) by letting

$$a_W(i) = i, \text{ for } i = 1, 2, ..., N,$$

in the linear rank statistic. This test is the locally most powerful rank test for detecting shift in logistic distribution, and it also has good power for detecting shifts in medium tailed symmetric distribution.

For a light tailed symmetric model, use the rank test (L) corresponding to scores given by

$$a_L(i) = \begin{cases} i - [(N+1)/4] & \text{if } i \leq (N+1)/4 \\ 0 & \text{if } (N+1)/4 < i < 3(N+1)/4 \\ i - [3(N+1)/4] & \text{if } i \geq 3(N+1)/4, \end{cases}$$

where [J] denotes the greatest integer less than or equal to J.

If the data indicate a model which is heavy tailed and symmetric, then use the rank test (H) defined by the scores

$$a_H(i) = \begin{cases} -[(N+1)/4] & \text{if } i < (N+1)/4 \\ i - [(N+1)/2] & \text{if } (N+1)/4 \leq i \leq 3(N+1)/4 \\ [(N+1)/4] & \text{if } i > 3(N+1)/4. \end{cases}$$

We first propose an adaptive two-sample distribution-free test having a deterministic approach using a fairly easy classification scheme. We use the data to assess the tailweight of the underlying distributions. An interesting measure based on quantiles, used in Crow and Siddiqui (1967), is given by

$$TW = \frac{F^{-1}(1-\beta_1) - F^{-1}(\beta_1)}{F^{-1}(1-\beta_2) - F^{-1}(\beta_2)},$$

where $0 < \beta_1 < \beta_2 < 0.5$, with

$$F^{-1}(u) = inf\{x : F(x) \geq u\}, 0 < u < 1.$$

Their choices for β_1 and β_2 are 0.025 and 0.25, respectively. Henceforth, we write $\hat{F}_n(x)$ to represent the empirical d.f. based on a sample of size n and $\hat{F}_n^{-1}(u)$ to represent the corresponding quantile of order u, $0 < u < 1$. Then the statistic

$$\widehat{TW} = \frac{\hat{F}_N^{-1}(0.975) - \hat{F}_N^{-1}(0.025)}{\hat{F}_N^{-1}(0.75) - \hat{F}_N^{-1}(0.25)},$$

is used to determine whether the tailweight of the underlying distribution is heavy or light. The proposed deterministic approach (AD3) will accordingly use the following classification scheme. We compute $\widehat{TW}$ using the combined sample of all N observations. If $\widehat{TW} > c_2$, we use the H test, if $c_1 \leq \widehat{TW} \leq c_2$, we decide to use the W test and finally if $\widehat{TW} < c_1$, we use the L test. We choose that value of c_1 and c_2 using the simulation study for which the AD3 test seems to be the most powerful. The choice $c_1 = 2.2$ and $c_2 = 5.3$ is found to be the best in terms of power of the test.

The probabilistic approach is based on p-values calculated from some preliminary tests of tailweights. In the probabilistic approach we again make use of the same measure of tailweight. We obtain the tailweight measures for the two samples as

$$\widehat{TW}_k = \frac{\hat{F}_{n_k}^{-1}(0.975) - \hat{F}_{n_k}^{-1}(0.025)}{\hat{F}_{n_k}^{-1}(0.75) - \hat{F}_{n_k}^{-1}(0.25)}, \quad k = 1, 2,$$

where $\hat{F}_{n_1}^{-1}$ and $\hat{F}_{n_2}^{-1}$ denote, respectively, the sample quantiles based on X and Y samples. Now we have to conduct some preliminary tests. For this we frame the following testing problems. To test the null hypothesis of light tailed model we take

TW = 1.9, the TW value for the uniform distribution and consider the problem of testing

$$H_{01} : TW = 1.9$$

against

$$H_{11} : TW > 1.9.$$

To test the null hypothesis regarding medium tailed model we take TW = 3.33, the TW value for the logistic distribution and set the testing problem here as

$$H_{02} : TW = 3.33$$

against

$$H_{12} : TW > 3.33.$$

We obtain the tailweight measures for the two samples and combine them to obtain the test statistic for the preliminary tests as

$$\widehat{TW}_c = \frac{n_1}{N}\,\widehat{TW}_1 + \frac{n_2}{N}\,\widehat{TW}_2.$$

The preliminary tests are then based on the following two asymptotically normally distributed statistics

$$U = \frac{\sqrt{N}}{\hat{\sigma}_{TW}}\,(\widehat{TW}_c - 1.9),$$

and

$$V = \frac{\sqrt{N}}{\hat{\sigma}_{TW}}\,(\widehat{TW}_c - 3.33),$$

where $\hat{\sigma}_{TW}$ is a consistent estimator of the asymptotic variance of $\widehat{TW}_c$.

We now introduce the proposed adaptive rule AD4. Let p_u^+ denote the p-value corresponding to an observed U for testing H_{01} against H_{11} i.e. $p_u^+ = P_{H_{01}}(U \geq u)$ and p_v^+ denote the p-value corresponding to an observed V for testing H_{02} against H_{12} i.e. $p_v^+ = P_{H_{02}}(V \geq v)$. Define $\pi_1 = p_u^+$, $\pi_2 = p_v^+(1 - p_u^+)$ and $\pi_3 = (1 - p_u^+)(1 - p_v^+)$.

The adaptive rule is: Reject H_0 with probability π_1 if $A_L > A_L(\alpha, n_1, n_2)$, with probability π_2 if $A_W > A_W(\alpha, n_1, n_2)$ and with probability π_3 if $A_H > A_H(\alpha, n_1, n_2)$, where $A_L(\alpha, n_1, n_2)$, $A_W(\alpha, n_1, n_2)$ and $A_H(\alpha, n_1, n_2)$ are the upper α-critical values for L, W and H tests, respectively.

***Chapter 4*: Modified Adaptive Nonparametric Tests for Two-Sample Location Problem Under Symmetry:**

The purpose of this chapter is to propose two modifications of the adaptive probabilistic test, proposed in the previous chapter, from two different viewpoints and then suggest an adaptive procedure combining the two ideas. Note that, the power of the AD4 test converges to the asymptotic power of the best component under a sequence of local alternatives excepting the two boundary cases, viz., TW = 1.9 and TW = 3.33. We now propose a modified adaptive procedure whose power converges to the power of the best component at all points under a sequence of local alternatives, including the above mentioned boundary cases.

In addition to the null hypotheses, *viz.*, H_{01} and H_{02}, discussed in the previous chapter, we consider the null hypothesis regarding heavy tailed model by taking TW = 4.32, the TW value for the Laplace distribution and set the testing problem here as

$$H_{03} : TW = 4.32$$

against

$$H_{13} : TW < 4.32,$$

and hence the following asymptotically normally distributed statistic

$$D = \frac{\sqrt{N}}{\hat{\sigma}_{TW}} (\widehat{TW}_c - 4.32).$$

We find the level actually attained or p-value, at the observed values u, v and d for U, V and D respectively. We then write

$$p_u^+ = P_{H_{01}}(U \geq u), p_v^+ = P_{H_{02}}(V \geq v), p_d^- = P_{H_{03}}(D \leq d)$$

for the respective p-values, and

$$p_u = min(2p_u^+, 1), p_v = min(2p_v^+, 1), p_d = min(2p_d^-, 1)$$

for the modified p-values. After getting the p-values or modified p-values, we find the triple (π_1, π_2, π_3), called classification probabilities, such that

$$0 < \pi_1, \pi_2, \pi_3 < 1 \text{ and } \pi_1 + \pi_2 + \pi_3 = 1.$$

We can write our adaptive test statistic as

$$AD = L.I(U^* < \pi_1) + W.I(\pi_1 \leq U^* \leq \pi_1 + \pi_2) + H.I(U^* > \pi_1 + \pi_2),$$

where U^* is uniformly distributed over (0,1) and is independent of $\{X_1, X_2, ..., X_{n_1}, Y_1, Y_2, ..., Y_{n_2}\}$. A possible choice of $\pi = (\pi_1, \pi_2, \pi_3)$ is given by $\pi_1 = p_u(1 - p_d), \pi_2 = (1 - p_u)(1 - p_d), \pi_3 = p_d$, for which all π values reach either 0 or 1 at all TW values when the sample size becomes large.

Again $p_u^+ = 0.05$ and $p_d^- = 0.05$ can be treated as complete dilemma. Thus the π values should be defined on the basis of the probabilities $P_u^+ = k_1(p_u^+)$ and $P_d^- = k_2(p_d^-)$, where $k_i, i = 1, 2$, are real valued functions satisfying (a)k_i is monotone, non-decreasing, (b)$k_i(0) = 0$, (c)$k_i(0.05) = 0.5$ and (d)$k_i(1) = 1$. The use of beta distribution with indices $(1, 13.513406)$ fits our present situation. π values are then defined as $\pi_1 = P_u^+(1 - P_d^-), \pi_2 = (1 - P_u^+)(1 - P_d^-), \pi_3 = P_d^-$.

Combining the above two modifications we define the probabilities $P_u = k_3(p_u)$ and $P_d = k_4(p_d)$, where $k_i, i = 3, 4$, are real valued functions satisfying (a)k_i is monotone, non-decreasing, (b)$k_i(0) = 0$, (c)$k_i(0.1) = 0.5$ and (d)$k_i(1) = 1$. The use of beta distribution with indices (0.30103, 1) fits the present situation. The corresponding π values are defined as $\pi_1 = P_u(1 - P_d), \pi_2 = (1 - P_u)(1 - P_d), \pi_3 = P_d$.

***Chapter 5*: Adaptive Nonparametric Tests for Generalized Behrens-Fisher Problem:**

Let $X_1, X_2, ..., X_{n_1}$ and $Y_1, Y_2, ..., Y_{n_2}$ be independent random samples corresponding to the populations with continuous d.f.'s F(x) and G(y), respectively. Let θ_X and θ_Y denote, respectively, the unique medians of the X and Y populations. The problem considered here is to test

$$H_0 : \theta_X = \theta_Y$$

against a suitable composite alternative. For simplicity, we consider the one-sided alternative

$$H_1 : \theta_X < \theta_Y.$$

We consider the modified Wilcoxon-Mann-Whitney statistic (T_U), suggested by Fligner and Policello(1981), and the modified Mood's median test (T_M), due to Fligner and Rust (1982), for this generalized Behrens-Fisher problem.

For the proposed deterministic approach we use the following measure of symmetry

$$Q(x) = (\bar{U}_\gamma(x) - \bar{M}_{0.5}(x))/(\bar{M}_{0.5}(x) - \bar{L}_\gamma(x)),$$

$$Q(y) = (\bar{U}_\gamma(y) - \bar{M}_{0.5}(y))/(\bar{M}_{0.5}(y) - \bar{L}_\gamma(y)),$$

where $\bar{U}_\gamma(x), \bar{M}_\gamma(x), \bar{L}_\gamma(x)$ denote, respectively, the $\gamma.n_1$ largest, middle and smallest order statistics corresponding to the X sample, and $\bar{U}_\gamma(y)$, $\bar{M}_\gamma(y)$, $\bar{L}_\gamma(y)$ denote that of the Y sample. Here we take $\gamma = 0.10$. When the data indicate that both the populations are not symmetric, i.e. at least one of Q(x) and Q(y) does not belong to some interval J, use the T_M test, otherwise use the T_U test. Different choices of the interval J are examined and J = (0.5, 2.3) is found from the simulation studies to be the best choice in terms of the robustness and the power of the adaptive procedure.

For the probabilistic approach we again consider the triples test proposed by Randles *et al.* (1980). Let p_{1t} and p_{2t} denote, respectively, the p-values corresponding to observed $\hat{\eta}$-values *viz.*, $\hat{\eta}_1$ and $\hat{\eta}_2$, for the X and Y samples. Whenever p_{1t} and p_{2t} are observed, perform a Bernoullian trial with probability of success $p_{0t} = min(p_{1t}, p_{2t})$. If success occurs, use the T_U test; otherwise, use the T_M test.

Here again $p_{0t} = \alpha = 0.05$ can be treated as complete dilemma. Thus we can modify the proposed probabilistic approach by performing the random experiment with probability of success $p^*_{0t} = min(p^*_{1t}, p^*_{2t})$, where $p^*_{1t} = k_5(p_{1t})$, $p^*_{2t} = k_5(p_{2t})$ with $k_5(\cdot)$ is a real-valued function satisfying the following conditions: i) k_5 is monotone, non-decreasing, ii) $k_5(0) = 0$, iii) $k_5(0.05) = 0.5$ and iv) $k_5(1) = 1$. The test rule will remain same as in the probabilistic approach with p_{0t} replaced by p^*_{0t}.

Chapter 6: **Adaptive Nonparametric Tests for Two-Sample Scale Problem Under Symmetry:**

Let $X_1, X_2, ..., X_{n_1}$ and $Y_1, Y_2, ..., Y_{n_2}$ be two independent random samples from populations with continuous distribution functions (d.f.s) $F(x - \theta_X)$ and $F((x - \theta_Y)/\tau)$, respectively, where θ_X and θ_Y are the location parameters, and τ is the scale parameter. We consider the problem of testing

$$H_0 : \tau = 1$$

against some composite alternative.

For testing the two-sample scale problem when the location parameters are equal to some common but unknown value we consider the statistic proposed by Fligner and Killeen (1976). Suppose $\widetilde{M} = \widetilde{M}(X_1, ..., X_{n_1}, Y_1, ..., Y_{n_2})$ be the combined sample median. Define $V_i = |X_i - \widetilde{M}|$ for i = 1, 2, ..., n_1, $W_j = |Y_j - \widetilde{M}|$ for j = 1, 2, ..., n_2, and let R_j be the rank of W_j in the combined sample $\{V_1, V_2, ..., V_{n_1}; W_1, W_2, ..., W_{n_2}\}$ of N = $n_1 + n_2$ observations. Then the statistic

$$S_1^+ = \frac{1}{N+1}\sum_{j=1}^{n_2} R_j$$

is an analog of the Ansari-Bradley statistic.

For the general two-sample scale problem, when no assumption is made regarding the equality of the medians of the two symmetric populations, we consider a modification of the procedure of Fligner and Killeen (1976). Let $\widetilde{M}_1$ denote the median of the sample $X_1, X_2, ..., X_{n_1}$, and $\widetilde{M}_2$ denote that of the sample $Y_1, Y_2, ..., Y_{n_2}$. Define $V_i^* = |X_i - \widetilde{M}_1|$ for i = 1, 2, ..., n_1 and $W_j^* = |Y_j - \widetilde{M}_2|$ for j = 1, 2, ..., n_2. Denote by R_j^* the rank of W_j^* in the combined sample. Here again we select the analog of the Ansari-Bradley statistic and denote the corresponding test statistic by

$$S_2^+ = \frac{1}{N+1}\sum_{j=1}^{n_2} R_j^*.$$

The null hypothesis for the preliminary test is that the two populations have equal median against the alternative that the medians are not equal with fewer assumptions on the shapes of the populations. For this we consider the modification of the Wilcoxon-Mann-Whitney statistic. Denote such modification by $\hat{T}_U$. Let p_{MW} denote the p-value corresponding to an observed $\hat{T}_U$. Our adaptive rule is: Reject H_0 with probability p_{MW} if $S_1^+ > s^+_{1(\alpha;n_1,n_2)}$ and with probability (1-p_{MW}) if $S_2^+ > s^+_{2(\alpha;n_1,n_2)}$, where $s^+_{1(\alpha;n_1,n_2)}$ and $s^+_{2(\alpha;n_1,n_2)}$ are the upper α-critical values for the S_1^+ and the S_2^+ tests respectively. We may require randomization to get the exact size α for each of the tests.

The proposed deterministic rule is defined as follows:

(I) When the data indicate that the two populations have the same median, use the S_1^+ test *i.e.* use the S_1^+ test if $|\hat{T}_U| \leq \tau_{\alpha/2}$;

(II) When the data indicate that the two populations have different location parameter use the S_2^+ test *i.e.* use the S_2^+ test if $|\hat{T}_U| > \tau_{\alpha/2}$.

***Chapter 7*: Two-Sample Adaptive Nonparametric Tests Under Progressive Censoring:**

Let $X_1, X_2, ..., X_{n_1}$ be a random sample from a continuous distribution with distribution function (d.f.) F(x), where n_1 is a fixed positive integer and $Y_1, Y_2, ...$ be a sequence of independent observations drawn one-by-one from another continuous distribution with d.f. G(y). For the two-sample location problem, F(x) corresponds to any continuous distribution and the second population is simply shifted to the d.f. $G(x) = F(x - \theta), -\infty < \theta < \infty$. Here we assume that $F(x) + F(-x) = 1$ for all x. Then the objective is to test

$$H_0 : \theta = 0$$

against some composite alternative.

The associated test statistics are simply

$$A_{T,k} = \sum_{j=1}^{k} a_T(R_j), \quad T = W, L, H,$$

where R_j is the rank of Y_j in the combined sample $(X_1, X_2, ...X_{n_1}, Y_1, Y_2, ..., Y_k)$. We are primarily interested in the sequence

$$\{A_{T,k} : n_0 \leq k \leq n_2\}, n_0 \geq 1,$$

where n_0 denotes the initial sample size with which the statistical monitoring procedure is installed on the experiment and n_2 is a pre-fixed upper bound of the second sample size. If at any stage the accumulated statistical evidence indicates a clear decision in favor of the alternative hypothesis, experimentation is terminated at that stage along with the rejection of H_0; otherwise, we continue up to the target sample size n_2.

Let us denote by

$$\mu_{T,l} = \frac{l}{n_1 + l} \sum_{i=1}^{n_1+l} a_T(i) \quad \text{and} \quad \sigma^2_{T,n_2} = \frac{n_1 n_2}{N(N-1)} \sum_{i=1}^{N} (a_T(i) - \bar{a}_T)^2$$

with $N\bar{a}_T = \sum_{i=1}^{N} a_T(i)$, $N = n_1+n_2$ for T = W, L and H. We consider here Kolmogorov-Smirnov type statistics defined by

$$S_{T,k} = \max_{n_0 \le l \le k} \left(\frac{A_{T,l} - \mu_{T,l}}{\sigma_{T,n_2}} \right) \quad \text{or} \quad \max_{n_0 \le l \le k} \left| \frac{A_{T,l} - \mu_{T,l}}{\sigma_{T,n_2}} \right|$$

for $n_0 \le k \le n_2$.

At the first stage we use the observed historical data of length n_1 to assess the tailweight of the underlying distributions. We consider the same preliminary tests as considered in Chapters 3 and 4 and obtain the p-values and the modified p-values as in Chapter 4. Then we set our adaptive test rule as follows: Perform a random experiment having three possible outcomes with probabilities π_1, π_2 and π_3. Note that, by definition, for every $\alpha \in (0, 1)$, there exists an $S_{T,n_2}(\alpha)$ such that

$$P_{H_0}\{S_{T,n_2} > S_{T,n_2}(\alpha)\} \le \alpha \le P_{H_0}\{S_{T,n_2} \ge S_{T,n_2}(\alpha)\},$$

where α is the desired level of significance. Reject H_0 after i drawings with probability π_1, π_2 and π_3 if

$$S_{T,k} \le S_{T,n_2}(\alpha),\ k = n_0, n_0+1, ..., i-1,\quad S_{T,i} > S_{T,n_2}(\alpha),\ n_0 \le i \le n_2, \qquad (A)$$

for T=L, W and H, respectively. Accept H_0 if no such 'i' exists.

We also suggest a possible way of extending the proposed procedure for the two-sample location problem without the assumption of symmetry. Then, in addition to the rank tests already considered, we take into account two more linear rank statistics.

The median test (M) is effective in detecting a shift if the observations appear to be from a heavy tailed distribution. If the data indicate a model which is skewed to the right, then we use the rank test (RS) due to Hogg-Fisher-Randles.

Here again we consider the triples test statistic (T) proposed by Randles *et al.* (1980) to test the null hypothesis that the underlying population is symmetric against the alternative that it is asymmetric. Let p_t denote the p-value corresponding to an observed T. The modified p-value is obtained as $P_t = k_8(p_t)$ where k_8 is a real-valued function satisfying the following conditions: i) k_8 is monotone, non-decreasing, ii) $k_8(0) = 0$, iii) $k_8(0.05) = 0.5$ and iv) $k_8(1) = 1$. Define the classification probabilities as $\Pi_1 = P_t P_u(1 - P_d)$, $\Pi_2 = P_t(1 - P_u)(1 - P_d)$, $\Pi_3 = P_t P_d$, $\Pi_4 = (1 - P_t)(1 - P_d)$ and $\Pi_5 = (1 - P_t)P_d$. Whenever P_u, P_d and P_t are observed, perform a random experiment having five possible outcomes with probabilities $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ and Π_5, where $\sum_{i=1}^{5} \Pi_i = 1$. Reject H_0 after i drawings with probability $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ and Π_5 if (A) is satisfied for T=L, W, H, RS and M, respectively. Accept H_0 if no such 'i' exists.

In all these chapters we illustrate the proposed adaptive procedures using real data sets. Extensive simulation study is presented in each chapter to compare the relative performance of the proposed adaptive procedures with the existing non-adaptive tests. We also discuss some relevant asymptotic properties of the proposed tests, which include the asymptotic null distribution of the standardized forms of the adaptive test statistics and the asymptotic power of the proposed adaptive test procedures under sequence of local alternatives.

1.5 Conclusions

Here we like to make the following observations about our work in the present dissertation.

1.5.1 In Chapter 2, we have considered adaptive nonparametric tests for the single sample location problem without any assumption regarding the skewness of the underlying distribution. It may be possible to extend this idea to develop adaptive test procedures for the bivariate single sample location problem as well. We would like to propose some adaptive procedures which allow the more powerful bivariate signed rank test to be used for fairly diagonally symmetric distributions, but otherwise prescribes the bivariate sign test. We like to proceed on this in future.

1.5.2 In Chapter 2, it is discussed that the selector statistic for the deterministic approach is based on a very simple measure of symmetry which may be easily affected by the presence of outliers in the data. We can consider some better measures of symmetry for the deterministic approach. One such alternative measure may be

$$Q_m = \sum_{j=1}^{m} \frac{|X_{(n+1-j)} - 2\widetilde{X} + X_{(j)}|}{X_{(n+1-j)} - X_{(j)}},$$

where $m = [n/2]$.

1.5.3 In Chapter 6, it is assumed that the underlying populations are symmetric. In some situations this assumption is not satisfied or questionable. Hence an adaptive test for the general two-sample scale problem should be developed. Presently, we have left it as future work.

1.5.4 It has already been pointed out that adaptive tests should be carefully constructed in order to maintain their desired level of significance. In this regard note that we do not consider any monotonic non-decreasing transformation of the p-values on (0,1) in Chapters 2 and 6 since in these cases adaptive tests based on such transformed p-values become anti-conservative.

1.5.5 Regarding publications it should be mentioned that a major part of Chapter 2 is published in **Statistical Methodology (2007)**; an Elsevier Journal. The matter of Chapter 6 is accepted in **Statistical Methods and Applications**, a journal from Springer group. Rest of this dissertation is being communicated to suitable journals.