Chapter 3: 
Approaches and Models for 
Credit Risk Management
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This Chapter reviews the different approaches and models for Credit Risk management.

3.1 Different Approaches for Credit Risk Modeling

There are two primary types of models in the literature that attempt to describe default processes for debt obligations and other defaultable financial instruments, usually referred to as structural and reduced-form (or intensity) models.

3.1.1 Structural Approach

Structural models use the evolution of firms’ structural variables, such as asset and debt values, to determine the time of default. Merton’s model (1974) was the first modern model of default and is considered the first structural model. In Merton’s model, a firm defaults if, at the time of servicing the debt, its assets are below its outstanding debt. A second approach, within the structural framework, was introduced by Black and Cox (1976). In this approach defaults occur as soon as firm’s asset value falls below a certain threshold. In contrast to the Merton approach, default can occur at any time.

Reduced form models do not consider the relation between default and firm value in an explicit manner. Intensity models represent the
most extended type of reduced form models. In contrast to structural models, the time of default in intensity models is not determined via the value of the firm, but it is the first jump of an exogenously given jump process. The parameters governing the default hazard rate are inferred from market data.

Structural default models provide a link between the credit quality of a firm and the firm’s economic and financial conditions. Thus, defaults are endogenously generated within the model instead of exogenously given as in the reduced approach.

Another difference between the two approaches refers to the treatment of recovery rates: whereas reduced models exogenously specify recovery rates, in structural models the value of the firm’s assets and liabilities at default will determine recovery rates.

The structural literature on credit risk starts with the paper by Merton (1974), who applies the option pricing theory developed by Black and Scholes (1973) to the modeling of a firm’s debt. In Merton’s model, the firm’s capital structure is assumed to be composed by equity and a zero-coupon bond with maturity T and face value of D. The firm’s equity is simply a European call option with maturity T and strike price D on the asset value and, therefore, the firm’s debt value is just the asset value minus the equity value. This approach assumes a very simple and unrealistic capital structure and implies that default can only happen at the maturity of the zero-coupon bond.

The paper by Black and Cox (1976) is the first of the so-called First Passage Models (FPM). First passage models specify default as the first time the firm’s asset value hits a lower barrier, allowing default to take place at any time. When the default barrier is exogenously fixed, as in Black and Cox (1976) and Longstaff and Schwartz (1995), it acts as a safety covenant to protect bondholders. Alternatively it can be endogenously fixed as a result of the stockholders’ attempt to
choose the default threshold which maximizes the value of the firm (cf. Leland 1994 and Leland and Toft 1996.)


In First Passage Models, by definition, default occurs the first time the asset value goes below a certain lower threshold, i.e. the firm is liquidated immediately after the default event. In contrast with First Passage Models, a new set of models is being put forward, supported by recent theoretical and empirical research, where a default event does not immediately cause liquidation but it represents the beginning of a process, the liquidation process, which might or might not cause liquidation after it is completed. This practice is consistent, for example, with the US Bankruptcy Law, where firms filling for bankruptcy are granted a court-supervised grace period (up to several years) aimed at sorting out their financial problems in order to, if possible, avoid liquidation. Those models are labeled as Liquidation Process Models (LPM).

State Dependent Models (SDM) represents, together with LPM, two recent efforts to incorporate into structural models different real-life phenomena. Although theoretically they make good sense, they lack empirical research testing its performance.

SDM assume that some of the parameters governing the firm’s ability to generate cash flows or its funding costs are state dependent, where states can represent the business cycle (recession vs. expansion) or the firm’s external rating.

I. Merton’s Model

Merton (1974) makes use of the Black and Scholes (1973) option pricing model to value corporate liabilities. This is a straightforward
application only if we adapt the firm’s capital structure and the
default assumptions to the requirements of the Black-Scholes model.
Let us assume that the capital structure of the firm is comprised by
equity and by a zero-coupon bond with maturity $T$ and face value of
$D$, whose values at time $t$ are denoted by $E_t$ and $z(t, T)$ respectively,
for $0 \leq t \leq T$. The firm’s asset value $V_t$ is simply the sum of equity
and debt values. Under these assumptions, equity represents a call
option on the firm’s assets with maturity $T$ and strike price of $D$. If at
maturity $T$ the firm’s asset value $V_T$ is enough to pay back the face
value of the debt $D$, the firm does not default and shareholders
receive $V_T - D$. Otherwise ($V_T < D$) the firm defaults, bondholders
take control of the firm, and shareholders receive nothing. Implicit in
this argument is the fact that the firm can only default at time $T$. This
assumption is important to be able to treat the firm’s equity as a
vanilla European call option, and therefore apply the Black-Scholes
pricing formula.

The rest of assumptions Merton (1974) adopts are the inexistence
of transaction costs, bankruptcy costs, taxes or problems with
indivisibilities of assets; continuous time trading; unrestricted
borrowing and lending at a constant interest rate $r$; no restrictions on
the short selling of the assets; the value of the firm is invariant under
changes in its capital structure (Modigliani-Miller Theorem) and that
the firm’s asset value follows a diffusion process.

The firm’s asset value is assumed to follow a diffusion process given
by

$$dV_t = rV_t dt + \sigma_V V dW_t,$$

Where $\sigma_V$ is the (relative) asset volatility and $W_t$ is a Brownian
motion.

The payoffs to equity holders and bondholders at time $T$ under the
assumptions of this model are respectively, $\max \{V_T - D, 0\}$ and
$V_T - E_T$, i.e.

$$E_T = \max\{V_T - D, 0\},$$
\[ z(T,T) = V_T - E_T. \] (3)

Applying the Black-Scholes pricing formula, the value of equity at time \( t \) \( (0 \leq t \leq T) \) is given by

\[ E_t(V, \sigma_V, T-t) = e^{-r(T-t)} \left[ e^{r(T-t)} V_t \Phi(d_1) - D \Phi(d_2) \right] \] (4)

Where \( \Phi(\cdot) \) is the distribution function of a standard normal random variable and \( d_1 \) and \( d_2 \) are given by

\[ d_1 = \frac{\ln \left( \frac{e^{r(T-t)} V_t}{D} \right) + \frac{1}{2} \sigma_V^2 (T-t)}{\sigma_V \sqrt{T-t}}, \] (5)

\[ d_2 = d_1 - \sigma_V \sqrt{T-t}. \] (6)

The probability of default at time \( T \) is given by

\[ P \left[ V_T < D \right] = \Phi(-d_2). \] (7)

Therefore, the value of the debt at time \( t \) is \( z(t, T) = V_t - E_t \).

In order to implement Merton’s model, the firm’s asset value \( V_t \) and its volatility \( \sigma_V \) must be estimated (both unobservable processes), and the debt structure of the firm into a zero-coupon bond with maturity \( T \) and face value \( D \) should be transformed.

The maturity \( T \) of the zero-coupon bond can be chosen either to represent the maturity structure of the debt, for example as the Macaulay duration of all the liabilities, or simply as a required time horizon.

The main advantage of Merton’s model is that it allows to directly apply the theory of European options pricing developed by Black and Scholes (1973). But to do so the model needs to make the necessary assumptions to adapt the dynamics of the firm’s asset value process, interest rates, and capital structure to the requirements of the Black-Scholes model. There is a trade off between realistic assumptions and ease of implementation and Merton’s model opts for the latter one. All extensions to this model introduce more realistic assumptions.
trying to end up with a model not too difficult to implement and with closed, or at least numerically feasible, solutions for the expressions of the debt value and the default probabilities. Merton himself (Merton 1974) presents some extensions to the model, in order to account for coupon bonds, callable bonds, stochastic interest rates, and relaxing the assumption that the Modigliani-Miller Theorem holds.

One problem of Merton’s model is the restriction of default time to the maturity of the debt, ruling out the possibility of an early default, no matter what happens with the firm’s value before the maturity of the debt. If the firm’s value falls down to minimal levels before the maturity of the debt but is able to recover and meet the debt’s payment at maturity, the default would be avoided in Merton’s approach.

Another handicap of the model is that the usual capital structure of a firm is much more complicated than a simple zero-coupon bond. Geske (1977, 1979) considers the debt structure of the firm as a coupon bond, in which each coupon payment is viewed as a compound option and a possible cause of default. At each coupon payment, the shareholders have the option either to make the payment to bondholders, obtaining the right to control the firm until the next coupon, or not to make the payment, in which case the firm defaults. Geske also extends the model to consider characteristics such as sinking funds, safety covenants, debt subordination, and payout restrictions.

The assumption of a constant and flat term structure of interest rates is other major criticism the model has received. Jones et al. (1984) suggest that there exists evidence that introducing stochastic interest rates, as well as taxes, would improve the model’s performance.” Stochastic interest rates allow to introduce correlation between the firm’s asset value and the short rate, and have been considered, among others, by Ronn and Verma (1986), Kim, Ramaswamy and Sundaresan (1993), Nielsen et al. (1993), Longstaff

Another characteristic of Merton's model, which will also be present in some of the FPM, is the predictability of default. Since the firm's asset value is modeled as a geometric Brownian motion and default can only happen at the maturity of the debt, it can be predicted with increasing precision as the maturity of the debt comes near. As a result, in this approach default does not come as a surprise, which makes the models generate very low short-term credit spreads. As it reviewed, introducing jumps in the process followed by the asset value has been one of the solutions considered to this problem.

Delianedis and Geske (2001) study the proportion of the credit spread that, in a corporate bond data set, is explained by default risk, using the Merton (1974) and Geske (1977) frameworks. They conclude that it only explains a small fraction of the credit spreads; the rest is attributable to taxes, jumps, and liquidity and market risk factors. They also include a jump component in the Merton model finding that "while jumps may explain a portion of the residual spread, it is unlikely that jumps can explain it entirely”.

II. First passage model

First Passage Models (FPM) were introduced by Black and Cox (1976) extending the Merton model to the case when the firm may default at any time, not only at the maturity date of the debt.

Consider, as in the previous section, that the dynamics of the firm's asset value under the risk neutral probability measure $P$ are given by the diffusion process.

$$dV_t = rV_t dt + \sigma V_t dW_t$$

And that there exists a lower level of the asset value such that the firm defaults once it reaches this level. Although Black and Cox (1976) considered a time dependent default threshold, let us assume first a constant default threshold $K > 0$. If we are at time $t \geq 0$ and
default has not been triggered yet and $V_t > K$, then the time of default $\tau$ is given by

$$\tau = \inf\{s \geq t \mid V_s \leq K\}$$

(9)

Using the properties of the Brownian motion $W_t$, in particular the reflection principle, we can infer the default probability from time $t$ to time $T$:

$$P[\tau \leq T \mid \tau \geq t] = \Phi(h_1) + \exp\left\{2\left(r - \frac{\sigma^2}{2}\right) \ln\left(\frac{K}{V_t}\right) - \frac{1}{2}\frac{\sigma^2}{V}\right\} \Phi(h_2)$$

(10)

Where

$$h_1 = \frac{\ln\left(\frac{K}{e^{r(T-t)V_t}}\right) + \frac{\sigma^2}{2} (T-t)}{\sigma\sqrt{T-t}}$$

(11)

$$h_2 = h_1 - \sigma\sqrt{T-t}$$

(12)

FPM have been extended to account for stochastic interest rates, bankruptcy costs, taxes, debt subordination, strategic default, time dependent and stochastic default barrier, jumps in the asset value process, etc. Although these extensions introduce more realism into the model, they increment its analytical complexity.

The default threshold, always positive, can be interpreted in various ways. We can think of it as a safety covenant of the firm’s debt which allows the bondholders to take control of the company once its asset value has reached this level. The safety covenant would act as a protection mechanism for the bondholders against an unsatisfactory corporate performance. In this case, the default threshold would be deterministic, although possibly time dependent, and exogenously fixed when the firm’s debt is issued. Kim, Ramaswamy and Sundaresan (1993) and Longstaff and Schwartz (1995) assume an exogenously given constant default threshold K. Black and Cox
(1976) consider a time dependent default barrier given by $e^{-\gamma(T-t)}K$. A particular case of Black and Cox default threshold specification is to consider $\gamma = r$, i.e. to consider a default barrier equal to the face value of the debt discounted at the risk-free interest rate. In that case, the default threshold can be made stochastic if the model considers a stochastic process for the interest rate, as in Briys and de Varenne (1997). Longstaff and Schwartz (1995) choose a constant default threshold and point out that “since it is the ratio of $V_t$ to $K$, rather than the actual value of $K$, that plays the major role in our analysis, allowing a more general specification for $K$ to simply make the model more complex without providing additional insight into the valuation of risky debt.”

Hsu, Saa-Requejo and Santa-Clara (2004) suggest that $V_t$ and $K$ do not matter directly to the valuation of default risky bonds but only through their ratio, which is a measure of the solvency of the firm. They model the default threshold as a stochastic process, which together with the stochastic process assumed for the firm’s asset value, allows them to obtain the stochastic process of the ratio $V_t/K$. The dynamics of the ratio $V_t/K$ are used to price corporate bonds.

The default threshold can also be chosen endogenously by the stockholders to maximize the value of the equity. See for example Mello and Parsons (1992), Nielsen et al. (1993), Leland (1994), Anderson and Sundaresan (1996), Leland and Toft (1996), Mella-Barral and Perraudin (1997), and François and Morellec (2004).

The literature has also considered the possibility of negotiation processes between stockholders and bondholders when the firm goes near the point of financial distress, from which the default threshold is determined.

Similarly as how we described the choice of the face-value of the zero-coupon in the Merton model, in FPM the default threshold can be calculated as a weighted average of short and long-term debts.
Interest rates can be considered either as a constant or as a stochastic process. The stochasticity of interest rates allows the model to introduce correlation between asset value and interest rates, and to make the default threshold stochastic, in the cases it is specified as the discounted value of the face value of the debt. Nielsen et al. (1993) and Longstaff and Schwartz (1995) consider a Vasicek process for the interest rate, correlated with the firms’ asset value:

\[ dV_t = (c - d) V_t dt + \sigma V_t dW_t, \]

\[ dr_t = (a - b r_t) dt + \sigma \sqrt{r_t} d\tilde{W}_t, \]

\[ d\tilde{W}_t dW_t = pdt. \]

Where \( W_t \) and \( \tilde{W}_t \) are correlated Brownian motions. Other specifications for the stochastic process of the short rate have been considered. For example Kim, Ramaswamy and Sundaresan (1993) suggest a CIR process

\[ dr_t = (a - b r_t) dt + \sigma \sqrt{r_t} dW_t, \]

and Briys and de Varenne (1997) a generalized Vasicek process

\[ dr_t = (a(t) - b(t) r_t) dt + \sigma(t) dW_t. \]

Hsu, Saá-Requejo and Santa-Clara (2004) consider both the case of independence between risk-free interest rates and the default generating mechanism (given by the dynamics of the ratio \( V_t/K_t \)) and the case of correlation between both processes, specifying the risk-free rate as a CIR process. They present an interesting empirical illustration of the model, covering the calibration of the risk-free rate process and the estimation of the model’s parameter through the Generalized Method of Moments.
The principal drawback of FPM is the analytical complexity that they introduce, which is increased if we consider stochastic interest rates or endogenous default thresholds.

This mathematical complexity makes it difficult to obtain closed form expressions for the value of the firm’s equity and debt, or even for the default probability, forcing us to make use of numerical procedures.

The empirical testing of FPM and structural models in general has not been very successful. Eom, Helwege and Huang (2003), who carry out an empirical analysis of five models (Merton, Geske, Leland and Toft, Longstaff and Schwartz, and Collin-Dufresne and Goldstein), conclude that: “Using estimates from the implementations we consider most realistic, we agree that the five structural bond pricing models do not accurately price corporate bonds. However, the difficulties are not limited to the under prediction of spreads. ... they all share the same problem of inaccuracy, as each has a dramatic dispersion of predicted spreads.”

Zhou (1997) indicates that “the empirical application of a diffusion approach has yielded very disappointing results.” Another drawback of the structural models presented before is the so-called predictability of defaults. Generally, structural models consider continuous diffusion processes for the firm’s asset value and complete information about asset value and default threshold. In this setting, the actual distance from the asset value to the default threshold tells us the nearness of default, in such a way that if we are far away from default the probability of default in the short-term is close to zero, because the asset value process needs time to reach the default point. The knowledge of the distance of default and the fact that the asset value follows a continuous diffusion process makes default a predictable event, i.e. default does not come as a surprise.

This predictability of defaults makes the models generate short-term credit spreads close to zero. In contrast, it is observed in the
market that even short-term credit spreads are bounded from below, incorporating the possibility of an unexpected default or deterioration in the firm’s credit quality.

The same characteristics of the structural models that imply the predictability of default also imply predictability of recovery. In models which do not consider strategic defaults, the bondholders get the remaining value of the firm in case of default, which is precisely the value of the default threshold at default. Thus, if it is assumed complete information about asset value and default threshold, the recovery rate is also a predictable quantity.

Essentially, two ways out of these predictability effects of structural models have been proposed in the literature. The predictability of default comes from the assumption of investors’ perfect knowledge of the firm’s asset value and default threshold.

In practice, it is not possible to deduce from the capital structure of the firm neither the value of the firm \( V_t \), its volatility \( \sigma V \), nor the level of the default threshold. If we consider incomplete information about either the firm value process, the default threshold (or both), investors can only infer a distribution function for these processes, which makes defaults impossible to predict. These considerations can be found, among others, in Duffie and Lando (2001), Giesecke (2005) and Jarrow and Protter (2004).

The second way consists of incorporating jumps in the dynamics of the firm value, which implies that the asset value of the firm can suddenly drop, reducing drastically the distance of default (between asset value and default threshold), or even causing a default if the drop is sufficiently high. Thus, default is not a predictable event any more, the default probabilities for short maturities do not tend to zero and so the credit spreads generated. Zhou (1997, 2001a) and Hilberink and Rogers (2002) deal with structural models in which the firm’s asset value incorporates a jump component. While Zhou extends Longstaff and Schwartz (1995) model considering a
lognormally distributed jump component, Hilberink and Rogers (2002) opt for an extension of Leland (1994) and Leland and Toft (1996) using Levy processes which only allow for downward jumps in the firm’s value. Both models avoid the problem of default predictability implying positive credit spreads for short maturities. Another characteristic of jump models is that they convert the recovery payment at default in a random variable, since the value of the firm can drop suddenly below the default threshold, whereas if the firm’s value follows a diffusion process without jumps, the value of the firm at default, i.e. what bondholders get is always equal to the default threshold because of the continuity of the firm’s value path.

Fouque, Sircar and Solna (2005) consider the effect of introducing stochastic volatility in FPM, finding that it increases short-term spreads.

Davydenko (2005) criticizes existing structural models because they obviate liquidity reasons as the main determinants of default for some firms, particularly the ones with high external financing costs:

"Several default triggers have been proposed in structural models of debt pricing. Most models assume that a firm defaults when the market value of its assets falls below a certain boundary (Black and Cox, 1976; Leland, 1994). This default boundary may correspond to an exogenous net-worth covenant, or to the endogenously determined threshold at which equity holders are no longer willing to service debt obligations. Should the firm find itself in a liquidity crisis while its asset value is still above the boundary, equity holders in these models will always be willing and able to avoid default by raising outside financing. This approach contrasts with the assumption that firms default when current assets fall short of current obligations, due to either a minimum cash-flow covenant, or market frictions precluding the firm from raising sufficient new external financing (Kim et al., 1993; Anderson and Sundaresan, 1996). Models incorporating both value- and liquidity-based defaults
are rare, and little empirical evidence is available to motivate the choice of the default trigger. If, in reality, default is triggered by different factors for different firms, existing models are likely to lack accuracy in predictions.”

Davydenko (2005), using a sample of US (speculative rating-grade) bond issuers from 1996 to 2003, shows that the importance of liquidity shortages in triggering default for a particular firm depends on the firm’s cost of external financing:

“Firms with low costs of external financing default when the continuation value of assets is low. By contrast, if external funds are costly, a liquidity crisis may force reorganization even if the going-concern surplus is still substantial.” Moreover, the author presents empirical evidence against the view that default is triggered when the asset value crosses a particular threshold.

Therefore, empirical evidence suggests that structural models need to be theoretically extended in order to incorporate the possibility of the firms defaulting because of liquidity shortages and high funding costs.

III. Estimation and Calibration

The literature provides several ways of calibrating $V_t$ and $\sigma_V$. The first method makes use of Itô’s Lemma to obtain a system of two equations in which the only two unknown variables are $V_t$ and $\sigma_V$. Assume the firm’s equity value follows a geometric Brownian motion under $P$, with volatility $\sigma_E$:

$$dE_t = rE_t dt + \sigma_E E_t dW_t$$

(18)

Since the value of the equity is a function of time and of the value of the assets, $E_t = f(V_t, t)$, we can apply Itô’s Lemma to get

$$dE_t = \left[ \frac{\partial f(V_t, t)}{\partial t} + \frac{\partial f(V_t, t)}{\partial V_t} V_t r + \frac{1}{2} \frac{\partial^2 f(V_t, t)}{\partial V_t^2} (V_t \delta_V)^2 \right] dt + \frac{\partial f(V_t, t)}{\partial V_t} V_t \delta_V dW_t.$$  

(19)
Comparing the coefficients multiplying the Brownian motion in the two previous equations, the following identity are obtained

$$\sigma_t \Delta E_t = \frac{\delta}{\delta \gamma_t} F(V_t, t) V_t \delta \gamma_t$$  \hspace{1cm} (20)

Noting that $$\frac{\delta}{\delta \gamma_t} F(V_t, t) = \frac{\delta E_t}{\delta \gamma_t} = \Phi(d_1)$$ and rearranging the first equation of the system are obtained:

$$\sigma_{\gamma} = \frac{E_t}{V_t} \sigma_E \Phi(d_1).$$  \hspace{1cm} (21)

The second equation results simply from matching the theoretical value of equity with the observed market price ($$E_t$$):

$$E_t(V_t, \sigma_{\gamma}, T - t) = \hat{E}_t.$$  \hspace{1cm} (22)

As it mentioned before, the only two unknowns in the system formed by the last two equations are $$V_t$$ and $$\sigma_{\gamma}$$.

Duan (1994) points out some drawbacks of the previous method. First, the method considers the equity volatility as constant and independent of the firm’s asset value and time. Second, he claims that the first equation is redundant since it is used to derive the second equation. And third, the traditional method does not provide us with distribution functions, or even confidence intervals, for the estimates of $$V_t$$ and $$\sigma_{\gamma}$$.

Duan (1994) proposes another method of estimating $$V_t$$ and $$\sigma_{\gamma}$$, based on maximum likelihood estimation using equity prices and the one-to-one relationship between equity and asset levels given by (4). Duan et al. (2003) follow the maximum likelihood approach introduced by Duan (1994) but, unlike previous works, they take into account the survivorship issue, by incorporating into the likelihood function the fact that a firm survived. They argue that “In the credit risk setting, it is imperative for analysts to recognize the fact that a firm in operation has by definition survived so far. Estimating a credit
risk model using the sample of equity prices needs to reflect this reality, or runs the risk of biasing the estimator.”

Duan and Fulop (2005) extend Duan’s (1994) maximum likelihood estimation method to account for the fact that observed equity prices might be contaminated by trading noises. They find that taking into account trading noises generates lower estimates for the asset volatility $\sigma_V$ and therefore overestimates the firms’ default probabilities.

Bruche (2005) describes how structural models can be estimated using a simulated maximum likelihood procedure, which allows to use data on any of the firm’s traded claims (bonds, equity, CDS, ...) as well as balance sheet information to improve the efficiency of the estimation. The paper explores the possibility of considering that not only equity, but the rest of the claims used in the estimation procedure can be priced with noise, showing that “even small amounts of noise can have serious consequences for estimation results when they are ignored.”

A different way of estimating $V_t$ and $\sigma_V$, which can be found on Jones et al. (1984), consists simply of estimating the asset value as the sum of the equity market value, the market value of traded debt and the estimated value of non-traded debt.

Provided with a time series for $V_t$ we can estimate its volatility $\sigma_V$.

Hull, Nelken and White (2004) propose a way to estimate the model’s parameters from implied volatilities of options on the company’s equity, avoiding to estimate $\sigma_E$ and to transform the firm’s debt structure into a zero-coupon bond. Using as inputs two equity implied volatilities and an estimate of the firm’s debt maturity $T$, their model provides us with an estimate of $\sigma_V$ and the leverage ratio $\frac{De^{-r(T-t)}}{V_t}$, which allows us to calculate $E_t$ and the probability of default.

To calculate the value of the debt $z(t,T)=V_t-E_t$ it is still needed an estimate for $V_t$. 

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We still have to estimate the default threshold $K$. Sundaram (2001) indicates that "default tends to occur in practice when the market value of the firm's assets drops below a critical point that typically lies below the book value of all liabilities, but above the book value of short-term liabilities." Thus, one approach is to choose a value for $D$ between those two limits. Davydenko (2005) estimates the default threshold to be around 72% of the firm's face value of debt.

IV. Liquidation Process Model

In FPM default occurs the first time the asset value goes below a certain lower threshold, i.e. the firm is liquidated immediately after the default event; the default event corresponding to the crossing of the asset value through the lower barrier. In contrast with FPM, a new set of models has emerged where the default event does not immediately cause liquidation but it represents the beginning of a process, the liquidation process, which might or might not cause liquidation after it is completed. We label those models Liquidation Process Models LPM.

The distinction between the terms default event and liquidation must be clear to understand LPM and their differences with FPM. A default event takes place when the firm's asset value $V_t$ goes below the lower threshold $K$ (which can be exogenous, constant, time dependent, stochastic or endogenously derived). A default event signals the beginning of a financially distressed period, which will not necessarily lead to liquidation. Liquidation takes place when the firm is actually liquidated, its activity stops and its remaining distributed among its claimholders.

In FPM described above the default event does coincide with liquidation. However, as pointed out by Couderc and Renault (2005), most liquidations "do not arise suddenly but are rather the conclusion of a long lasting process." As pointed out by Moraux (2004): Empirical studies in USA have found that additional 'survival' periods
beyond the main default event last up to 3 years (Altman-Eberhart (1994), Betker (1995), Hotchkiss (1995)). Helwege (1999) reports that the longest default of modern US junk bond market is seven years long.

The fact that the liquidation process can take quite a while implies that when empirically studying the causes of liquidation past information shows up as a significant explanatory variable, together, of course, with contemporaneous information, because it comprises information about the liquidation process. Information here referring to the firms’ financial variables as well as financial markets, business cycle, credit markets and default cycle indicators. Couderc and Renault (2005) use a database containing the rating history of over ten thousand firms for the period 1981-2003 and analyze, using duration models, whether past values of several financial markets (business cycle, credit markets and default cycle) are relevant in explaining default probabilities in addition to their contemporaneous values. Their results show the critical importance of past information in default probabilities.

LPM extend FPM to account for the fact that the liquidation time takes place after (sometimes quite a lot after) the occurrence of a default event.

François and Morellec (2004), Moraux (2004), and Galai, Raviv and Wiener (2005) put forward (the first to the best of our knowledge) theoretical LPM.

François and Morellec (2004) argue that while in most of FPM the default event leads to an immediate liquidation of the firm’s assets, firms in financial distress have several options to deal with their distress. First, under Chapter 7 of the US Bankruptcy Code, they can liquidate its assets straight away. This possibility would fit FPM. However firms can also file for bankruptcy under Chapter 11 of the US bankruptcy Code and start a court-supervised liquidation process. The authors refer to existing literature to provide some evidence
about the relevance of Chapter 11: “Upon default, the court grants
the firm a period of observation during which the firm can renegotiate
its claims. At the end of this period, the court decides whether the
firm continues as a going concern or not. Empirical studies show that
most firms emerge from Chapter 11. Only a few firms (5%, according
to Gilson, John, and Lang [1990] and Weiss [1990], and between
15% and 25%, according to Morse and Shaw [1988]) are eventually
liquidated under Chapter 7 after filing Chapter 11. Why do some
firms recover while others do not? It is generally acknowledged that
there exist two types of defaulting firms. First, firms that
economically sound promptly recover under Chapter 11. Default was
only due to a temporary financial distress. Second, firms that are
economically unsound keep on losing value under Chapter 11.”

François and Morellec consider that, after a default event, i.e. after
the asset value $V_t$ goes below the lower threshold $K$, a firm is
liquidated if and only if $V_t$ remains below $K$ consecutively during a
period of time of a given length $d$ (which in their numerical
simulations they take to be 2 years). If a default event happens and
the asset value remains under the lower threshold for a period lower
than $d$ the liquidation process finishes and the firm continues in
business as usual. The term consecutively in the definition of
liquidation above means that the number of successfully managed
past default events and liquidation periods the firms have
experienced does not affect the maximum length $d$ of future
liquidation periods.

The authors provide closed-form solutions for corporate debt and
equity values and analyze the implications of the model for optimal
leverage and credit spreads. Numerical simulations show that credit
spreads are an increasing function of the length $d$.

including an additional cause of liquidation to François and Morellec’s
one (which they call liquidation procedure A). Under his proposed
liquidation procedure, procedure B, liquidation happens when the total, i.e. cumulative, time the firm assets value stands under the default threshold exceeds d. The difference between procedures A and B lies in the words consecutively and cumulative, and Moraux (2004) explains it clearly: “Under the procedure A, each time the firm value process passes through and above K, the liquidation procedure is closed and the hypothetical distress counter is set to zero. The next time a default event occurs, an identical procedure is run and an equal period of time d is granted. ... Under the procedure B, the distress counter is never set to zero. Subsequent granted periods (and therefore tolerance) will be lower and lower as more default events and long financial distress will be observed. In fact, the granted time is lowered (each time) by the duration just used.”

Financial distress refers to the situation in which \( V_t < K \). A firm can be liquidated by either one or the other liquidation procedures. Moraux (2004) shows that any liquidation procedure based on the time spent by the firm in financial distress is bounded by the procedures A and B in the sense that its implied liquidation date will be higher (lower) than the liquidation date implied by procedure B (A).

The author derives closed form solutions for different claims such as equity, different seniority debts and convertible debt. In particular, the value of equity is derived as a Down and Out Parisian option written on the firm assets under liquidation procedure A and as a Down and Out cumulative call option under liquidation procedure B. Numerical simulations show that the value of equity is an increasing function of d, and that, unlike in François and Morellec (2004), credit spreads increase or decrease with d depending on the seniority of the debt.

Galai, Raviv and Wiener (2005) represent a step forward in the refinement of LPM, proposing a model extending and including the two previous ones. Galai, Raviv and Wiener argue that in the two
previous models, the only thing that matters for a firm to be liquidated is the amount of time it spends in financial distress (either successively or cumulatively), but they fail to “capture the following two common features of bankruptcy procedures: (i) Recent distress events may have a greater effect on the decision to liquidate a firm’s assets than old distress events. ... (ii) Severe distress events may have greater effect on the decision to liquidate a firm than mild distress events.” To account for such two stylized facts, the authors propose a structural model in which a firm is liquidated when a state variable representing the cumulative weighted time period spent by the firm in distress exceeds d. At each time, the cumulative weighted time period is computed as a weighted average of the total time spent by the firm in distress, weighted by (i) how far away in the past such distress occurred and (ii) how severe was such a distress, where distress severity is measured as an increasing function of $\max \{0, K - V \}$.

Galai, Raviv and Wiener’s model have as special cases models such as Merton (1974), Black and Cox (1976), Leland (1994), Fan and Sundaresan (2001), François and Morellec (2004) and Moraux (2004). As a consequence it represents the more general LPM so far. The authors solve the model numerically using Monte Carlo simulation based on Parisian options and Parisian contracts techniques to value debt and equity. They provide a very intuitive comparison of the liquidation mechanics in their general model with François and Morellec’s and Moraux’s ones, showing that Moraux’s cumulative liquidation procedure (B) have too strong memory because far away distress periods have the same impact on liquidation triggering than current ones.

V. State dependent model

Another avenue for (so far) theoretical research within the structural approach consists of extending standard models with regime switching: some of the model parameters are state-
contingent. As we review below, states can represent the state of the business cycle or simply the firm’s external rating. Cash-flows, bankruptcy costs and funding costs might be state-dependent.

This branch of structural models is able to reduce the problems of predictability of defaults (and recovery) suffered by standard models because the firm is subject to exogenous changes of parameters which affect its ability to generate cash flows or its funding costs, which are the main drivers of default probabilities.

Hackbarth, Miao and Morellec (2004) and Elizalde (2005) put forward two different models illustrating the previous ideas. In both cases the authors provide closed form expressions for the value of equity and debt, whose solution imply solving systems of ordinary differential equations.

In Hackbarth, Miao and Morellec (2004) cash flows and recovery rates depend on the state of the business cycle. Cash flows $x_t$ follow a geometric Brownian motion and are scaled by a business cycle scalar factor: they are higher in expansions $yHx_t$ than in recession’s $yLx_t$, $yH > yL$. In the same way, bankruptcy costs are expressed as a state-dependent fraction $1 - \alpha$ of the firm’s assets; again, the recovery rate in expansions $\alpha_H$ is higher than in recessions $\alpha_L$, $\alpha_H > \alpha_L$. At each point in time, there is an exogenous probability of switching between recession and expansion. The default threshold is endogenously chosen by equity holders to maximize the value of equity, and it turns out to be higher in recessions: the firm defaults earlier in recessions than in expansions. Numerical examples illustrate the implications of the model for default thresholds, default clustering, optimal leverage (countercyclical) and credit spreads. As argued above the model is able to generate non-trivial short term spreads.

Elizalde (2005) develops a structural model which, although originally applied to banks, can be extended to any firm. In contrast with previous models, the firms’ asset value is assumed to be
unobserved by debt holders. Debt holders rely on the ratings published by rating agencies to set the debt’s coupon as a function of those ratings. As a consequence, the firms’ funding costs are contingent on their ratings. Rating agencies perform timely audits to firms, with a given frequency, to find out their risk and asset levels, which determine the rating. Switching from one rating to another implies changes in the cost of debt and, as a consequence, in the ability of the firm to repay it. As in Hackbarth, Miao and Morellec (2004) the default threshold is chosen endogenously by equity holders and it is rating-dependent.

As described by Duffie (2005), “It has become increasingly common for bond issuers to link the size of the coupon rate on their debt with their credit rating, offering a higher coupon rate at lower ratings, perhaps in an attempt to appeal to investors based on some degree of hedging against a decline in credit quality. This embedded derivative is called a ‘ratings-based step-up.’” The author illustrates an example of a ratings-based step-up bond issued by Deutsche Telecom in 2002 with coupon payments linked to the firm’s rating. While Elizalde (2005) derives the price of such a bond using a structural model, Duffie provides its pricing formula using an intensity model.

Like LPM, State Dependent Models (SDM) have only been developed theoretically and their future success in credit risk modeling (if any) lies in their empirical applicability and their ability to replicate and predict credit spreads and default probabilities.

VI. Cyclical Default Correlation

The most natural way to introduce default dependences between firms in structural models is by correlating the firms’ asset processes. Suppose we have $i = 1, \ldots, I$ different firms with asset value processes given by

$$dV_{i,t} = rV_{i,t} \, dt + \sigma V_{i,t} \, dW_{i,t},$$

(23)
For \( i = 1, \ldots, I \), where \( W_{t,i}, \ldots, W_{t,t} \) are correlated Brownian motions. As in the single firm case, these models imply predictable defaults. One way of getting rid of the default predictability would be to introduce jump components in the firms’ asset processes. Those jump components could be either correlated or uncorrelated across firms. Correlated jump components, besides making defaults unpredictable, would also account for credit risk contagion effects. The main problem lies in the calibration of those jump components.

**VII. Contagion default correlation**

Cyclical default correlation does not account for all the credit risk dependence between firms. Giesecke (2004) and Giesecke and Goldberg (2004) extend structural models for default correlation to incorporate credit risk contagion effects. The default of one firm can trigger the default of related firms. Furthermore, default times tend to concentrate in some periods of time in which the probability of default of all firms is increased and which can not be totally, or even partially, explained by the firms’ common dependence on some macroeconomic factors.

Contagion effects can arise in this setting by direct links between firms in terms of, for example, commercial or financial relationships. The news about the default of one firm have a big impact on the credit quality of other related firms, which is immediately, reflected in their default probabilities.

In structural FPM we assume that investors have complete information about both asset processes and default thresholds, so they always know the nearness of default for each firm, i.e. the distance between the actual level of the firm’s assets and its default threshold.

Giesecke (2004) and Giesecke and Goldberg (2004) introduce contagion effects in the model by relaxing the assumption that investors have complete information about the default thresholds of
the firms. In Giesecke (2004), bond holders do not have perfect information neither about such thresholds nor about their joint distribution. However, they form a prior distribution which is updated at any time one of such thresholds is revealed, which only happens when the corresponding firm defaults. In Giesecke (2004) investors have incomplete information about the firms’ default thresholds but complete information about their asset processes. Giesecke and Goldberg (2005) extend that framework to one in which investors do not have information neither about the firms’ asset values nor about their default thresholds. In this case, default correlation is introduced through correlated asset processes and, again, investors receive information about the firms’ asset and default barrier only when they default. Such information is used to update their priors about the distribution of the remaining firms’ asset values and default thresholds.

The incomplete information about the level of the default thresholds and the fact that those levels are dependent among firms (through a copula function) generate the source of credit risk contagion. Investors form a belief about the level of the firms’ default thresholds. Each time one of the firms defaults, the true level of its default threshold is revealed, and investors use this new information to update their beliefs about the default thresholds of the rest of the firms. This sudden updating of the investors’ perceptions about the default thresholds of the firm, and thus about the nearness of default for each firm, introduces the default contagion effects in the models.

This model allows the introduction of default correlation both through dependences between firms’ asset values, cyclical default correlations, and through dependences between firms’ default barriers, contagion effects.

The major problem of this approach is to calibrate and estimate the default threshold copula. See Giesecke (2003) for some remarks on how to choose and calibrate that copula.
VIII. **Factor mode**

Factor models consider the firms' asset values as a function of a group of common factors, which introduce the default correlation in the model, plus a firm's specific factor:

\[ V_{i,t} = \sum_{j=1}^{J} w_{i,j} Z_{j,t} + \varepsilon_{i,t}, \]  

(23)

Where \( Z_{1}, \ldots, Z_{J} \) represent the common factors, \( \varepsilon_{1}, \ldots, \varepsilon_{I} \) the firms' specific factors (independent of \( Z_{1}, \ldots, Z_{J} \)), and the correlation structure is given by the coefficients \( w \). Once we know the realization of the common factors, the firms' asset value and thus the firms' default probabilities are independent.

The calibration of factor models is usually carried out by a logit or probit regression, depending on the assumptions about the distribution of the factors. Schönbucher (2000), Finger (1999), and Frey, McNeil and Nyfeler (2001) present illustrations of these models.

### 3.1.2 Intensity (Reduced-Form) Approach

This part analyzes reduced-form credit risk models, and reviews the three main approaches to incorporate credit risk correlation among firms within the framework of reduced models. There are three distinguished approaches to model default correlation in the literature of intensity credit risk modeling.

The first approach, *conditionally independent defaults (CID) models*, introduces credit risk dependence of the firms' default intensity processes on a common set of state variables. *Contagion models* extend the CID approach to account for default clustering (periods in which the firms' credit risk is increased and in which the majority of the defaults take place). Finally, default dependencies can also be accounted for using *copula functions*. The copula approach takes as given the marginal default probabilities of the different firms and
I. Conditionally Independent Default model

This approach introduces correlation in the firms’ default intensities making them dependent on a set of common variables $X_t$ and on a firm specific factor. These models have received the name of conditionally independent defaults (CID) models, because conditioned to the realization of the state variables $X_t$ the firm’s default intensities are independent as are the default times that they generate. Apparently, the main drawback of these models is that they do not generate sufficiently high default correlations. However, Yu (2002) indicates that this is not a problem of the model per se, but rather an indication of the lack of sophistication in the choice of the state variables.

Two direct extensions of the CID approach try to introduce more default correlation in the models. One is the possibility of joint jumps in the default intensities (Duffie and Singleton 1999b) and the other is the possibility of default-event triggers that cause joint defaults (Duffie and Singleton 1999b, Kijima 2000, and Kijima and Muromachi 2000).

From now on, we consider $i = 1, \ldots, I$ different firms and denote by $\lambda_{i,t}$ and $T_i$ their default intensities and default times respectively.

In CID models, firms’ default intensities are independent once we fix the realization of the state variables $X_t$. The default correlation is introduced through the dependence of each firm’s intensity on the random vector $X_t$. A firm specific factor of stochasticity $\lambda_{i,t}^*$, independent across firms, completes the specification of each firm’s Default intensity:

$$\lambda_{i,t} = a_{0,i} + a_{1,i} X_{1,t} + \ldots + a_{J,i} X_{J,t} + \lambda_{i,t}^*,$$  \hspace{1cm} (25)
Where \( a_j, \lambda_i \) are some deterministic coefficients, for \( j=1, \ldots, J \) and \( i=1, \ldots, I \).

Since default times are continuously distributed, this specification implies that the probability of having two or more simultaneous defaults is zero.

Now consider an example of a CID model based on Duffee (1999). The default free interest rate is given by

\[
    r_t = a_0 + X_{1,t} + X_{2,t},
\]

(26)

Where \( a_0 \) is a constant coefficient, and \( X_{1,t} \) and \( X_{2,t} \) are two latent factors (unobservable, interpreted as the slope and level of the default-free yield curve). After having estimated the latent factors \( X_{1,t} \) and \( X_{2,t} \) from default-free bond data, Duffee (1999) uses them to model the intensity process of each firm \( i \) as

\[
    \lambda_{i,t} = a_{0,i} + a_{1,i} (X_{1,t} - \bar{X}_1) + a_{2,i} (X_{2,t} - \bar{X}_2) + \lambda^*_{i,t},
\]

(27)

\[
    d\lambda^*_{i,t} = \kappa_i + (\theta_i - \lambda^*_{i,t}) dt + \sigma_i \sqrt{\lambda^*_{i,t}} dW_{i,t}.
\]

(28)

where \( W_{1,t}, \ldots, W_{I,t} \) are independent Brownian motions, \( a_{0,i}, a_{1,i}, a_{2,i} \) and \( \kappa_i \) are constant coefficients and \( \bar{X}_1 \) and \( \bar{X}_2 \) are the sample means of \( X_{1,t} \) and \( X_{2,t} \).

The intensity of each firm \( i \) depends on the two common latent factors \( X_{1,t} \) and \( X_{2,t} \), and on an idiosyncratic factor \( \lambda^*_{i,t} \), independent across firms. The coefficients \( a_{0,i}, a_{1,i}, a_{2,i}, \kappa_i, \theta_i \) and \( \sigma_i \) are different for each firm. In Duffee’s model \( \lambda^*_{i,t} \) captures the stochasticity of intensities and the coefficients \( a_{i,j} \) and \( a_{2,i} \), \( i = 1, \ldots, I \), capture the correlations between intensities themselves, and between intensities and interest rates.

Duffee (1999), Zhang (2003), Driessen (2005), and Elizalde (2005b) propose, and estimate, different CID models.

The literature on credit risk correlation has criticized the CID approach, arguing that it generates low levels of default correlation.
when compared with empirical default correlations. However, Yu (2002a) suggests that this apparent low correlation is not a problem of the approach itself but a problem of the choice of state or latent variables, owing to the inability of a limited set of state variables to fully capture the dynamics of changes in default intensities. In order to achieve the level of correlation seen in empirical data, a CID model must include among the state variables, the evolution of the stock market, corporate and default-free bond markets, as well as various industry factors.

According to Yu, the problem of low correlation in Duffee’s model may arise because of the insufficient specification of the common factor structure, which may not capture all the sources of common variation in the model, leaving them to the idiosyncratic component, which in turn would not be independent across firms. In fact, Duffee finds that idiosyncratic factors are statistically significant and correlated across firms. As long as the firms’ credit risks depend on common factors different from the interest rate factors, Duffee’s specification is not able to capture all the correlation between firms’ default probabilities. Xie, Wu and Shi (2004) estimate Duffee’s model for a sample of US corporate bonds and perform a careful analysis of the model pricing errors. A principal component analysis reveals that the first factor explains more than 90% of the variation of pricing errors. Regressing bond pricing errors with respect to several macroeconomic variables, they find that returns on S&P 500 index explain around 30% of their variations. Therefore, Duffee’s model leaves out some important aggregate factors that affect all bonds.

Driessen (2005) proposes a model in which the firms’ hazard rate is a linear function of two common factors, two factors derived from the term structure of interest rates, a firm idiosyncratic factor, and a liquidity factor. Yu also examines the model of Driessen (2005), finding that the inclusion of two new common factors elevates the default correlation.
Finally, Elizalde (2005b) shows that any firm’s credit risk is, to a very large extent, driven by common risk factors affecting all firms. The paper decomposes the credit risk of a sample of corporate bonds (14 US firms, 2001-2003) into different unobservable risk factors. A single common factor accounts for more than 50% of all (but two) of the firms’ credit risk levels, with an average of 68% across firms. Such factor represents the credit risk levels underlying the US economy and is strongly correlated with main US stock indexes. When three common factors are considered (two of them coming from the term structure of interest rates) the model explains an average of 72% of the firms’ credit risk.

In the CID approach, to simulate default times we proceed as we did in the single entity case. Once we know the realization of the state variables \( X_t \), we simulate a set of \( I \) independent unit exponential random variables \( \eta_I, ..., \eta_I \), which are also independent of \( (G_{X,t}) \). The default time of each firm \( i = 1, ..., I \) is defined by

\[
\tau_i = \inf\left\{ t > 0 \mid \int_0^t \lambda_{i,s} \, ds \geq \eta_i \right\}. \tag{29}
\]

Thus, once we have simulated \( \eta_i \), \( \tau_i \) will be such that

\[
\int_0^{\tau_i} \lambda_{i,s} \, ds = \eta_i \tag{30}
\]

Duffie and Singleton (1999) proposed two ways out of the low correlation problem.

One is the possibility of joint jumps in the default intensities, and the other is the possibility of default-event triggers that cause joint defaults.

Duffie and Singleton develop an approach in which firms experience correlated jumps in their default intensities. Assume that the default intensity of each firm follows the following process:

\[
d\lambda_{i,t} = \kappa_i (\theta_i - \lambda_{i,t}) \, dt + dq_{i,t}, \tag{31}
\]
which consists of a deterministic mean reversion process plus a pure jump process \( q_{i,t} \) whose intensity of arrival is distributed as a Poisson random variable with parameter \( \gamma_i \) and whose jump size follows an exponential random variable with mean \( \mu \) (equal for all firms \( i = 1, \ldots, I \)). Duffie and Singleton introduce correlation to the firm's jump processes, keeping unchanged the characteristics of the individual intensities. They postulate that each firm's jump component consists of two kinds of jumps, joint jumps and idiosyncratic jumps. The joint jump process has a Poisson intensity \( \gamma_c \) and an exponentially distributed size with mean \( \mu \). Individual default intensities experience a joint jump with probability \( p_i \). That is, a firm suffers a joint jump with Poisson intensity of arrival of \( p_i \gamma_c \). In order to keep the total jump in each firm's default intensity with intensity of arrival \( \gamma_i \) and size \( \mu_i \), the idiosyncratic jump (independent across firms) is set to have an exponentially distributed size \( \mu_i \) and intensity of arrival \( h_i \), such that \( \gamma_i = p_i \gamma_c + h_i \).

Note that if \( p_i = 0 \) the jumps are only idiosyncratic jumps, implying that default intensities and hence default times are independent across firms. If \( p_i = 1 \) and \( h_i = 0 \) all firms have the same jump intensity, which does not mean that default times are perfectly correlated, since the size of the jump is independent across firms. Only if we additionally assume that \( \mu \) goes to infinity do we obtain identical default times.

The second alternative considers the possibility of simultaneous defaults triggered by common credit events, at which several obligors can default with positive probability. Imagine there exist \( m = 1, \ldots, M \) common credit events, each one modeled as a Poisson process with intensity \( \lambda^c m,t \). Given the occurrence of a credit event \( m \) at time \( t \), each firm \( i \) defaults with probability \( p_{i,m,t} \). If, given the occurrence of a common shock, the firm's default probability is less than one; this common shock is called non-fatal shock, whereas if this probability is one, the common shock is called fatal shock. In addition to the
common credit events, each entity can experience default through an
idiosyncratic Poisson process with intensity \( \lambda^*_{i,t} \), which is independent
Across firms. Therefore, the total intensity of firm \( i \) is given by

\[
\lambda_{i,t} = \lambda^*_{i,t} + \sum_{m=1}^{M} \phi_{i,m,t} \lambda^c_{m,t}.
\]  

(32)

Consider a simplified version of this setting with two firms, constant
idiosyncratic intensities \( \lambda^*_1 \) and \( \lambda^*_2 \), and one common and fatal
event with constant intensity \( \lambda^c \). In this case firm \( i \)'s survival probability is
given by

\[
S_i(t,T) = \exp\left(-\left(\lambda^*_i + \lambda^c\right)(T - t)\right)
\]  

(33)

Denoting by \( s(t; T_1, T_2) \) the joint survival probability, given no
default until time \( t \), that firm 1 does not default before time \( T_1 \) and
firm 2 does not default before time \( T_2 \), then

\[
s(t; T_1, T_2) = \exp\left(-\lambda^*_1(T_1 - t) - \lambda^*_2(T_2 - t) - \lambda^c \max\{T_1 - t, T_2 - t\}\right) = \exp\left(-\left(\lambda^*_1 + \lambda^c\right)(T_1 - t) - \left(\lambda^*_2 + \lambda^c\right)(T_2 - t) + \lambda^c \min\{T_1 - t, T_2 - t\}\right),
\]  

(34)

Which can be expressed as

\[
S(t; T_1, T_2) = S_1(t, T)S_2(t, T)\min\{\exp\left(\lambda^c (T_1 - t)\right), \exp\left(\lambda^c (T_2 - t)\right)\}.
\]  

(35)

This expression for the joint survival probability explicitly includes
individual survival probabilities and a term which introduces the
dependence structure. This is the approach followed by copula
functions, which couple marginal probabilities into joint probabilities.
In fact, the above example is a special case of copula function, called
Marshall-Olkin copula.

The relationship between joint survival and default probabilities is
given by

\[
S(t; T_1, T_2) = 1 - p_1(t, T_1) - p_2(t, T_2) + p(t; T_1, T_2),
\]  

(36)

where \( p(t; T_1, T_2) \) represents the joint default probability, given no
default until time \( t \), that firm 1 defaults before time \( T_1 \) and firm 2
defaults before time \( T_2 \). Obviously the case with multiple common
shocks is more troublesome in terms of notation and calibration because, for every possible common credit event, an intensity must be specified and calibrated.

Duffie and Singleton (1999) propose algorithms to simulate default times within these two frameworks. The criticisms that the joint credit event approach has received stem from the fact that it is unrealistic that several firms default at exactly the same time, and also from the fact that after a common credit event that makes some obligors default, the intensity of other related obligors that do not default does not change at all.

Although theoretically appealing, the main drawback of these two last models has to be with their calibration and implementation. To the best of my knowledge there is not a single paper which carries out an empirical calibration and implementation of a model like the ones presented in this section. The same applies to the contagion models presented in the next section.

II. Contagion Mechanisms

Contagion models take CID models one step further, introducing into the model two empirical facts: that the default of one firm can trigger the default of other related firms and that default times tend to concentrate in certain periods of time, in which the default probability of all firms is increased. The last model examined in the previous section (joint credit events) differs from contagion mechanisms in that if an obligor does not experience a default, its intensity does not change due to the default of any related obligor. The literature of default contagion includes two approaches:

The infectious defaults model of Davis and Lo (1999), and the model proposed by Jarrow and Yu (2001), which we shall refer to as propensity model. The main issues to be resolved concerning these two models are associated with difficulties in their calibration to market prices.
The Davis and Lo model has two versions, a static version that only considers the number of defaults in a given time period, and a dynamic version in which the timing of default is also incorporated.

In the dynamic version of the model, each firm has an initial hazard rate of \( \lambda_{i,t} \), for \( i = 1, \ldots, I \), which can be constant, time dependent or follow a CID model. When a default occurs, the default intensity of all remaining firms is increased by a factor \( \alpha > 1 \), called enhancement factor, to \( \alpha \lambda_{i,t} \). This augmented intensity remains for an exponentially distributed period of time, after which the enhancement factor disappears (\( \alpha = 1 \)). During the period of augmented intensity, the default probabilities of all firms increase, reflecting the risk of default contagion.

In order to account for the clustering of default in specific periods, Jarrow and Yu (2001) extend CID models to account for counterparty risk, i.e. the risk that the default of a firm may increase the default probability of other firms with which it has commercial or financial relationships. This allows them to introduce extra-default dependence in CID models to account for default clustering. In a first attempt, Jarrow and Yu assume that the default intensity of a firm depends on the status (default/not default) of the rest of the firms, i.e. symmetric dependence. However, symmetric dependence introduces a circularity in the model, which they refer to as looping defaults, which makes it extremely difficult and troublesome to construct and derive the joint distribution of default times.

Jarrow and Yu restrict the structure of the model to avoid the problem of looping defaults. They distinguish between primary firms \( (1, \ldots, K) \) and secondary firms \( (K+1, \ldots, I) \). First, they derive the default intensity of primary firms, using a CID model.

The primary firm intensities \( \lambda_{1,t}, \ldots, \lambda_{K,t} \) are \( (G_X,t) \)-adapted and do not depend on the default status of any other firm. If a primary firm defaults, this increases the default intensities of secondary firms, but
not the other way around (asymmetric dependence). Thus, secondary firms’ default intensities are given by

\[ \lambda_{i,t} = \lambda_{i,1} + \sum_{j=1}^{K} a_{i,j} 1\{\tau_j \leq t\}, \]

for \( i = K + 1, \ldots, I \) and \( j = 1, \ldots, K \), where \( \lambda_{i,1} \) and \( a_{i,j} \) are \((G_{X,t})\)-adapted. \( \lambda_{i,1} \) represents the part of secondary firm \( i \)'s hazard rate independent of the default status of other firms.

Default intensities of primary firms \( \lambda_{1,t}, \ldots, \lambda_{K,t} \) are \((G_{X,t})\)-adapted, whereas default intensities of secondary firms \( \lambda_{K+1,t}, \ldots, \lambda_{I,t} \) are adapted with respect to the filtration \((G_{X,t}) V (G_{I,t}) V \ldots V (G_{K,t})\).

This model introduces a new source of default correlation between secondary firms, and also between primary and secondary firms, but it does not solve the drawbacks of low correlation between primary firms, which CID models apparently imply, because the setting for primary firms is, after all, only a CID model.

III. Copula Model

In CID and contagion models the specification of the individual intensities includes all the default dependence structure between firms. In contrast, the copula approach separates individual default probabilities from the credit risk dependence structure. The copula function takes as inputs the marginal probabilities and introduces the dependence structure to generate joint probabilities.

Copulas were introduced in 1959 and have been extensively applied to model, among others, survival data in areas such as actuarial science.

In the rest of this section we review copula theory and its use in the credit risk literature. To make notation simple, assume we are at time \( t = 0 \) and take \( s_i(t) \) and \( p_i(t) \) (or \( F_i(t) \)) to be the survival and default probabilities, respectively, of firm \( i = 1, \ldots, I \) from time 0 to time \( t > 0 \). Then:

\[ F_i(t) = P[\tau_i \leq t] = 1 - s_i(t) = 1 - P[\tau_i > t], \]

(38)
Where $T_i$ denotes the default time of firm $i$.

A copula function transforms marginal probabilities into joint probabilities. In case we model default times, the joint default probability is given by

$$F(t_1, \ldots, t_I) = P[t_1 \leq t_i \ldots \leq t_I] = C^d(F_1(t_1), \ldots, F_I(t_I)),$$

(39)

and if we model survival times, the joint survival probability takes the form

$$s(t_1, \ldots, t_I) = P[t_i > t_1 \ldots > t_I] = C^s(S_1(t_1), \ldots, S_I(t_I)),$$

(40)

Where $C^d$ and $C^s$ are two different copulas.

The copula function takes as inputs the marginal probabilities without considering how we have derived them. Thus, the intensity approach is not the only framework with which we can use copula functions to model the default dependence structure between firms. Any other approach to model marginal default probabilities, such as the structural approach, can use copula theory to model joint probabilities.

An intuitive definition of a copula function is as follows:

**Definition 1:** A function $C: [0, 1]^I \rightarrow [0, 1]$ is a copula if there are uniform random variables $U_1, \ldots, U_I$ taking values in $[0, 1]$ such that $C$ is their joint distribution function.

A copula function $C$ has uniform marginal distributions, i.e.

$$C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i,$$

(41)

For all $i = 1, \ldots, I$ and $u_i \in [0,1]$

This definition is used, for example, by Schönbucher (2003). The copula function $C$ is the joint distribution of a set of $I$ uniform random variables $U_1, \ldots, U_I$. Copula functions allow one to separate the modeling of the marginal distribution functions from the modeling of the dependence structure. The choice of the copula does not constrain the choice of the marginal distributions. Sklar (1959) showed that any multivariate distribution function $F$ can be written in the form of a copula function.
The following theorem is known as **Sklar's Theorem**:

Let $Y_1, \ldots, Y_I$ be random variables with marginal distribution functions $F_1, \ldots, F_I$ and joint distribution function $F$. Then there exists an $I$-dimensional copula $C$ such that $F(y_1, \ldots, y_I) = C(F_1(y_1), \ldots, F_I(y_I))$ for all $(y_1, \ldots, y_I)$ in $R^I$. Moreover, if each $F_i$ is continuous, then the copula $C$ is unique.

We shall consider the default times of each firm $T_1, \ldots, T_I$ as the marginal random variables whose joint distribution function will be determined by a copula function. If $Y$ is a random variable with distribution function $F$ then the random variable $U$, defined as $U=F(Y)$, is a uniform $[0, 1]$ random variable. Denoting by $t_i$ the realization of each $T_i$,

$$F(t_1, \ldots, t_I) = P[\tau_1 \leq t_1, \ldots, \tau_I \leq t_I] = C(F_1(t_1), \ldots, F_I(t_I)).$$ \hspace{1cm} (42)

The marginal distribution of the default time $T_i$ will be given by

$$F_i(t_i) = F(\infty, \ldots, \infty, t_i, \infty, \ldots, \infty) = P[\tau_i \leq \infty, \ldots, \tau_i \leq t_i, \ldots, \tau_I \leq \infty] = C(F_1(\infty), \ldots, F_i(t_i), \ldots, F_I(\infty)) = C(1, \ldots, F_i(t_i), \ldots, 1).$$ \hspace{1cm} (43)

In the bi-variate case, the relationship between the copula $C_d$ and the survival copula $C_s$, which satisfies $s(t_1, t_2) = C_s(s_1(t_1), s_2(t_2))$, is given by

$$C_s(u_1, u_2) = u_1 + u_2 - 1 + C_d(1-u_1, 1-u_2).$$ \hspace{1cm} (44)

Nelsen (1999) points out that $C_s$ is a copula and that it couples the joint survival function $s(\cdot, \ldots, \cdot)$ to its univariate margins $s_1(\cdot), \ldots, s_I(\cdot)$ in a manner completely analogous to the way in which a copula connects the joint distribution function $F(\cdot, \ldots, \cdot)$ to its margins $F_1(\cdot), \ldots, F_I(\cdot)$. When modeling credit risk using the copula framework we can specify a copula for either the default times or the survival times.

The dependence between the marginal distributions linked by a copula is characterized entirely by the choice of the copula.
If $C_1$ and $C_2$ are two 1-dimensional copula functions we say that $C_1$ is smaller than $C_2$, denoted by $C_1 \prec C_2$, if $C_1(u) \leq C_2(u)$ for all $u \in [0,1]^\prime$.

The Fréchet-Hoeffding copulas, $C^-$ and $C^+$, are two reference copulas given by

\begin{align}
C^- &= \max\{u_1 + \ldots + u_i + 1 - I,0\}, \\
C^+ &= \min\{u_1 \ldots u_i\},
\end{align}

Satisfying $C^- \prec C \prec C^+$ for any copula $C$. However, this is a partial ordering in the sense that not every pair of copulas can be compared in this way. In order to compare any two copulas, it would be interesting to find an index to measure the dependence structure between two random variables introduced by the choice of the copula function. Linear (Pearson) correlation coefficient $\rho$ is the most used measure of dependence, however it harbors several drawbacks which makes it not very suitable to compare copula functions. For example, linear correlation depends not only on the copula but also on the marginal distributions.

We focus on four dependence measures that depend only on the copula function, not in the marginal distributions: Kendall’s tau, Spearman’s rho and upper/lower tail dependence coefficients.

First, we introduce the concept of concordance:

**Definition 2**: Let $(y_1, y_2)$ and $(\tilde{y}_1, \tilde{y}_2)$ be two observations from a vector $(Y_1, Y_2)$ of continuous random variables. Then, $(y_1, y_2)$ and $(\tilde{y}_1, \tilde{y}_2)$ are said to be concordant if $(y_1 - \tilde{y}_1)(y_2 - \tilde{y}_2) > 0$ and discordant if $(y_1 - \tilde{y}_1)(y_2 - \tilde{y}_2) < 0$.

Kendall’s tau and Spearman’s rho are two measures of concordance:

**Definition 3**: Let $(Y_1, Y_2)$ and $(\tilde{Y}_1, \tilde{Y}_2)$ be i.i.d. random vectors of continuous random variables with the same joint distribution function given by the copula $C$ (and with marginal $F_1$ and $F_2$). Then, Kendall’s tau of the vector $(Y_1, Y_2)$ (and thus of the copula $C$) is defined as the probability of concordance minus the probability of discordance, i.e.
\[ \tau = P\left( (Y_1 - Y_1')(Y_2 - Y_2') > 0 \right) - P\left( (Y_1 - Y_1')(Y_2 - Y_2') < 0 \right) \quad (47) \]

**Definition 4:** Let \((Y_1, Y_2), (Y'_1, Y'_2)\) and \((Y''_1, Y''_2)\) be i.i.d. random vectors of continuous random variables with the same joint distribution function given by the copula \(C\) (and with marginal \(F_1\) and \(F_2\)). Then, Spearman’s rho of the vector \((Y_1, Y_2)\) (and thus of the copula \(C\)) is defined as

\[ \rho_s = 3 \left( P\left( (Y_1 - Y_1')(Y_2 - Y_2') > 0 \right) - P\left( (Y_1 - Y_1')(Y_2 - Y_2') < 0 \right) \right). \quad (48) \]

Both, Kendall’s tau and Spearman’s rho take values in the interval \([0, 1]\) and can be defined in terms of the copula function by

\[ \tau = 4 \int_{0,1} C(u,v) dC(u,v) \]

\[ \rho_s = 12 \int_{0,1} uvdC(u,v) - 3 = 12 \int_{0,1} C(u,v) dudv - 3. \quad (49) \]

The Fréchet-Hoeffding copulas take the two extreme values of Kendall’s tau and Spearman’s rho: if the copula of the vector \((Y_1, Y_2)\) is \(C^-\) then \(\tau = \rho_s = -1\), and if it has copula \(C^+\) then \(\tau = \rho_s = 1\). The product copula \(C^p\) represents independent random variables, i.e. if \(Y_1, \ldots, Y_1\) are independent random variables, their copula is given by \(C^p\), such that \(C^p(u_1, \ldots, u_1) = u_1 \ldots u_1\). For a vector \((Y_1, Y_2)\) of independent random variables \(\tau = \rho_s = 0\). Kendall’s tau and Spearman’s rho are equal for a given copula \(C\) and its associated survival copula \(C^s\).

Kendall’s tau and Spearman’s rho are measures of global dependence. In contrast, tail dependence coefficients between two random variables \((Y_1, Y_2)\) are local measures of dependence, as they refer to the level of dependence between extreme values, i.e. values at the tails of the distributions \(F_1(Y_1)\) and \(F_2(Y_2)\).

**Definition 5:** Let \((Y_1, Y_2)\) be a random vector of continuous random variables with copula \(C\) (and with marginal \(F_1\) and \(F_2\)). Then, the
coefficient of upper tail dependence of the vector \((Y_1, Y_2)\) (and thus of the copula \(C\)) is defined as

\[
\lambda_U = \lim_{u \to 1} P\left[ Y_1 > F_1^{-1}(u) | Y_2 > F_2^{-1}(u) \right],
\]

(50)

Where \(F_i^{-1}\) represents the inverse function of \(F_i\), provided the limit exists. We say that the random vector (and thus the copula \(C\)) has upper tail dependence if \(\lambda_U > 0\). Similarly, the coefficient of lower tail dependence of the vector \((Y_1, Y_2)\) (and thus of the copula \(C\)) is defined as

\[
\lambda_L = \lim_{u \to 0} P\left[ Y_1 < F_1^{-1}(u) | Y_2 < F_2^{-1}(u) \right]
\]

(51)

We say that the random vector (and thus the copula \(C\)) has lower tail dependence if \(\lambda_L > 0\).

Upper (lower) tail dependence measures the probability that one component of the vector \((Y_1, Y_2)\) is extremely large (small) given that the other is extremely large (small). As in the case of Kendall’s tau and Spearman’s rho, tail dependence is a copula property and can be expressed as

\[
\lambda_U = \lim_{u \to 1} \frac{1 + C(u, u) - 2u}{1 - u},
\]

(52)

\[
\lambda_L = \lim_{u \to 0} \frac{C(u, u)}{u}.
\]

(53)

The upper (lower) coefficient of tail dependence of the copula \(C\) is the lower (upper) coefficient of tail dependence of its associated survival copula \(C^S\).

Consider the random vector \((\tau_1, \tau_2)\) of default times for two firms, the coefficient of upper (lower) tail dependence represents the probability of long term survival (immediate joint death). The existence of default clustering periods implies that a copula to model joint default (survival) probabilities should have lower (upper) tail dependence to capture those periods.
3.2 Credit Risk Models

During the last two years a number of initiatives have been made public. CreditMetrics from JP Morgan, first published and well publicized in 1997, is reviewed in the next section. CreditMetrics’ approach is based on credit migration analysis, i.e. the probability of moving from one credit quality to another, including default, within a given time horizon, which is often taken arbitrarily as 1 year. CreditMetrics models the full forward distribution of the values of any bond or loan portfolio, say 1 year forward, where the changes in values are related to credit migration only, while interest rates are assumed to evolve in a deterministic fashion. Credit-VaR of a portfolio is then derived in a similar fashion as for market risk. It is simply the percentile of the distribution corresponding to the desired confidence level.

KMV Corporation, a firm specialized in credit risk analysis, has developed over the last few years a credit risk methodology, as well as an extensive database, to assess default probabilities and the loss distribution related to both default and migration risks. KMV’s methodology differs somewhat from CreditMetrics as it relies upon the "Expected Default Frequency", or EDF, for each issuer, rather than upon the average historical transition frequencies produced by the rating agencies, for each credit class.

Both approaches rely on the asset value model originally proposed by Merton (1974), but they differ quite substantially in the simplifying assumptions they require in order to facilitate its implementation. How damaging are, in practice, these compromises to a satisfactory capture of the actual complexity of credit measurement stays an open issue. It will undoubtedly attract many new academic developments in the years to come.

At the end of 1997, Credit Suisse Financial Products (CSFP) released a new approach, CreditRisk+, which only focuses on default. CreditRisk+ assumes that default for individual bonds, or loans,
follows a Poisson process. Credit migration risk is not explicitly modeled in this analysis. Instead, CreditRisk+ allows for stochastic default rates which partially account, although not rigorously, for migration risk.

Finally, CreditPortfolioView, which, like CreditRisk+, measures only default risk, has introduced by McKinsey. It is a discrete time multi-period model, where default probabilities are a function of macro-variables such as unemployment, the level of interest rates, the growth rate in the economy, government expenses, foreign exchange rates, which also drive, to a large extent, credit cycles.

3.2.1 CreditMetrics™ and CreditVaR I

CreditMetrics / CreditVaR I are methodologies based on the estimation of the forward distribution of the changes in value of a portfolio of loan and bond type products at a given time horizon, usually 1 year. The changes in value are related to the eventual migrations in credit quality of the obligor, both up and downgrades, as well as default.

In comparison to market-VaR, credit-VaR poses two new challenging difficulties. First, the portfolio distribution is far from being normal, and second, measuring the portfolio effect due to credit diversification is much more complex than for market risk.

While it was legitimate to assume normality of the portfolio changes due to market risk, it is no longer the case for credit returns which are by nature highly skewed and fat-tailed as shown in next Figure. Indeed, there is limited upside to be expected from any improvement in credit quality, while there is substantial downside consecutive to downgrading and default. The percentile levels of the distribution cannot be any longer estimated from the mean and variance only. The calculation of VaR for credit risk requires simulating the full distribution of the changes in portfolio value.
To measure the effect of portfolio diversification we need to estimate the correlations in credit quality changes for all pairs of obligors. But, these correlations are not directly observable. CreditMetrics / CreditVaR I base their evaluation on the joint probability of asset returns, which itself results from strong simplifying assumptions on the capital structure of the obligor, and on the generating process for equity returns. This is clearly a key feature of CreditMetrics / CreditVaR I on which we will elaborate in the next section.

Finally, CreditMetrics / CreditVaR I, as the other approaches reviewed in this paper, assumes no market risk since forward values and exposures are simply derived from deterministic forward curves. The only uncertainty in CreditMetrics / CreditVaR I relates to credit migration, i.e. the process of moving up or down the credit spectrum. In other words, credit risk is analyzed independently of market risk, which is another limitation of this approach.

I. Model’s Framework

CreditMetrics / CreditVaR I risk measurement framework is best summarized by Figure (3-2) which shows the two main building blocks, i.e. “value-at-risk due to credit” for a single financial instrument, then value-at-risk at the portfolio level which accounts for portfolio diversification effects (“Portfolio Value-at-Risk due to
There are also two supporting functions, "correlations" which derives the asset return correlations which are used to generate the joint migration probabilities, and "exposures" which produces the future exposures of derivative securities, like swaps.

II. Credit-Var for a bond (building block #1)

The first step is to specify a rating system, with rating categories, together with the probabilities of migrating from one credit quality to another over the credit risk horizon. This transition matrix is the key component of the credit-Var model proposed by JP Morgan. It can be Moody’s, or Standard & Poor’s, or the proprietary rating system internal to the bank. A strong assumption made by CreditMetrics/CreditVaR I is that all issuers are credit-homogeneous within the same rating class,
With the same transition probabilities and the same default probability, KMV departs from CreditMetrics/CreditVaR I in the sense that in KMV's framework each issuer is specific, and is characterized by his own asset returns distribution, its own capital structure and its own default probability.

Second, the risk horizon should be specified. It is usually 1 year, although multiple horizons could be chosen, like 1±10 years, when one is concerned by the risk profile over a longer period of time as it is needed for long dated illiquid instruments.

The third phase consists of specifying the forward discount curve at the risk horizon(s) for each credit category, and, in the case of default, the value of the instrument which is usually set at a percentage, named the "recovery rate", of face value or "par".

In the final step, this information is translated into the forward distribution of the changes in portfolio value consecutive to credit migration.

The following example taken from the technical document of CreditMetrics illustrates the four steps of the credit-VaR model.

**Example:** Credit-VaR for a senior unsecured BBB rated bond maturing exactly in 5 years, and paying an annual coupon of 6%.

**Step 1: Specify the transition matrix.**

The rating categories, as well as the transition matrix, are chosen from a rating system (Table (3-1))

In the case of Standard & Poor’s there are 7 rating categories, the highest credit quality being AAA, and the lowest, CCC; the last state is default. Default corresponds to the situation where an obligor cannot make a payment related to a bond or a loan obligation, whether it is a coupon or the redemption of principal. "Pari passu" clauses are such that when an obligor defaults on one payment related to a bond or a loan, he is technically declared in default on all debt obligations.
The bond issuer has currently a BBB rating, and the italicized line corresponding to the BBB initial rating in Table 1 shows the probabilities estimated by Standard & Poor’s for a BBB issuer to be, in 1 year from now, in one of the 8 possible states, including default. Obviously, the most probable situation is for the obligor to stay in the same rating category, i.e. BBB, with a probability of 86.93%. The probability of the issuer defaulting within 1 year is only 0.18%, while the probability of being upgraded to AAA is also very small, i.e. 0.02%. Such transition matrix is produced by the rating agencies for all initial ratings. Default is an absorbing state, i.e. an issuer who is in default stays in default.

Moody’s also publishes similar information. These probabilities are based on more than 20 years of history of firms, across all industries, which have migrated over a 1 year period from one credit rating to another. Obviously, this data should be interpreted with care since it represents average statistics across a heterogeneous sample of firms, and over several business cycles. For this reason many banks prefer to rely on their own statistics which relate more closely to the composition of their loan and bond portfolios.

Moody’s and Standard & Poor’s also produce long-term average cumulative default rates, as shown in Table 2 in a tabular form and in Fig. 4 in a graphical form. For example, a BBB issuer has a probability of 0.18% to default within 1 year, 0.44% to default in 2 years, 4.34% to default in 10 years.

Tables (3-1) and (3-2) should in fact be consistent with one another. From Table (3-2) we can back out the transition matrix which best replicates, in the least square sense, the average cumulative default rates. Indeed, assuming that the process for default is Markovian and stationary, then multiplying the 1-year transition matrix n times generates the n-year matrix. The n-year default probabilities are simply the values in the last default column.
of the transition matrix, and should match the column in year \( n \) of Table (3-2).

<table>
<thead>
<tr>
<th>Initial rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>90.81</td>
<td>8.33</td>
<td>0.68</td>
<td>0.06</td>
<td>0.12</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AA</td>
<td>0.70</td>
<td>90.65</td>
<td>7.79</td>
<td>0.64</td>
<td>0.06</td>
<td>0.14</td>
<td>0.02</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>0.09</td>
<td>2.27</td>
<td>91.05</td>
<td>5.52</td>
<td>0.74</td>
<td>0.26</td>
<td>0.01</td>
<td>0.06</td>
</tr>
<tr>
<td>BBB</td>
<td>0.02</td>
<td>0.33</td>
<td>5.95</td>
<td>86.93</td>
<td>5.30</td>
<td>1.17</td>
<td>0.12</td>
<td>0.18</td>
</tr>
<tr>
<td>BB</td>
<td>0.03</td>
<td>0.14</td>
<td>0.67</td>
<td>7.73</td>
<td>80.53</td>
<td>8.84</td>
<td>1.00</td>
<td>1.06</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0.11</td>
<td>0.24</td>
<td>0.43</td>
<td>6.48</td>
<td>83.46</td>
<td>4.07</td>
<td>5.20</td>
</tr>
<tr>
<td>CCC</td>
<td>0.22</td>
<td>0</td>
<td>0.22</td>
<td>1.30</td>
<td>2.38</td>
<td>11.24</td>
<td>64.86</td>
<td>19.79</td>
</tr>
</tbody>
</table>

Source: Standard & Poor’s Credit Week (15 April 96)

<table>
<thead>
<tr>
<th>Term</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5...</th>
<th>7...</th>
<th>10...</th>
<th>15...</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.00</td>
<td>0.00</td>
<td>0.07</td>
<td>0.15</td>
<td>0.24...</td>
<td>0.66...</td>
<td>1.40...</td>
<td>1.40</td>
</tr>
<tr>
<td>AA</td>
<td>0.00</td>
<td>0.02</td>
<td>0.12</td>
<td>0.25</td>
<td>0.43...</td>
<td>0.89...</td>
<td>1.29...</td>
<td>1.48</td>
</tr>
<tr>
<td>A</td>
<td>0.06</td>
<td>0.16</td>
<td>0.27</td>
<td>0.44</td>
<td>0.67...</td>
<td>1.12...</td>
<td>2.17...</td>
<td>3.00</td>
</tr>
<tr>
<td>BBB</td>
<td>0.18</td>
<td>0.44</td>
<td>0.72</td>
<td>1.27</td>
<td>1.78...</td>
<td>2.99...</td>
<td>4.34...</td>
<td>4.70</td>
</tr>
<tr>
<td>BB</td>
<td>1.06</td>
<td>3.48</td>
<td>6.12</td>
<td>8.68</td>
<td>10.97...</td>
<td>14.46...</td>
<td>17.73...</td>
<td>19.91</td>
</tr>
<tr>
<td>B</td>
<td>5.20</td>
<td>11.00</td>
<td>15.95</td>
<td>19.40</td>
<td>21.88...</td>
<td>25.14...</td>
<td>29.02...</td>
<td>30.65</td>
</tr>
<tr>
<td>CCC</td>
<td>19.79</td>
<td>26.92</td>
<td>31.63</td>
<td>35.97</td>
<td>40.15...</td>
<td>42.64...</td>
<td>45.10...</td>
<td>45.10</td>
</tr>
</tbody>
</table>

Source: S&P Credit Week, Apr. 15, 1996

Actual transition and default probabilities vary quite substantially over the years, depending whether the economy is in recession, or in expansion. When implementing a model which relies on transition probabilities, one may have to adjust the average historical values as shown in Table (3-1), to be consistent with one’s assessment of the current economic environment.
Moody’s study by Carty and Lieberman (1996) provides historical default statistics; both the mean and standard deviation, by rating category for the population of obligors they have rated during the period 1920-1996 (see Table (3-3)).

**Step 2: Specify the credit risk horizon.**

The risk horizon is usually 1 year, and is consistent with the transition matrix shown in Table 1. But this horizon is purely arbitrary, and is mostly dictated by the availability of the accounting data and financial reports processed by the rating agencies. In KMV’s framework, which relies on market data as well as accounting data, any horizon can be chosen from a few days to several years. Indeed, market data can be updated daily while assuming the other firm characteristics stay constant until new information becomes available.

**Step 3: Specify the forward pricing model.**

The valuation of a bond is derived from the zero-curve corresponding to the rating of the issuer. Since there are 7 possible credit qualities, 7 “spread” curves are required to price the bond in all possible states, all obligors within the same rating class being marked-to-market with the same curve. The spot zero curves is used to determine the current spot value of the bond. The forward price of the bond in 1 year from now is derived from the forward zero-curve, 1 year ahead, which is then applied to the residual cash flows from year one to the maturity of the bond. Table (3-4) gives the 1-year forward zero-curves for each credit rating.

Empirical evidence shows that for high grade investment bonds the spreads tend to increase with time to maturity, while for low grade, like CCC the spread tends to be wider at the short end of the curve than at the long end, as shown in Figure (3-3).
Table (3-3) One-year default rates by rating, 1970-1995

<table>
<thead>
<tr>
<th>Credit rating</th>
<th>One-year default rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average (%)</td>
</tr>
<tr>
<td>Aaa</td>
<td>0.00</td>
</tr>
<tr>
<td>Aa</td>
<td>0.03</td>
</tr>
<tr>
<td>A</td>
<td>0.01</td>
</tr>
<tr>
<td>Baa</td>
<td>0.13</td>
</tr>
<tr>
<td>Ba</td>
<td>1.42</td>
</tr>
<tr>
<td>B</td>
<td>7.62</td>
</tr>
</tbody>
</table>


Table (3-4) One-year forward zero-curves for each credit rating (%)

<table>
<thead>
<tr>
<th>Category</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>3.60</td>
<td>4.17</td>
<td>4.73</td>
<td>5.12</td>
</tr>
<tr>
<td>AA</td>
<td>3.65</td>
<td>4.22</td>
<td>4.78</td>
<td>5.17</td>
</tr>
<tr>
<td>A</td>
<td>3.72</td>
<td>4.32</td>
<td>4.93</td>
<td>5.32</td>
</tr>
<tr>
<td>BBB</td>
<td>4.10</td>
<td>4.67</td>
<td>5.25</td>
<td>5.63</td>
</tr>
<tr>
<td>BB</td>
<td>5.55</td>
<td>6.02</td>
<td>6.78</td>
<td>7.27</td>
</tr>
<tr>
<td>B</td>
<td>6.05</td>
<td>7.02</td>
<td>8.03</td>
<td>8.52</td>
</tr>
<tr>
<td>CCC</td>
<td>15.05</td>
<td>15.02</td>
<td>14.03</td>
<td>13.52</td>
</tr>
</tbody>
</table>

Source: CreditMetrics, JP Morgan.

The 1-year forward price of the bond, if the obligor stays BBB, is then:

\[ V_{BBB} = 6 + \frac{6}{1.0410} + \frac{6}{(1.0467)^2} + \frac{6}{(1.0525)^3} + \frac{106}{(1.0563)^4} = 107.55 \]
If we replicate the same calculations for each rating category we obtain the values shown in Table (3-5).

If the issuer defaults at the end of the year, we assume that not everything is lost. Depending on the seniority of the instrument, a recovery rate of par value is recuperated by the investor. These recovery rates are estimated from historical data by the rating agencies. Table (3-6) shows the recovery rates for bonds by different seniority classes as estimated by Moody’s. In our example the recovery rate for senior unsecured debt is estimated to be 51.13%, although the estimation error is quite large and the actual value lies in a fairly large confidence interval.

In the Monte Carlo simulation used to generate the loss distribution, it is assumed that the recovery rates are distributed according to a beta distribution with the same mean and standard deviation as shown in Table (3-6).

*Figure (3-3) Spread curves for different credit qualities*
Step 4: Derive the forward distribution of the changes in portfolio value.

The distribution of the changes in the bond value, at the 1-year horizon, due to an eventual change in credit quality is shown Table (3-7) and Figure (3-4). This distribution exhibits long downside tails. The first percentile of the distribution of ΔV, which corresponds to credit-VaR at the 99% confidence level, is -23.91.

It is much larger than if we computed the first percentile assuming a normal distribution for ΔV. In that case credit-VaR at the 99% confidence level would be only -7.43.

### Table (3-6) Recovery rates by seniority class (% of face value, i.e., “par”)

<table>
<thead>
<tr>
<th>Seniority class</th>
<th>Mean (%)</th>
<th>Standard deviation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior secured</td>
<td>53.80</td>
<td>26.86</td>
</tr>
<tr>
<td>Senior unsecured</td>
<td>51.13</td>
<td>25.45</td>
</tr>
<tr>
<td>Senior subordinated</td>
<td>38.52</td>
<td>23.81</td>
</tr>
<tr>
<td>Subordinated</td>
<td>32.74</td>
<td>20.18</td>
</tr>
<tr>
<td>Junior subordinated</td>
<td>17.09</td>
<td>10.90</td>
</tr>
</tbody>
</table>

Table (3-7) Distribution of the bond values, and changes in value of a BBB bond, in 1 year

<table>
<thead>
<tr>
<th>Year-end rating</th>
<th>Probability of state: p(%)</th>
<th>Forward price: V($)</th>
<th>Change in value: ΔV($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.02</td>
<td>109.37</td>
<td>1.82</td>
</tr>
<tr>
<td>AA</td>
<td>0.33</td>
<td>109.19</td>
<td>1.64</td>
</tr>
<tr>
<td>A</td>
<td>5.95</td>
<td>108.66</td>
<td>1.11</td>
</tr>
<tr>
<td>BBB</td>
<td>86.93</td>
<td>107.55</td>
<td>0</td>
</tr>
<tr>
<td>BB</td>
<td>5.30</td>
<td>102.02</td>
<td>-5.53</td>
</tr>
<tr>
<td>B</td>
<td>1.17</td>
<td>98.10</td>
<td>-9.45</td>
</tr>
<tr>
<td>CCC</td>
<td>0.12</td>
<td>83.64</td>
<td>-23.91</td>
</tr>
<tr>
<td>Default</td>
<td>0.18</td>
<td>51.13</td>
<td>-56.42</td>
</tr>
</tbody>
</table>

Source: CreditMetrics, JP Morgan.

Figure (3-4) Histogram of the 1-year forward prices and changes in value of a BBB bond.
First, consider a portfolio composed of 2 bonds with an initial rating of BB and A, respectively. Given the transition matrix shown in Table (3-1), and assuming no correlation between changes in credit quality, we can then derive easily the joint migration probabilities shown in Table 8. Each entry is simply the product of the transition probabilities for each obligor. For example, the joint probability that obligor #1 and obligor #2 stay in the same rating class is

\[ 73.32\% = 80.53\% \times 91.05\% \]

Where 80.53\% is the probability that obligor #1 keeps his current rating BB, and 91.05\% is the probability that obligor #2 stays in rating class A.

Unfortunately, this table is not very useful in practice when we need to assess the diversification effect on a large loan or bond portfolio. Indeed, the actual correlations between the changes in credit quality are different from zero. And it will be shown in Section 5 that the overall credit-VaR is in fact quite sensitive to these correlations. Their accurate estimation is therefore determinant in portfolio optimization from a risk-return perspective.

Correlations are expected to be higher for firms within the same industry or in the same region, than for firms in unrelated sectors. In addition, correlations vary with the relative state of the economy in the business cycle. If there is a slowdown in the economy, or a recession, most of the assets of the obligors will decline in value and quality, and the likelihood of multiple defaults increases substantially. The contrary happens when the economy is performing well: default correlations go down. Thus, we cannot expect default and migration probabilities to stay stationary over time. There is clearly a need for a structural model that bridges the changes of default probabilities to fundamental variables whose correlations stay stable over time. Both CreditMetrics and KMV derive the default and migration probabilities.
from a correlation model of the firm’s assets that will be detailed in
the next section.

Contrary to KMV, and for the sake of simplicity, CreditMetrics / CreditVaR I have chosen the equity price as a proxy for the asset
value of the firm that is not directly observable. This is another
strong assumption in CreditMetrics that may affect the accuracy of
the method.

Table (3-8) Joint migration probabilities (%) with zero correlation for 2 issuers rated BB and A

<table>
<thead>
<tr>
<th>Obligor #1 (BB)</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.09</td>
<td>2.27</td>
<td>91.05</td>
<td>5.52</td>
<td>0.74</td>
<td>0.26</td>
<td>0.01</td>
<td>0.06</td>
</tr>
<tr>
<td>AAA</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>AA</td>
<td>0.14</td>
<td>0.00</td>
<td>0.00</td>
<td>0.13</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>A</td>
<td>0.67</td>
<td>0.00</td>
<td>0.02</td>
<td>0.61</td>
<td>0.40</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>BBB</td>
<td>7.73</td>
<td>0.01</td>
<td>0.18</td>
<td>7.04</td>
<td>0.43</td>
<td>0.06</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>BB</td>
<td>80.53</td>
<td>0.07</td>
<td>1.83</td>
<td>73.32</td>
<td>4.45</td>
<td>0.60</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>B</td>
<td>8.84</td>
<td>0.01</td>
<td>0.20</td>
<td>8.05</td>
<td>0.49</td>
<td>0.07</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>CCC</td>
<td>1.00</td>
<td>0.00</td>
<td>0.02</td>
<td>0.91</td>
<td>0.06</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Default</td>
<td>1.06</td>
<td>0.00</td>
<td>0.02</td>
<td>0.97</td>
<td>0.06</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

First, CreditMetrics estimates the correlations between the equity
returns of various obligors, and then the model infers the correlations
between changes in credit quality directly from the joint distribution
of equity returns.

The proposed framework is the option pricing approach to the
valuation of corporate securities initially developed by Merton (1974).
The firm’s assets value, $V_t$, is assumed to follow a standard
geometric Brownian motion, i.e.:

$$V_t = V_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \sqrt{t} Z_t \right)$$  \hspace{1cm} (54)

With $Z_t \sim (0; 1)$, $\mu$ and $\sigma^2$ being respectively the mean and
variance of the instantaneous rate of return on the assets of the firm,$\frac{dV_t}{V_t}$. $V_t$ is lognormal distributed with expected value at time $t$,

$$E(V_t) = V_0 \exp \{ \mu t \}.$$
It is further assumed that the firm has a very simple capital structure, as it is financed only by equity, $S_t$, and a single zero-coupon debt instrument maturing at time $T$, with face value $F$, and current market value $B_t$. The firm’s balance sheet can be represented as in Table (3-9).

In this framework, default only occurs at maturity of the debt obligation, when the value of assets is less than the promised payment, $F$, to the bond holders. Figure (3-5) shows the distribution of the assets’ value at time $T$, the maturity of the zero-coupon debt, and the probability of default which is the shaded area below $F$.

Merton’s model is extended by CreditMetrics to include changes in credit quality as illustrated in Figure (3-6). This generalization consists of slicing the distribution of asset returns into bands in such a way that, if we draw randomly from this distribution, we reproduce exactly the migration frequencies shown in the transition matrix. Figure (3-6) shows the distribution of the normalized assets’ rates of return, 1 year ahead, which is normal with mean zero and unit variance. The credit rating thresholds correspond to the transition probabilities in Table (3-1) for a BB rated obligor. The right tail of the distribution on the right hand side of $Z_{AAA}$ corresponds to the probability for the obligor of being upgraded from BB to AAA, i.e. 0.03%. Then, the area between $Z_{AA}$ and $Z_{AAA}$ corresponds to the probability of being upgraded from BB to AA, etc. The left tail of the distribution, on the left-hand side of $Z_{CCC}$, corresponds to the probability of default, i.e. 1.06%.

<table>
<thead>
<tr>
<th>Table (3-9) Balance sheet of Merton’s firm</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Assets</strong></td>
</tr>
<tr>
<td>Risky assets: $V_t$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Total</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
Figure (3-5) Distribution of the firm’s assets value at maturity of the debt obligation

Figure (3-6) Generalization of the Merton model to include rating changes

Standard normal distribution for a BB-rated firm

<table>
<thead>
<tr>
<th>Rating</th>
<th>Default</th>
<th>CCC</th>
<th>B</th>
<th>Z_{CCC}</th>
<th>Z_{B}</th>
<th>Z_{B}</th>
<th>Firm remains BB</th>
<th>Z_{BBB}</th>
<th>Z_{A}</th>
<th>Z_{AA}</th>
<th>Z_{AAA}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.06</td>
<td>1.00</td>
<td>8.84</td>
<td>-2.30</td>
<td>-2.04</td>
<td>-1.2</td>
<td>80.53</td>
<td>1.37</td>
<td>2.39</td>
<td>2.93</td>
<td>3.43</td>
</tr>
</tbody>
</table>
This generalization of Merton’s model is quite easy to implement. It assumes that the normalized log-returns over any period of time are normally distributed with mean 0 and variance 1, and it is the same for all obligors within the same rating category. If \( P_{Def} \) denotes the probability for the BB-rated obligor of defaulting, then the critical asset value \( V_{Def} \) is such that

\[
\rho_{Def} = \Pr[V_t \leq V_{Def}]
\]

This can be translated into a normalized threshold \( Z_{CCC} \), such that the area in the left tail below \( Z_{CCC} \) is \( P_{Def} \). Indeed, according to (54), default occurs when \( Z_t \) satisfies

\[
\rho_{Def} = \Pr \left[ \frac{\ln(V_{Def} / V_0) - (\mu - (\sigma^2 / 2))t}{\sigma \sqrt{t}} \geq Z_t \right]
\]

\[
\Pr \left[ Z_t < - \frac{\ln(V_0 / V_{Def}) + \left[ \frac{\mu - (\sigma^2 / 2)t}{\sigma \sqrt{t}} \right]}{\sigma \sqrt{t}} \right] = N(-d_2),
\]

Where the normalized return is \( N[0; 1] \)

\[
r = \frac{\ln(V_t / V_0) - (\mu - (\sigma^2 / 2))t}{\sigma \sqrt{t}}
\]

\( Z_{CCC} \) is simply the threshold point in the standard normal distribution corresponding to a cumulative probability of \( P_{Def} \). Then, the critical asset value \( V_{Def} \) which triggers default is such that \( Z_{CCC} = -d_2 \) where:

<table>
<thead>
<tr>
<th>Table (3-10) Transition probabilities and credit quality thresholds for BB and A rated obligors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rating in 1 year</td>
</tr>
<tr>
<td>------------------</td>
</tr>
<tr>
<td>AAA</td>
</tr>
<tr>
<td>AA</td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>BBB</td>
</tr>
<tr>
<td>BB</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>CCC</td>
</tr>
<tr>
<td>Default</td>
</tr>
</tbody>
</table>
\[ d_2 = \frac{\ln(V_0 / V_{Def}) + (\mu - (\sigma^2 / 2))t}{\sigma \sqrt{t}} \]  

(57)

And is also called "distance-to-default". Note that only the threshold levels are necessary to derive the joint migration probabilities, and they are calculated without the need to observe the asset value, and to estimate its mean and variance. Only to derive the critical asset value \( V_{Def} \) we need to estimate the expected asset return \( \mu \) and asset volatility \( \sigma \).

Accordingly \( Z_B \) is the threshold point corresponding to a cumulative probability of being either in default or in rating CCC, i.e., \( p_{Def} \). \( p_{CCC} \), etc.

Further, since asset returns are not directly observable, CreditMetrics/ CreditVaR I chose equity returns as a proxy, which is equivalent to assume that the firm's activities are all equity financed.

Now, for the time being, assume that the correlation between asset rates of return is known, and is denoted by \( \rho \), which is assumed to be equal to 0.20 in our example. The normalized log-returns on both assets follow a joint normal distribution:

\[ f(r_{BB}, r_A; \rho) = \frac{1}{2\pi \sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{r_{BB}^2 - 2\rho r_{BB} r_A + r_A^2}{r_{BB}^2 - 2\rho r_{BB} r_A + r_A^2}\right]\right\}. \]  

(58)

We can then easily compute the probability for both obligors of being in any combination of ratings, e.g. that they remain in the same rating classes, i.e. BB and A, respectively:

\[ Pr(-1.23 < r_{BB} < 1.37, -1.51 < r_A < 1.98) = \int_{-1.23}^{1.37} \int_{-1.51}^{1.98} f(r_{BB}, r_A; \rho) dr_{BB} dr_A = 0.7365 \]  

(59)

If we implement the same procedure for the other 63 combinations we obtain Table (3-11). We can compare Table (3-11) with Table (3-8), the later being derived assuming zero correlation, to notice that the joint probabilities are different.

Figure (3-7) illustrates the effect of asset return correlation on the joint default probability for the rated BB and A obligors.
### Table (3-11) Joint rating probabilities (%) for BB and A rated Obligors when correlation between asset returns is 20%

<table>
<thead>
<tr>
<th>Rating of first company (BB)</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Def</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.00</td>
<td>0.00</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.03</td>
</tr>
<tr>
<td>AA</td>
<td>0.00</td>
<td>0.01</td>
<td>0.13</td>
<td>0.13</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.14</td>
</tr>
<tr>
<td>A</td>
<td>0.00</td>
<td>0.04</td>
<td>0.61</td>
<td>0.01</td>
<td>0.40</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.67</td>
</tr>
<tr>
<td>BBB</td>
<td>0.02</td>
<td>0.35</td>
<td>7.10</td>
<td>7.20</td>
<td>0.02</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>7.69</td>
</tr>
<tr>
<td>BB</td>
<td>0.07</td>
<td>1.79</td>
<td>73.65</td>
<td>4.24</td>
<td>0.56</td>
<td>0.18</td>
<td>0.01</td>
<td>0.04</td>
<td>80.53</td>
</tr>
<tr>
<td>B</td>
<td>0.00</td>
<td>0.08</td>
<td>7.80</td>
<td>0.79</td>
<td>0.13</td>
<td>0.05</td>
<td>0.00</td>
<td>0.01</td>
<td>8.87</td>
</tr>
<tr>
<td>CCC</td>
<td>0.00</td>
<td>0.01</td>
<td>0.85</td>
<td>0.11</td>
<td>0.02</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Default</td>
<td>0.00</td>
<td>0.01</td>
<td>0.90</td>
<td>0.13</td>
<td>0.02</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>1.07</td>
</tr>
<tr>
<td>Total</td>
<td>0.09</td>
<td>2.29</td>
<td>91.06</td>
<td>5.48</td>
<td>0.75</td>
<td>0.26</td>
<td>0.01</td>
<td>0.06</td>
<td>100</td>
</tr>
</tbody>
</table>

Source: CreditMetrics, JP Morgan.

### Figure (3-7). Probability of joint defaults as a function of asset returns correlation

![Figure showing probability of joint defaults as a function of asset returns correlation](image)

Source: CreditMetrics, JP Morgan.

To be more specific, consider two obligors whose probabilities of default are \( P_1 (P_{\text{Def1}}) \) and \( P_2 (P_{\text{Def2}}) \), respectively. Their asset return correlation is \( \rho \). The events of default for obligors 1 and 2 are denoted \( \text{DEF1} \) and \( \text{DEF2} \), respectively, and \( P (\text{DEF1}; \text{DEF2}) \) is the joint probability of default. Then, it can be shown that the default correlation is

\[
\text{corr}(\text{DEF1}, \text{DEF2}) = \frac{P(\text{DEF1}, \text{DEF2}) - P_1 \cdot P_2}{\sqrt{P_1(1-P_1) \cdot P_2(1-P_2)}}.
\] (60)
The joint probability of both obligors defaulting is, according to Merton's model,

\[ P(\text{DEF}_1, \text{DEF}_2) = Pr[V_1 \leq V_{\text{Def}_1}, V_2 \leq V_{\text{Def}_2}] \quad (61) \]

where \( V_1 \) and \( V_2 \) denote the asset values for both obligors at time \( t \), and \( V_{\text{Def}_1} \) and \( V_{\text{Def}_2} \) are the corresponding critical values which trigger default. Expression (61) is equivalent to

\[ P(\text{DEF}_1, \text{DEF}_2) = Pr[r_1 \leq -d_1^1, r_2 \leq -d_2^2] = N_2(-d_1^1, -d_2^2, \rho), \quad (62) \]

Where \( r_1 \) and \( r_2 \) denote the normalized asset returns as defined in (56) for obligors 1 and 2, respectively, and \( d_1^1 \) and \( d_2^2 \) are the corresponding distant to default as in (57). \( N_2(x, y, \rho) \) denotes the cumulative standard bivariate normal distribution where \( \rho \) is the correlation coefficient between \( x \) and \( y \). Figure (3-7) is simply the graphical representation of (61) for the asset return correlation varying from 0 to 1.

**Example:**

\[ \rho = 20\%, \]

\[ P(\text{DEF}_1, \text{DEF}_2) = N_2(-d_2^1, -d_2^2, \rho) = N_2(-3.24, -2.30, 0.20) \]

\[ = 0.000054, \]

\[ P(\text{A}) = 0.06\%, \]

\[ P(\text{BB}) = 1.06\%; \]

If then follows:

\[ \text{Corr (DEF}_1, \text{DEF}_2) = 0.019 = 1.9\%. \]

The ratio of asset returns correlations to default correlations is approximately 10-1 for asset correlations in the range of 20-60%. This shows that the joint probability of default is in fact quite sensitive to pair wise asset return correlations, and it illustrates the necessity to estimate correctly these data to assess precisely the diversification effect within a portfolio. In last Section we show that, for the benchmark portfolio we selected for the comparison of credit models, the impact of correlations on credit-VaR is quite large. It is larger for low credit quality than for high grade portfolios. Indeed, when the credit quality of the portfolio deteriorates the expected
number of defaults increases, and this number is magnified by an increase in default correlations.

The statistical procedure to estimate asset return correlations is discussed in the next section dedicated to KMV.

IV. Analysis of credit diversification (building block #2)

The analytic approach that we just sketched out for a portfolio with bonds issued by 2 obligors is not doable for large portfolios. Instead, CreditMetrics/ CreditVaR I implement a Monte Carlo simulation to generate the full distribution of the portfolio values at the credit horizon of 1 year. The following steps are necessary.

1. Derivation of the asset return thresholds for each rating category.
2. Estimation of the correlation between each pair of obligors' asset returns.
3. Generation of asset return scenarios according to their joint normal distribution. A standard technique to generate correlated normal variables is the Cholesky decomposition. Each scenario is characterized by $n$ standardized asset returns, one for each of the $n$ obligors in the portfolio.
4. For each scenario, and for each obligor, the standardized asset return is mapped into the corresponding rating, according to the threshold levels derived in step 1.
5. Given the spread curves which apply for each rating, the portfolio is revalued.
6. Repeat the procedure a large number of times, say 100,000 times, and plot the distribution of the portfolio values to obtain a graph which looks like Figure (3-1).
7. Then, derive the percentiles of the distribution of the future values of the portfolio.
V. Credit-VaR and calculation of the capital charge

Economic capital stands as a cushion to absorb unexpected losses related to credit events, i.e. migration and/or default. Figure (3-8) shows how to derive the capital charge related to credit risk.

\[ V_{(p)} = \text{value of the portfolio in the worst case scenario at the p\% confidence level.} \]

\[ FV = \text{forward value of the portfolio} = V_0 (1 + PR). \]

\[ V_0 = \text{current mark-to-market value of the portfolio.} \]

\[ PR = \text{promised return on the portfolio.} \]

\[ EV = \text{expected value of the portfolio} = V_0 (1 + ER). \]

\[ ER = \text{expected return on the portfolio.} \]

\[ EL = \text{expected loss} = FV - EV. \]

The expected loss does not contribute to the capital allocation, but instead goes into reserves and is imputed as a cost into the RAROC calculation. The capital charge comes only as a protection against unexpected losses:

*Figure (3-8) Credit-VaR and calculation of economic capital.*

*Capital = EV - V_{(p)}*
VI. Marginal risk measures (building block #2, continuation)

In addition to the overall credit-VaR analysis for the portfolio, CreditMetrics/CreditVaR I offer the interesting feature of isolating the individual marginal risk contributions to the portfolio. For example, for each asset, CreditMetrics/CreditVaR I calculate the marginal standard deviation, i.e. the impact of each individual asset on the overall portfolio standard deviation. By comparing the marginal standard deviation to the stand-alone standard deviation for each loan, one can assess the extent of the benefit derived from portfolio diversification when adding the instrument in the portfolio. Figure (3-9) shows the marginal standard deviation for each asset, expressed in percentage of the overall standard deviation, plotted against the marked-to-market value of the instrument. This is an important pro-active risk management tool as it allows one to identify trading opportunities in the loan/bond portfolio where concentration, and as a consequence overall risk, can be reduced without affecting expected profits. Obviously, for this framework to become fully operational it needs to be complemented by a RAROC model which provides information on the adjusted return on capital for each deal. The same framework can also be used to set up credit risk limits, and monitor credit risk exposures in terms of the joint combination of market value and marginal standard deviation, as shown in figure (3-10).

Figure (3-9) Risk versus size of exposures within a typical credit portfolio
VII. Estimation of asset correlations (building block #3)

Since asset values are not directly observable, equity prices for publicly traded firms are used as a proxy to calculate asset correlations. For a large portfolio of bonds and loans, with thousand of obligors, it would still require the computation of a huge correlation matrix for each pair of obligors. To reduce the dimensionality of the this estimation problem, CreditMetrics/CreditVaR I use a multi-factor analysis. This approach maps each obligor to the countries and industries that most likely determine its performance. Equity returns are correlated to the extent that they are exposed to the same industries and countries. In CreditMetrics/CreditVaR I the user specifies the industry and country weights for each obligor, as well as the "firm-specific risk", which is idiosyncratic to each obligor and neither correlated to any other obligor nor any index.

Figure (3-10) Example of risk limits for a portfolio

(Source: CreditMetrics, JP Morgan).
VIII. Exposures (building block #4)

What is meant by "exposures" in CreditMetrics/CreditVaR I is somewhat misleading since market risk factors are assumed constant. This building block is simply the forward pricing model that applies for each credit rating. For bond-type products like bonds, loans, receivables, commitments to lend, letters of credit, exposure simply relates to the future cash flows at risk, beyond the 1-year horizon. Forward pricing is derived from the present value model using the forward yield curve for the corresponding credit quality. The example presented in earlier Sections illustrates how the exposure distribution is calculated for a bond.

For derivatives, like swaps and forwards, the exposure is conditional on future interest rates. Contrary to a bond, there is no simple way to derive the future cash flows at risk without making some assumptions on the dynamics of interest rates. The complication arises since the risk exposure for a swap can be either positive if the swap is in-the-money for the bank, or negative if it is out-of-the-money. In the later case it is a liability and it is the counterparty that is at risk. Figure (3-11) shows the exposure profiles of an interest rate swap for different interest rate scenarios, assuming no change in the credit ratings of the counterparty, and of the bank. The bank is at risk only when the exposure is positive.

At this stage we assume the average exposure of a swap given and it is supposed to have been derived from an external model. In CreditMetrics/CreditVaR I interest rates being deterministic, the calculation of the forward price distribution relies on an ad hoc procedure:

\[
\text{Value of swap in 1 year; in rating } R = \text{Forward risk-free value in 1 year} - \text{Expected loss in years 1 to maturity for the given rating } R
\]

Where:
Expected loss in years 1 to maturity for the given rating $R$

\[ E = \text{Average exposure from year 1 to maturity} \times \text{Probability of default in years 1 through maturity} \]

For the given rating $R \times (1 - \text{recovery rate})$

The forward risk-free value of the swap is calculated by discounting the future net cash flows of the swap, based on the forward curve, and discounting them using the forward Government yield curve. This value is the same for all credit ratings.

The probability of default in year 1 through maturity either comes directly from Moody's or Standard & Poor's, or can be derived from the transition matrix as previously discussed in last Sections. The recovery rate comes from the statistical analyses provided by the rating agencies.

*Figure (3-11) Risk exposure of an interest rate swap*

Source: CreditMetrics, JP Morgan
Table (3-12) Distribution of the 1-year forward values of a 3-year interest rate swap

<table>
<thead>
<tr>
<th>Year-end rating</th>
<th>Two-year default likelihood (</th>
<th>Forward value ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.00</td>
<td>1</td>
</tr>
<tr>
<td>AA</td>
<td>0.02</td>
<td>6</td>
</tr>
<tr>
<td>A</td>
<td>0.15</td>
<td>46</td>
</tr>
<tr>
<td>BBB</td>
<td>0.48</td>
<td>148</td>
</tr>
<tr>
<td>BB</td>
<td>2.59</td>
<td>797</td>
</tr>
<tr>
<td>B</td>
<td>10.41</td>
<td>3209</td>
</tr>
<tr>
<td>CCC</td>
<td>33.24</td>
<td>10,304</td>
</tr>
<tr>
<td>Default</td>
<td>-</td>
<td>50,860</td>
</tr>
</tbody>
</table>

Example:
Consider a 3-year interest rate swap with a $10 million notional Value. The average expected exposure between year 1 and 3 is supposed to be $61627. Given the 2-year probability of default, the distribution of 1-year forward values for the swap can be calculated according to the above formulas (57) and (60). The results are shown in Table (3-12), where $FV$ denotes the forward risk-free value of the swap.

Obviously, this ad hoc calculation of the exposure of an interest rate swap is not satisfactory. Only a model with stochastic interest rates will allow a proper treatment of exposure calculations for swaps as well as other derivative securities.

3.2.2 KMV model

The major weakness of CreditMetrics/CreditVaR I is not the methodology, which is rather appealing, but the reliance on transition probabilities based on average historical frequencies of defaults and credit migration. The accuracy of CreditMetrics/CreditVaR I calculations rely upon two critical assumptions:

- First, all firms within the same rating class have the same default rate, and second, the actual default rate is equal to the historical average default rate. The same assumptions also apply to the other transition probabilities. In other words, credit rating changes and
credit quality changes are identical, and credit rating and default rates are synonymous, i.e. the rating changes when the default rate is adjusted, and vice versa.

This view has been strongly challenged by KMV. Indeed, this cannot be true since default rates are continuous, while ratings are adjusted in a discrete fashion, simply because rating agencies take time to upgrade or downgrade companies whose default risk have changed. KMV has shown through a simulation exercise that the historical average default rate and transition probabilities can deviate significantly from the actual rates. In addition, KMV has demonstrated that substantial differences in default rates may exist within the same bond rating class, and the overlap in default probability ranges may be quite large with, for instance, some BBB and AA rated bonds having the same probability of default. KMV has replicated 50 000 times, through a Monte Carlo simulation, Moody’s study of default over a 25-year period. For each rating they have assumed a fixed number of obligors which is approximately the same as in Moody’s study. For each rating they have assumed that the true probability of default is equal to the reported Moody’s average default rate over the 25-year period. KMV has also run the simulation for several levels of correlation among the asset returns, ranging from 15% to 45%. A typical result is illustrated in Figure (3-12) for a BBB obligor. Given an exact default probability of 13 bp, the 25-year average historical default rate ranges between 4 and 27 bp at the 95% confidence level, for an asset correlation of 15%.

The distribution is quite skewed so that the mean default rate usually exceeds the typical (median) default rate for each credit class. Thus the average historical default probability overstates the default rate for a typical obligor.

Unlike CreditMetrics/CreditVaR I, KMV does not use Moody’s or Standard & Poor’s statistical data to assign a probability of default which only depends on the rating of the obligor. Instead, KMV derives
the actual probability of default, the Expected Default Frequency (EDF), for each obligor based on a Merton (1974)’s type model of the firm. The probability of default is thus a function of the firm’s capital structure, the volatility of the asset returns and the current asset value. The EDF is firm-specific, and can be mapped into any rating system to derive the equivalent rating of the obligor. EDFs can be viewed as a “cardinal ranking" of obligors relative to default risk, instead of the more conventional "ordinal ranking" proposed by rating agencies and which relies on letters like AAA, AA, etc. Contrary to CreditMetrics/CreditVaR I, KIEV’S model does not make any explicit reference to the transition probabilities which, in KIEV’S methodology, are already imbedded in the EDFs. Indeed, each value of the EDF is associated with a spread curve and an implied credit rating.

As for CreditMetrics/CreditVaR I, KMV’s model is also based on the option pricing approach to credit risk as originated by Merton (1974). Thus, credit risk is essentially driven by the dynamics of the asset value of the issuer. Given the current capital structure of the firm, i.e. the composition of its liabilities: equity, short-term and long-term debt, convertible bonds, etc., once the stochastic process for the asset value has been specified, then the actual probability of default for any time horizon, 1 year, 2 years, etc. can be derived. Figure (3-5) in the previous section depicts how the probability of default relates to the distribution of asset returns and the capital structure of the firm, in the simple case where the firm is financed by equity and a zero-coupon bond.

K MV best applies to publicly traded companies for which the value of equity is market determined. The information contained in the firm’s stock price and balance sheet can then be translated into an implied risk of default as shown in the next section.
I. Actual probabilities of default: EDFs (expected default frequencies)

The derivation of the probabilities of default proceeds in 3 stages which are discussed below: estimation of the market value and volatility of the firm’s assets; calculation of the distance-to-default, which is an index measure of default risk; and scaling of the distance-to-default to actual probabilities of default using a default database.

A: Estimation of the asset value, $V_A$, and the volatility of asset return, $\sigma_A$

In the contingent claim approach to the pricing of corporate securities, the market value of the firm’s assets is assumed to be lognormal distributed, i.e. the log-asset return follows a normal distribution. This assumption is quite robust and, according to KMV’s own empirical studies, actual data conform quite well to this hypothesis. In addition the distribution of asset return is stable over time, i.e. the volatility of asset return stays relatively constant.
If all the liabilities of the firm were traded and marked-to-market every day, then the task of assessing the market value of the firm’s assets and their volatility would be straightforward. The firm’s assets value would be simply the sum of the market values of the firm’s liabilities, and the volatility of the asset return could be simply derived from the historical time series of the reconstituted assets value.

In practice, however, only the price of equity for most public firms is directly observable, and in some cases part of the debt is actively traded. The alternative approach to assets valuation consists in applying the option pricing model to the valuation of corporate liabilities as suggested in Merton (1974).

In order to make the model tractable, KMV assumes that the capital structure is only composed of equity, short-term debt which is considered equivalent to cash, long-term debt which is assumed to be perpetuity, and convertible preferred shares. With these simplifying assumptions it is then possible to derive analytical solutions for the value of equity, VE, and its volatility, σE:

\[
V_E = f(V_A, \sigma_A, K, c, r),
\]

\[
\sigma_E = g(V_A, \sigma_A, K, c, r),
\]

(65)

Where K denotes the leverage ratio in the capital structure, c is the average coupon paid on the long-term debt and r the risk-free interest rate.

If σE were directly observable, like the stock price, we could resolve, simultaneously (65) and (66) for VA and σA. But the instantaneous equity volatility, σE, is relatively unstable, and is in fact quite sensitive to the change in assets value, and there is no simple way to measure precisely σE from market data. Since only the value of equity, VE, is directly observable, we can back out VA from (65) which becomes a function of the observed equity value, or stock price, and the volatility of asset returns:

\[
V_A = h(V_E, \sigma_A, K, c, r).
\]

126
To calibrate the model for $\sigma_A$, KMV uses an iterative technique.

**B: Calculation of the DD**

In the option pricing framework default, or equivalently bankruptcy, occurs when assets value falls below the value of the firm's liabilities. In practice, default is distinct from bankruptcy which corresponds to the situation where the firm is liquidated, and the proceeds from the assets sale is distributed to the various claim holders according to pre-specified priority rules. Default is the event when a firm misses a payment on a coupon and/or the reimbursement of principal at debt maturity. Cross-default clauses on debt contracts are such that when the firm misses a single payment on a debt, it is declared in default on all its obligations. Figure (3-13) shows the number of bankruptcies and defaults for the period 1973-1994.

*Figure (3-13) Bankruptcies and defaults, quarterly from 1973 to 1997*

US Bankruptcies and Defaults

- **Bankruptcies**
- **Default**

(Source: KMV Corporation).
KMV has observed from a sample of several hundred companies that firms default when the asset value reaches a level somewhere between the value of total liabilities and the value of short-term debt. Therefore, the tail of the distribution of asset value below total debt value may not be an accurate measure of the actual probability of default. Loss of accuracy may also result from other factors such as the non-normality of asset return distribution, the simplifying assumptions about the capital structure of the firm. This can be further aggravated by the fact that there are unknown indrawn commitments (lines of credit) which, in case of distress, will be used and as a consequence may unexpectedly increase liabilities while providing the necessary cash to honor promised payments.

For all these reasons, KMV implements an intermediate phase before computing the probabilities of default. As shown in Figure (3-14), which is similar to Figure (3-5), KMV computes an index called "distance-to-default" (DD). DD is the number of standard deviations between the mean of the distribution of the asset value, and a critical threshold, the "default point", set at the par value of current liabilities including short term debt to be serviced over the time horizon, plus half the long-term debt. Formally DD is defined as follows:

\[
DD = \text{distance-to-default} = \frac{STD + 1/2LTD - E(V_t)}{\text{STD}}
\]

where:
- \( \text{STD} \) sort-term debt,
- \( \text{LTD} \) long-term debt,
- \( \text{DPT} \) default point = \( \text{STD} + 1/2\text{LTD} \),
- \( \text{DD} \) distance-to-default which is the distance between the expected Asset value in 1-year, \( E(V_t) \), and the default point, \( \text{DPT} \) expressed in standard deviation of future asset returns:
Given the lognormal assumption of asset values as specified in (54) then, according to (57), the DD expressed in unit of asset return standard deviation at time horizon $T$, is

$$DD = \frac{\ln(V_0/DPT_T) + (\mu - (1/2)\sigma^2)T}{\sigma\sqrt{T}},$$

(68)

Where $V_0$ is current market value of assets, $DPT_T$ the default point at time horizon $T$, $\mu$, the expected net return on assets, $\sigma$ the annualized asset volatility.

It follows that the shaded area below the default point is equal to $N(-DD)$. 
C: Derivation of the probabilities of default from the DD

This last phase consists of mapping the DD to the actual probabilities of default, for a given time horizon. These probabilities are called by KMV, EDFs, for Expected Default Frequencies.

Based on historical information on a large sample of firms, which includes firms which defaulted one can estimate, for each time horizon, the proportion of firms of a given ranking, say DD=4, which actually defaulted after 1-year.

This proportion, say 40 bp, or 0.4%, is the EDF as shown in Figure (3-15).

**Example:**

Current market value of assets \( V_0 = 1000 \)
Net expected growth of assets per annum \( 20\% \)
Expected asset value in 1 year \( V_0 \times (1.20) = 1200 \)
Annualized asset volatility, \( \sigma \) \( 100 \)
Default point \( 800 \)

Then:

\[
DD = \frac{(1200-800)}{100} = 4
\]

*Figure (3-15) Mapping of the "distance-to-default" into the "expected default frequencies", for a given time horizon.*
Assume that among the population of all the firms with a DD of 4 at one point in time, say 5000 firms, 20 defaulted 1 year later, then:

\[
\text{EDF}_{1 \text{ yr}} = \frac{20}{5000} = 0.004 = 0.4\% \text{ or } 40 \text{ bp.}
\]

The implied rating for this probability of default is BB\(^+\).

The next example is provided by KMV and relates to Federal Express on two different dates: November 1997 and February 1998.

**Example:**

Federal Express ($ figures are in billions of US$).

<table>
<thead>
<tr>
<th></th>
<th>November 1997</th>
<th>February 1998</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market capitalization</td>
<td>$7.7</td>
<td>$7.3</td>
</tr>
<tr>
<td>Book liabilities</td>
<td>$4.7</td>
<td>$4.9</td>
</tr>
<tr>
<td>Market value of assets</td>
<td>$12.6</td>
<td>$12.2</td>
</tr>
<tr>
<td>Asset volatility</td>
<td>15%</td>
<td>17%</td>
</tr>
<tr>
<td>Default point</td>
<td>$3.4</td>
<td>$3.5</td>
</tr>
</tbody>
</table>

\[
\text{DD} = \frac{(12.6 - 3.4)}{(0.15 \times 12.6)} = 4.9 \quad \text{and} \quad \frac{(12.2 - 3.5)}{(0.17 \times 12.2)} = 4.2
\]

\[
\text{EDF} = 0.06\% (6 \text{ bp}) \quad \text{and} \quad 0.11\% (11 \text{ bp})
\]

\[
= \text{AA}^- \quad \text{and} \quad = A^-
\]

This last example illustrates the main causes of changes for an EDF, i.e. the variations in the stock price, the debt level (leverage ratio), and the asset volatility which is the expression of the perceived degree of uncertainty on the business value.

**D: EDF as a predictor of default**

KMV has provided the service "Credit Monitor" of estimated EDFs since 1993. EDFs have proved to be a useful leading indicator of default, or at least of the degradation of the creditworthiness of issuers. When the financial situation of a company starts to deteriorate, EDFs tend to shoot up quickly until default occurs as shown in Figure (3-16). Figure (3-17) shows the evolution of equity.
value, asset value, as well as the default point during the same period. On the vertical axis of both graphs the EDF in percent and the corresponding Standard & Poor’s rating are shown.

KMV has analyzed more than 2000 US companies that have defaulted or entered into bankruptcy over the last 20 years, these firms belonging to a large sample of more than 100,000 company-years with data provided by Compustat. In all cases KMV has shown a sharp increase in the slope of the EDF between 1 and 2 years prior to default.

Changes in EDFs tend also to anticipate at least by 1 year the downgrading of the issuer by rating agencies like Moody’s and Standard & Poor’s, as shown in Figure (3-18). Contrary to Moody’s and Standard & Poor’s historical default statistics, EDFs are not biased by periods of high or low defaults. Distant-to-default can be observed to decrease during recession periods.

Figure (3-16) EDF of a firm which actually defaulted versus EDFs of firms in various quartiles and the lower decile. (The quartiles and decile represent a range of EDFs for a specific credit class.)
Figure (3-17) Assets value, equity value, short-term debt and long-term debt of a firm which actually defaulted.

Asset Value

ST+LT

Equity value

12/93 08/93 02/94 08/94 02/95 02/96 08/96 02/97 08/97

Figure (3-18) EDF of a firm which actually defaulted versus Standard & Poor's rating.

EDF

Default

S & P

12/93 08/93 02/94 08/94 02/95 02/96 08/96 02/97 08/97
E: EDFs and ratings

Standard & Poor's risk ratings represent default probabilities only, while Moody's factors also include a measure of the probability of loss, i.e. EDF X LGD. Table (3-13) shows the correspondence between EDFs and the ratings of Standard & Poor's, Moody's, as well as the internal ratings of CIBC, Nationbank and Swiss Bank Corp. The ratings of Nationbank and Swiss Bank were published in their recent CLO transactions.

Within any rating class the default probabilities of issuers are clustered around the median. However, as we discussed earlier, the average default rate for each class is considerably higher than the default rate of the typical firm.

<table>
<thead>
<tr>
<th>EOF (bp)</th>
<th>S&amp;P</th>
<th>Moody's</th>
<th>CIBS</th>
<th>Nation bank</th>
<th>SBC</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-4</td>
<td>&gt; AA</td>
<td>&gt; Aa2</td>
<td>1</td>
<td>AAA</td>
<td>C1</td>
</tr>
<tr>
<td>4-10</td>
<td>AA/A</td>
<td>A1</td>
<td>2</td>
<td>AA</td>
<td>C2</td>
</tr>
<tr>
<td>10-19</td>
<td>A/BBBB+</td>
<td>Baa1</td>
<td>3</td>
<td>A</td>
<td>C3</td>
</tr>
<tr>
<td>19-40</td>
<td>BBB+/BBB-</td>
<td>Baa3</td>
<td>4</td>
<td>A/BB</td>
<td>C4</td>
</tr>
<tr>
<td>40-72</td>
<td>BBB-/BB</td>
<td>Ba1</td>
<td>4.5</td>
<td>BBB/BB</td>
<td>C5</td>
</tr>
<tr>
<td>72-101</td>
<td>BB/BB-</td>
<td>Ba3</td>
<td>5</td>
<td>BB</td>
<td>C6</td>
</tr>
<tr>
<td>101-143</td>
<td>BB-/B+</td>
<td>B1</td>
<td>5.5</td>
<td>BB</td>
<td>C7</td>
</tr>
<tr>
<td>143-202</td>
<td>B+/B</td>
<td>B2</td>
<td>6</td>
<td>BB/B</td>
<td>C8</td>
</tr>
<tr>
<td>202-345</td>
<td>B/B-</td>
<td>B2</td>
<td>6.5</td>
<td>B</td>
<td>C9</td>
</tr>
</tbody>
</table>

This is because each rating class contains a group of firms which have much higher probabilities of default, due to the approximate exponential change in default rates as default risk increases. These are firms which should have been downgraded, but as of yet no downgrade has occurred. There are also firms that should have been upgraded. Table (3-14) shows the variation of the EDFs within each rating class.
Three conclusions follow from the previous analysis. First, since the rating agencies are slow to change their ratings, the historical frequency of staying in a rating class should overstate the true probability of keeping the same credit quality. Second, the average historical probability of default overstates the true probability of default for typical firms within each rating class, due to the difference between the mean and the median default rates. Third, if both the probability of staying in a given rating class, and the probability of default are too large, then the transition probabilities must be too small.

KMV has constructed a transition matrix based upon default rates rather than rating classes. They start by ranking firms into groups based on non-overlapping ranges of default probabilities that are typical for a rating class. For instance all firms with an EDF less than 2 bp are ranked AAA, then those with an EDF comprised between 3 and 6 bp are in the AA group, firms with an EDF of 7-15 bp belong to A rating class, and so on. Then using the history of changes in EDFs we can produce a transition matrix shown in Table (3-15) which is similar in structure to the one produced as Table 1 and reproduced as Table (3-16).

However, the difference in the various probabilities between the two tables is striking, but as expected. According to KMV, except for AAA, the probability of staying in the same rating class is between half and one-third of historical rates produced by the rating agencies. KMV’s probabilities of default are also lower, especially for the low grade
quality. Migration probabilities are also much higher for KMV, especially for the grade above and below the current rating class.

These differences may have a considerable impact on the VaR calculations such as those derived in the previous section related to CreditMetrics.

Table (3.15) KMV 1-year transition matrix based on non-overlapping EDF ranges

<table>
<thead>
<tr>
<th>Initial rating</th>
<th>Rating at year-end (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AAA</td>
</tr>
<tr>
<td>AAA</td>
<td>66.26</td>
</tr>
<tr>
<td>AA</td>
<td>21.66</td>
</tr>
<tr>
<td>A</td>
<td>2.76</td>
</tr>
<tr>
<td>BBB</td>
<td>0.30</td>
</tr>
<tr>
<td>BB</td>
<td>0.08</td>
</tr>
<tr>
<td>B</td>
<td>0.01</td>
</tr>
<tr>
<td>CCC</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Source: KMV Corporation

Table (3.16) Transition matrix based on actual rating changes

<table>
<thead>
<tr>
<th>Initial rating</th>
<th>Rating at year-end (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AAA</td>
</tr>
<tr>
<td>AAA</td>
<td>98.81</td>
</tr>
<tr>
<td>AA</td>
<td>0.70</td>
</tr>
<tr>
<td>A</td>
<td>0.09</td>
</tr>
<tr>
<td>BBB</td>
<td>0.02</td>
</tr>
<tr>
<td>BB</td>
<td>0.03</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
</tr>
<tr>
<td>CCC</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Source: Standard & Poor’s CreditWeek (April 15, 1996).

II. Valuation model for cash flows subject to default risk

In CreditMetrics/CreditVaR I the valuation model is quite simplistic and has already been described in previous Sections. If 1-year is the time horizon, then the forward value of a bond is the discounted value of the future cash flows beyond 1 year, where the discount factors are derived from the forward yield curve. To each credit rating is associated a specific spread curve, and the distribution of future values follows from the transition probabilities.

In KMV the approach is quite different, and is consistent with the option pricing methodology to the valuation of contingent cash flows.
Given the term structure of EDFs for a given obligor, we can derive the net present value of any stream of contingent cash flows. The final step, discussed in the next section, consists of deriving the loss distribution for the entire portfolio.

More specifically, KMV’s pricing model is based upon the “risk neutral” valuation model, also named the Martingale approach to the pricing of securities, which derives prices as the discounted expected value of future cash flows. The expectation is calculated using the so-called risk neutral probabilities and not the actual probabilities as they can be observed in the market place from historical data or the EDFs. Assuming, for the time being, that we know how to derive the “risk neutral probabilities" from the EDFs, then the valuation of risky cash flows proceeds in two steps, first the valuation of the default-free component, and second, the valuation of the component exposed to credit risk.

A: Case of a single cash flow

Example: Valuation of a zero coupon bond with a promised payment in 1 year of $100, with a recovery of \( (1-LGD) \) if the issuer default, i.e. \( LGD \) is the loss given default, assumed to be 40% in this example illustrated in Figure (3-19).

The risk-free component, \( $100 \ (1-LGD) \) is valued using the default-free discount curve, i.e.

\[
P V_1 = PV \ (risk-free \ cash \ flow) = 100 \ (1-LGD)/ \ (1+r) = $54.5,
\]

Where \( r \) denotes the 1-year risk-free rate assumed to be 10%.

The risky cash flow is valued using the Martingale approach, i.e.

\[
P V_2 \ (risky \ cash \ flow) = E_Q \ (discounted \ risky \ cash \ flow),
\]

Where the expectation is calculated using the risk neutral probability. Denote by \( Q \), the risk neutral probability that the issuer defaults in 1-year from now, and it is assumed to be 20%, then:
Figure (3-19) Valuation of a single cashflow subject to default risk.

\[
\begin{align*}
\text{No default} & \quad \text{\$100} \quad \text{\$100(1-LGD)} \quad \text{\$100\times LGD} \\
\text{default} & \quad \text{\$100(1-LGD)} \quad \text{\$100(1-LGD)} \quad \text{\$0}
\end{align*}
\]

Risky bond \quad Default free component \quad Risky component

\[
\text{PV}_2 = \text{PV(\text{risky cash flow})} = \frac{100 \cdot LGD \cdot (1-Q) + 0 \cdot Q}{1+r} \\
= \frac{100 \cdot LGD \cdot (1-Q)}{1+r} = \$29.1.
\]

The present value of this zero coupon bond subject to default risk is the sum of the default-free component and the risky component, i.e.

\[
PV = PV_1 + PV_2 = \$54.5 + \$29.1 = \$83.6
\]

If the zero coupon bond were default free, its present value would simply be its discounted value using the default-free interest rate, i.e.

\[
\$100 = (1+r) = \$90.9
\]

We can then compute the implicit discount rate, \( R \), which accounts for default risk, i.e.

\[
R = r + CS
\]

Where \( CS \) denotes the credit spread. It is solution of

\[
\frac{100 \cdot (1-LGD)}{1+r} + \frac{100 \cdot LGD \cdot (1-Q)}{1+r} = \frac{100}{1+r + CS}.
\]

Solving (69) for \( CS \) gives:

\[
CS = \frac{LGD \cdot Q \cdot (1+r)}{1-LGD \cdot Q}.  \tag{70}
\]
For this example, $R=19.6\%$, so that the 1-year credit spread for this issuer is 9.6\%.

**B: Generalized pricing model for a bond or a loan subject to default risk:**

The previous approach can be easily generalized to the valuation of a stream of cash flows $[C_1, C_i, \ldots, C_n]$:

$$PV = (1-LGD) \sum_{i=1}^{n} \frac{C_i}{(1+r_i)^{t_i}} + LGD \sum_{i=1}^{n} \frac{(1-Q_i)}{(1+r_i)^{t_i}},$$  \hspace{1cm} (71)

Or in continuous time notation,

$$PV = (1-LGD) \sum_{i=1}^{n} C_i e^{-r_i t_i} + LGD \sum_{i=1}^{n} (1-Q_i) C_i e^{-r_i t_i},$$ \hspace{1cm} (72)

Where $Q_i$ denotes the cumulative "risk neutral" EDF at the horizon $t_i$ and $r^*_i = \ln(1+r_i)$.

**Example:** What is the value of a 5-year bond with a face value of $100, which pays an annual coupon of 6.25\%? Let us assume that the risk-free interest rate is 5\%, the LGD is 50\% and the cumulative risk neutral probabilities are given in the table below.

<table>
<thead>
<tr>
<th>Time</th>
<th>$Q_i$ (%)</th>
<th>Discount Factor $1/(1+r_i)^{t_i}$</th>
<th>Cash flow</th>
<th>$PV_1$ (risk-free Cash flows)</th>
<th>$PV_2$ (risky Cash flows)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5) $= \frac{1}{2}(4) \times (3)$</td>
<td>(6) $= (5)[1-(2)]$</td>
</tr>
<tr>
<td>1</td>
<td>1.89</td>
<td>0.9512</td>
<td>6.25</td>
<td>2.97</td>
<td>2.92</td>
</tr>
<tr>
<td>2</td>
<td>4.32</td>
<td>0.9048</td>
<td>6.25</td>
<td>2.83</td>
<td>2.71</td>
</tr>
<tr>
<td>3</td>
<td>6.96</td>
<td>0.8607</td>
<td>6.25</td>
<td>2.69</td>
<td>2.50</td>
</tr>
<tr>
<td>4</td>
<td>9.69</td>
<td>0.8187</td>
<td>6.25</td>
<td>2.56</td>
<td>2.31</td>
</tr>
<tr>
<td>5</td>
<td>12.47</td>
<td>0.7788</td>
<td>106.25</td>
<td>41.37</td>
<td>36.21</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td>52.42</td>
<td>46.65</td>
</tr>
</tbody>
</table>

$$PV=PV_1+PV_2= 99.07,$$
This methodology also applies to simple credit derivatives like a default put:

Example: What is the premium of a 1-year default put which pays $1 in case the underlying bond defaults?

\[
\begin{array}{ccc}
\text{Buyer of put} & \text{Premium} & \text{Seller of put} \\
\text{protection} & $0 & \text{protection} \\
\text{No default} & \text{Default} \\
\end{array}
\]

Assume a risk neutral probability \( Q = 0.39\% \) and an interest rate \( r = 5.8\% \), then \( \text{Premium} = Q / (1 + r) = 0.0039 / 1.058 = 0.37\% \).

III. Derivation of the risk neutral EDFs

Under the risk neutral probability measure the expected return on all securities is the default free interest rate, \( r \), for any horizon, say \( T \). Therefore, the risk neutral EDF, or \( Q \), is defined as the probability of default, i.e. the probability that the value of the assets at time \( T \) falls below the default point \( DPT_T \), under the modified risk neutral process for the asset value, \( V_t^* \):

\[
Q = \Pr\left[ V_T^* \leq DPT_T \right] = \Pr\left[ \ln V_0 + \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} Z_r \leq \ln DPT_T \right] = \Pr\left[ Z \leq \frac{\ln \left( \frac{V_0}{DPT_T} \right) + \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right] = N(-d_2^*),
\]

Where \( N(\cdot) \) is the cumulative standard normal distribution and

\[
d_2^* = \frac{\ln \left( \frac{V_0}{DPT_T} \right) + \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}},
\]

With \( (d V_t^*/ V_t^*) = r dt + \sigma dW_t \) where \( W_t \) is a standard Brownian motion, and is normally distributed with zero mean and variance equal to \( T \).
If the EDF was precisely the shaded area under the default point in Figure (3-6), then we would have exactly: $\sqrt{TZ_T} = W_T - W_0$

$EDF_T = N(-d_2)$

Where $d_2$ has already been defined in (56), i.e.

$$d_2 = \frac{ln[V_0/D^{PT}] + \left( \frac{\mu - \left(1/2\right)\sigma^2}{\sigma} \right)T}{\sigma\sqrt{T}}.$$

Since $-d_2 + ((\mu - r)\sqrt{T}/\sigma = -d_2^*$, it thus follows that the cumulative risk neutral $EDF, Q_T$, at horizon $T$ can be expressed as

$$Q_T = N\left[N^{-1}(EDF) + \left(\frac{\mu - r}{\sigma}\right)\sqrt{T}\right]. \quad (74)$$

Since $\mu \geq r$ it follows that $Q_T \geq EDF_T$ i.e. the risk neutral probability of default, after adjusting for the price of risk, is higher than the actual probability of default.

According to the CAPM

$$\mu - 1 = \beta \pi \quad (75)$$

With

$$\beta = \text{beta of the asset with the market} = \frac{cov(R, R_M)}{var(R_M)} = \rho_{R, R_M} \frac{\sigma}{\sigma_M}, \quad (76)$$

Where $R$ and $R_M$ denote the return of the firm’s asset and the market portfolio, respectively; $\sigma$ and $\sigma_M$ are the volatility of the asset return and the market return, respectively; $\rho$ is the correlation between the asset’s return and the market’s return.

$$\pi = \text{market risk premium for a unit of beta risk} = \mu_M - r. \quad (77)$$

Where $\mu_M$ and $\mu$ denote the expected return on the market portfolio and the firm’s assets, respectively, and $r$ is the risk-free rate.

It follows that

$$\frac{\mu - 1}{\sigma} = \beta \pi = \rho_{R, R_M} \frac{\pi}{\sigma_M} = \rho \frac{\pi}{\sigma} = \rho U, \quad (78)$$
Where $U = \pi / \sigma_M$ denotes the market Sharpe ratio, i.e. the excess return per unit of market volatility for the market portfolio.

Substituting (78) into (74) we obtain:

$$Q_T = N \left[ N^{-1}(EDF_T) + \rho \frac{\pi}{\sigma_M} \sqrt{T} \right].$$

(79)

Rho is estimated by the linear regression of asset returns against market returns:

$$R = \alpha + \beta R_M + \epsilon,$$

(80)

Where $\alpha$ is the intercept of the regression and $\epsilon$ the error term. Rho is simply the square root of the R-squared of this regression.

In practice, $\pi$, the market risk premium is difficult to estimate statistically, and it varies over time. In addition, the EDF is not precisely the shaded area under the default point in Figure (3-5), and the asset return distribution is not exactly normal. For all these reasons, KMV estimates the risk neutral EDF, $Q_T$, by calibrating the market Sharpe ratio, $U$, and $\theta$ in the following relation, using bond data:

$$Q_T = N \left[ N^{-1}(EDF_T) + \rho U^0 \right].$$

(81)

Where $\theta$ is a time parameter which should be, in theory, equal to 1/2.

Assuming we have derived the zero-coupon curve for an obligor, then according to the pricing model (72) presented earlier:

$$e^{-R_{\tilde{t}_i}} = \left[ (1-LGD) + (1-Q_i)LGD \right] e^{-\tilde{r}_{\tilde{t}_i}}.$$

(82)

For $i = 1, \ldots, n$, where $R_{\tilde{t}_i}$ is the continuously compounded zero-coupon interest rate for the obligor, i.e. $R_{\tilde{t}_i} = \ln (1+R_{\tilde{t}_i})$, for maturity $t_i$, $r_{\tilde{t}_i}$ the continuously compounded zero-coupon risk-free rate i.e. $r_{\tilde{t}_i} = \ln(1 + ri)$, for maturity $t_i$, so that
Combining (81) and (83) we obtain:

\[ \tilde{R} - \tilde{r} = -\frac{1}{t_i} \ln [1 - Q_i \text{LGD}] . \tag{83} \]

Where \( \tilde{R}_i = \tilde{r}_i \) is the obligor’s corporate spread for maturity \( t_i \), which is directly extracted from corporate bond data. \( U \) and \( \theta \) are calibrated to produce the best fit of (84) in the least square sense.

IV. Credit-VaR and calculation of the capital charge for a Portfolio

KMV does not simulate the full forward distribution of the portfolio values at the credit horizon, \( H \). Instead, KMV derives analytically the loss distribution of the portfolio at this horizon. For the sake of simplicity assume that all bonds mature at time \( T \), greater than the credit horizon, \( H \).

Denote by \( V_{H/ND} \) the discounted value of the portfolio at time \( H \), assuming no default, and \( VH \), the equilibrium value of the portfolio at time \( H \), derived from the valuation model presented in previous sections. The portfolio loss at time \( H \) is defined as the difference between the risk-less value of the portfolio and its market value at that time:

\[ L = V_{H/ND} - VH \]

Note that \( VH \) is unknown at time 0, only its probability distribution can be derived so that the loss, \( L \), is a random variable.

Under some simplifying assumption it can be shown that the limiting distribution of the portfolio loss, when the portfolio is widely diversified across issuers, is a normal inverse for which it is relatively easy to compute the percentiles. The normal inverse distribution is highly skewed and leptokurtic.
Table (3-17) shows some values of the α-percentile, $L_\alpha$, expressed as the number of standard deviations from the mean, for several values of parameters. The α-percentiles of the standard normal distribution are shown for comparison.

$p$ is the probability of default of one bond in the portfolio (all bonds are assumed to have the same probability of default).

$\rho$ is the pair-wise asset correlation, constant across all issuers.

$\alpha$ is the confidence level.

The expected loss for the portfolio is $EL=p$ and its standard deviation is denoted by $s$.

### Table (3-17) Values of $(L_\alpha - p)/s$ for the normal inverse distribution

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\rho$</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 0.01$</th>
<th>$\alpha = 0.001$</th>
<th>$\alpha = 0.0001$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.1</td>
<td>1.19</td>
<td>3.8</td>
<td>7.0</td>
<td>10.7</td>
</tr>
<tr>
<td>0.01</td>
<td>0.4</td>
<td>0.55</td>
<td>4.5</td>
<td>11.0</td>
<td>18.2</td>
</tr>
<tr>
<td>0.001</td>
<td>0.1</td>
<td>0.98</td>
<td>4.1</td>
<td>8.8</td>
<td>15.4</td>
</tr>
<tr>
<td>0.001</td>
<td>0.4</td>
<td>0.12</td>
<td>3.2</td>
<td>13.2</td>
<td>31.7</td>
</tr>
<tr>
<td>Normal</td>
<td></td>
<td>1.28</td>
<td>2.3</td>
<td>3.1</td>
<td>3.7</td>
</tr>
</tbody>
</table>

These values manifest the extreme non-normality of the distribution. Suppose a lender holds a large portfolio of bonds issued by obligors where pair-wise asset correlation is $\rho=0.4$ and where probability of default is $p=0.01$ If the desired confidence level is $\alpha=0.001$ (10 bp), then the required capital should be enough to cover 11 times the portfolio loss standard deviation.

To be more specific the capital charge is

$\alpha$-percentile - expected spread revenue;

Where

$\alpha$-percentile is expressed in absolute value;

$Expected$ $spread$ $revenue = total$ $spread$ $revenue - expected$ $loss$

And
Total spread revenue = annualized expected revenue over funding cost.

V. Asset return correlation model

CreditMetrics/CreditVaR I and KMV derive asset return correlations from a structural model which links correlations to fundamental factors. By imposing a structure on the return correlations, sampling errors inherent in simple historical correlations are avoided, and a better accuracy in forecasting correlations is achieved. In addition, there is a practical need to reduce dramatically the number of correlations to be calculated. Assume that a bank is dealing with $N=1000$ different counterparties. Then, we have $N(N-1)/2$ different correlations to estimate, i.e., 499,500. This number is staggering. Multi-factor models of asset returns reduce correlations to be calculated to those between the limited numbers of common factors affecting asset returns.

It is assumed that the firm’s asset returns are generated by a set of common, or systematic risk factors, and idiosyncratic factors. The idiosyncratic factors are either firm, or country or industry-specific, and do not contribute to asset return correlations, since they are not correlated with each other and not correlated with the common factors. Asset return correlations between two firms are only explained by the common factors to all firms. Only the risks associated with the idiosyncratic risk factors can be diversified away through portfolio diversification, while the risk contribution of the common factors is, on the contrary, non diversifiable.

For the sake of illustration, assume the asset return generating process for all firms is

$$R_k = \alpha_k + \beta_{1k} I_1 + \beta_{2k} I_2 + \varepsilon_k \quad \text{for } k=1, \ldots, N, \quad (87)$$

Where $N$ is the number of obligors (firms), $r_k$ the asset return for firm $k$, $\alpha_k$ the component of asset return independent of common factors, $I_1, I_2$ are the common factors, $\beta_{1k}, \beta_{2k}$ are the
expected changes in \( r_k \), given a change in common factors 1 and 2, respectively, \( \varepsilon_k \) the idiosyncratic risk factor with zero mean, and assumed to be uncorrelated with all the common factors, as well as with the idiosyncratic risk factors of the other firms.

Then, from elementary statistics we can derive the well-known results in portfolio theory:

\[
\begin{align*}
\text{var}(r_k) &= \sigma_k^2 \\
&= \beta_{1k}^2 \text{var}(I_1) + \beta_{2k}^2 \text{var}(I_2) + \text{var}(\varepsilon_k^2) + 2\beta_{1k}\beta_{2k} \text{cov}(I_1,I_2), \\
\text{cov}(r_i,r_j) &= \sigma_{ij} \\
&= \beta_{ij}\beta_{2j} \text{var}(I_1) + \beta_{2i}\beta_{2j} \text{var}(I_2) + (\beta_{ii}\beta_{2j} + \beta_{2i}\beta_{1j}) \text{cov}(I_1,I_2).
\end{align*}
\]

If we denote by \( \rho_{ij} \) the asset return correlation between firm \( i \) and firm \( j \), then

\[
\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i\sigma_j}.
\]

To derive the asset return correlation between any number of firms we only need, according to (88)-(90), to estimate the \( \beta_{ik} \), i.e., \( 2N \) parameters, and the covariance matrix for the common factors, i.e., \( 3 \) parameters. In the previous example where we considered \( N=1000 \) firms, the implementation of this 2-factor model would only require an estimate of 2003 parameters instead of 499,500 different historical asset return correlations. For \( K \) common factors the number of parameters to be estimated is \( KN+K(K-1)/2 \). If \( K=10 \) then this number becomes 10,045. This result can be easily generalized to any number of common factors and idiosyncratic risk components.

The issue now is to specify the factor structure. CreditMetrics and KMV are proposing relatively similar models, and in the following we will only present KMV’s model which is more comprehensive and elaborated.
KMV constructs a three-layer factor structure model as shown in Figure (3-20):

**First level**: a composite company-specific factor, which is constructed individually for each firm based on the firm's exposure to each country and industry,

**Second level**: country and industry factors,

**Third level**: global, regional and industrial sector factors.

The first level of the structure divides between firm specific, or idiosyncratic risk, and common, or systematic risk. The first, systematic risk is captured by a single composite index, which is firm
specific, and which is constructed as a weighted sum of the firm’s exposure to country and industry factors defined at the second level of the structure:

Where \( r_k \) is asset return for firm \( k \), \( CF_k \) the composite factor for firm \( k \), \( \beta_k \) the firm \( k \)’s response to composite factor, i.e., expected change in \( r_k \) given a change in composite factor and \( \varepsilon_k \) firm \( k \)’s specific risk factor.

\[
r_k = \beta_k CF_k + \varepsilon_k \text{ for all firms } k = 1, \ldots, N,
\]

The composite factor is constructed as the sum of the weighted country and industry factors specified at the second level of the structure:

\[
CF_k = \sum_m \alpha_{km} C_m + \sum_n \alpha_{kn} I_n,
\]

Where \( C_m \) is rate of change on country risk factor \( m \), \( I_n \) the rate of change on industry risk factor \( n \), \( \alpha_{km} \) the weight of firm \( k \) in country \( m \), with the constraint that \( \sum_m \alpha_{km} = 1 \) and \( \alpha_{kn} \) the weight of firm \( k \) in industry \( n \), with the constraint that \( \sum_n \alpha_{kn} = 1 \).

For example, consider a Canadian firm which has two lines of business, and assume that the data is extracted from Compustat:

<table>
<thead>
<tr>
<th>Business line</th>
<th>SIC</th>
<th>Assets (%)</th>
<th>Sales (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lumber and forestry</td>
<td>2431</td>
<td>35</td>
<td>45</td>
</tr>
<tr>
<td>Paper production</td>
<td>2611</td>
<td>65</td>
<td>55</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>100</strong></td>
<td><strong>100</strong></td>
</tr>
</tbody>
</table>

In the above table SIC denotes the Standard Industrial Classification which is a US based business classification system.

To determine the weight by industry we average the asset and sales breakdowns. Thus for the above example the weight for Lumber and Forestry is:

\[
40\% = (35\% + 45\%)/2,
\]
And for Paper it is

\[ 60\% = \frac{65\% + 55\%}{2}, \]

Note that by construction the weights add up to 100%. The country exposures are calculated in a similar manner and should also sum up to 100%. In this example, we assume a 100% exposure to Canada. Then the composite factor can be written as

\[ CF = 1.0C_{Canada} + 0.6I_{paper} + 0.4I_{lumber}. \]

At the third level of the factor structure the risk of countries and industries is further decomposed into systematic and idiosyncratic components. The systematic component is captured by basic factors like: global economic effect, regional factor effect and sector factor effect. While the common factor is firm-specific, the third level factors are the same for all countries and all industries:

\[
\begin{align*}
\begin{bmatrix}
\text{Country} \\
\text{return}
\end{bmatrix} &= \begin{bmatrix}
\text{Global} \\
\text{economic} \\
\text{effect}
\end{bmatrix} + \begin{bmatrix}
\text{Regional} \\
\text{factor} \\
\text{effect}
\end{bmatrix} + \begin{bmatrix}
\text{Sector} \\
\text{Factor} \\
\text{effect}
\end{bmatrix} + \begin{bmatrix}
\text{Country} \\
\text{specific} \\
\text{risk}
\end{bmatrix}, \\
\begin{bmatrix}
\text{Industry} \\
\text{return}
\end{bmatrix} &= \begin{bmatrix}
\text{Global} \\
\text{economic} \\
\text{effect}
\end{bmatrix} + \begin{bmatrix}
\text{Regional} \\
\text{factor} \\
\text{effect}
\end{bmatrix} + \begin{bmatrix}
\text{Sector} \\
\text{factor} \\
\text{effect}
\end{bmatrix} + \begin{bmatrix}
\text{Industry} \\
\text{specific} \\
\text{risk}
\end{bmatrix}.
\end{align*}
\]

We can now express this factor structure into a form similar to (87) from which it is easy to derive the asset return correlations (90).

### 3.2.3 CreditRisk$^+$ model

CreditRisk$^+$ applies an actuarial science framework to the derivation of the loss distribution of a bond/loan portfolio. Only default risk is modeled, not downgrade risk. Contrary to KMV, default risk is not related to the capital structure of the firm. In CreditRisk$^+$ no assumption is made about the causes of default: an obligor $A$ is either in default with probability $P_A$, or it is not in default with probability $1 - P_A$. It is assumed that:
• For a loan, the probability of default in a given period, say 1 month, is the same for any other month;
• For a large number of obligors, the probability of default by any particular obligor is small, and the number of defaults that occur in any given period is independent of the number of defaults that occur in any other period.

Under those circumstances, the probability distribution for the number of defaults, during a given period of time (say 1 year) is well

\[ P(n \text{ defaults}) = \frac{\mu^n e^{-\mu}}{n!} \text{ for } n = 0, 1, 2, \ldots, \] (91)

Where

\[ \mu = \text{average number of defaults per year; } \]

\[ \mu = \sum P_A; \text{ where } P_A \text{ denotes the probability of default for obligor } A: \]

The annual number of defaults, \( n \), is a stochastic variable with mean \( \mu \), and standard deviation \( \sqrt{\mu} \). The Poisson distribution presents the nice property to be fully specified by only one parameter \( \mu \). For example, if we assume \( \mu = 3 \) then the probability of no default in the next year is

\[ P(0 \text{ default}) = \frac{3^0 e^{-3}}{0!} = 0.05 = 5\% \]

And the probability of exactly 3 defaults is

\[ P(3 \text{ defaults}) = \frac{3^3 e^{-3}}{3!} = 0.224 = 22.4\%. \]

I. CreditRisk+ framework

The distribution of default losses for a portfolio is derived in two stages, as shown in Figure (3-21).
A: Frequency of default events (building block #1)

So far we have assumed that a standard Poisson distribution approximates the distribution of the number of default events. Then we should expect the standard deviation of the default rate to be approximately equal to the square root of the mean, i.e., \( \sqrt{\mu} \), where \( \mu \) is the average default rate. According to Table (3-3), for obligors in rating category \( B \), we expect a standard deviation of the default rate to be close to \( \sqrt{7.27} \), i.e., 2.69, while Table (3-3) reports an actual standard deviation of 5.1. We derive similar observations for \( Baa \) and \( Ba \) obligors. Under those circumstances the Poisson distribution will underestimate the actual probability of default. This is not a surprising result when we observe the variability of default rates over time. As a matter of fact, we expect the mean default rate to change over time depending on the business cycle.

Still the Poisson distribution can be used to represent the default process, but with the additional assumption that the mean default rate is itself stochastic with mean \( I \) and standard deviation \( \sigma_\mu \).

Assuming a stochastic default rate makes the distribution of defaults more skewed with a fat right tail (see Figure (3-22)).
B: Severity of the losses (building block #2)

In the event of default of an obligor, the counterparty incurs a loss equal to the amount owned by the obligor (the exposure, i.e., the marked-to-market value if positive, and zero otherwise, at the time of default) less a recovery amount (see Table 3-6).

In CreditRisk⁺ the exposure for each obligor is adjusted by the anticipated recovery rate, in order to calculate the loss given default. These adjusted exposures are exogenous to the model, and are independent of market risk and downgrade risk.

C: Distribution of default losses for a portfolio (building block #3)

In order to derive the loss distribution for a well-diversified portfolio, the losses (exposures, net of the recovery adjustments) are divided into bands, with the level of exposure in each band being approximated by a single number.

*Figure (3-22) Distribution of default events*

(Source: CreditRisk⁺).
Example: Suppose the bank holds a portfolio of loans and bonds from 500 different obligors, with exposures between $50,000 and $1 million.

<table>
<thead>
<tr>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obligor</td>
</tr>
<tr>
<td>Exposure</td>
</tr>
<tr>
<td>Probability of default</td>
</tr>
<tr>
<td>Expected loss</td>
</tr>
</tbody>
</table>

Note: In CreditRisk+ the exposure is the forward value of the facility times the loss given default rate.

In the following table we only show the exposures for the first 6 obligors.

<table>
<thead>
<tr>
<th>Obligor A</th>
<th>Exposure ($) (loss giver Default) ( L_A )</th>
<th>Exposure (in $100,000) ( v_j )</th>
<th>Round-off Exposure (in $100,000) ( v_j )</th>
<th>Band j</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>150,000</td>
<td>1.5</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>460,000</td>
<td>4.6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>435,000</td>
<td>4.35</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>370,000</td>
<td>3.7</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>190,000</td>
<td>1.9</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>480,000</td>
<td>4.8</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

The unit of exposure is assumed to be \( L=100,000 \). Each band \( j \), \( j = 1; \ldots; m \), with \( m=10 \), has an average common exposure: \( v_j=100,000 \times j \).

In CreditRisk+ each band is viewed as an independent portfolio of loans/bonds, for which we introduce the following notation:

<table>
<thead>
<tr>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common exposure in band ( j ) in units of ( L )</td>
</tr>
<tr>
<td>Expected loss in band ( j ) in units of ( L )</td>
</tr>
<tr>
<td>Expected number of defaults in band ( j )</td>
</tr>
</tbody>
</table>
Then, by definition we have
\[ \varepsilon_j = v_j + \mu_j \]

Hence,
\[ \mu_j = \frac{\varepsilon_j}{v_j} \]  \hspace{1cm} (92)

Denote by \( \varepsilon_A \) the expected loss for obligor \( A \) in units of \( L \), i.e.,
\[ \varepsilon_A = \frac{\lambda_A}{L} \]

Then, the expected loss over a 1-year period in band \( j \), \( \varepsilon_j \), expressed in units of \( L \), is just the sum of the expected losses \( \varepsilon_A \) of all the obligors belonging to band \( j \), i.e.,
\[ \varepsilon_j = \sum_{A:v_A=v_j} \varepsilon_A. \]

From (92) it follows that the expected number of defaults per annum in band \( j \) is
\[ \mu_j = \frac{\varepsilon_j}{v_j} = \sum_{A:v_A=v_j} \frac{\varepsilon_A}{v_j} = \sum_{A:v_A=v} \frac{\varepsilon_A}{V_A}. \]

The table below provides an illustration of the results of those calculations:

<table>
<thead>
<tr>
<th>Band ( j )</th>
<th>Number of obligors</th>
<th>( \varepsilon_j )</th>
<th>( \mu_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>70</td>
<td>25.2</td>
<td>6.3</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>35</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>60</td>
<td>14.4</td>
<td>2.4</td>
</tr>
<tr>
<td>7</td>
<td>50</td>
<td>38.5</td>
<td>5.5</td>
</tr>
<tr>
<td>8</td>
<td>40</td>
<td>19.2</td>
<td>2.4</td>
</tr>
<tr>
<td>9</td>
<td>40</td>
<td>25.2</td>
<td>2.8</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>4</td>
<td>0.4</td>
</tr>
</tbody>
</table>
To derive the distribution of losses for the entire portfolio we proceed as follows:

**Step 1:** Probability generating function for each band.

Each band is viewed as a portfolio of exposures by itself. The probability generating function for any band, say band $j$, is by definition:

$$G_j(z) = \sum_{n=0}^{\infty} p(loss = nL)z^n = \sum_{n=0}^{\infty} p(n defaults)z^{nv_j},$$

Where the losses are expressed in the unit $L$ of exposure.

Since we have assumed that the number of defaults follows a Poisson distribution (see expression (91)) then:

$$G_j(z) = \sum_{n=0}^{\infty} \frac{e^{-\mu_j} \mu_j^n}{n!} z^{nv_j} = \exp\{-\mu_j + \mu_j z^{v_j}\}.$$  \(93\)

To derive the distribution of losses for the entire portfolio we proceed as follows:

**Step 2:** Probability generating function for the entire portfolio.

Since we have assumed that each band is a portfolio of exposures, independent from the other bands, the probability generating function for the entire portfolio is just the product of the probability generating function for each band:

$$G(z) = \prod_{j=1}^{m} \exp\{-\mu_j + \mu_j z^{v_j}\} = \exp\left\{-\sum_{j=1}^{m} \mu_j + \sum_{j=1}^{m} \mu_j z^{v_j}\right\},$$ \(94\)

Where $\mu = \sum_{j=1}^{m} \mu_j$ denotes the expected number of defaults for the entire portfolio.

**Step 3:** Loss distribution for the entire portfolio.

Given the probability generating function (94) it is straightforward to derive the loss distribution, since

$$P(loss of nL) = \frac{1}{n!} \frac{d^n (G(z))}{dz^n} \bigg|_{z=0} \text{ for } n = 1, 2, \ldots,$$

These probabilities can be expressed in closed form and depend only on 2 sets of parameters: $\varepsilon_j$ and $v_j$. 

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II. Extensions of the basic model

CreditRisk$^+$ proposes several extensions of the basic one period, one factor model. First, the model can be easily extended to a multi-period framework, and second, the variability of default rates can be assumed to result from a number of "background" factors, each representing a sector of activity. Each factor, $k$, is represented by a random variable, $X_k$, which is the number of defaults in sector $k$, and which is assumed to be Gamma distributed. The mean default rate for each obligor is then supposed to be a linear function of the background factors, $X_k$. These factors are further assumed to be independent.

In all cases CreditRisk$^+$ derives a closed form solution for the loss distribution of a bond/loan portfolio which makes this approach very attractive from a computational standpoint.

III. Advantages and limits of CreditRisk$^+$

CreditRisk$^+$ presents the advantage of being relatively easy to implement. First, closed form expressions are derived for the probability of portfolio bond/loan losses, which make CreditRisk$^+$ computationally attractive. In addition, marginal risk contributions by obligor can be easily computed. Second, CreditRisk$^+$ focuses only on default, requiring relatively few inputs to estimates. For each instrument only the probability of default and the exposure are required.

The same limitations as for CreditMetrics and KMV apply, i.e., the methodology assumes no market risk. In addition, CreditRisk$^+$ ignores migration risk so that the exposure for each obligor is fixed and does not depend on eventual changes in the credit quality of the issuer, as well as the variability of future interest rates. Even in its most general form where the probability of default depends upon several stochastic background factors, exposures are still constant and not related to changes in these factors.
Finally, like CreditMetrics and KMV, CreditRisk$^+$ does not deal with non-linear products such as, e.g., options and foreign currency swaps.

### 3.2.4 CreditPortfolioView Model

CreditPortfolioView is a multi-factor model which is used to simulate the joint conditional distribution of default and migration probabilities for various rating groups in different industries, for each country, conditional on the value of macroeconomic factors like the unemployment rate, the rate of growth in GDP, the level of long-term interest rates, foreign exchange rates, government expenditures and the aggregate savings rate.

CreditPortfolioView is based on the casual observation that default probabilities, as well as migration probabilities, are linked to the economy. When the economy worsens both downgrades as well as defaults increase. It is the contrary when the economy becomes stronger. In other words, credit cycles follow business cycles closely. Since the state of the economy is, to a large extent, driven by macroeconomic factors, CreditPortfolioView proposes a methodology to link those macroeconomic factors to the default and migration probabilities.

Provided that data is available this methodology can be applied in each country, to different sectors and various classes of obligors which react differently over the business cycle like construction, financial institutions, agriculture, services, etc.

### I. Default prediction model

Default probabilities are modeled as a logit function where the independent variable is a country speculative grade specific index which depends upon current and lagged macroeconomic variables:
\[ P_{j,t} = \frac{1}{1 + e^{-Y_{j,t}}}, \]  

(95)

Where \( P_{j,t} \) is the conditional probability of default in period \( t \), for speculative grade obligors in country/industry \( j \), \( Y_{j,t} \) is the index value derived from a multifactor model described below.

Note that the logit transformation ensures that the probability takes a value between 0 and 1.

The macroeconomics index, which captures the state of the economy in each country, is determined by the following multi-factor model:

\[ Y_{j,t} = \beta_{j,0} + \beta_{j,1} X_{j,t,1} + \beta_{j,2} X_{j,t,2} + \ldots + \beta_{j,m} X_{j,t,m} + \nu_{j,t}. \]  

(96)

Where \( Y_{j,t} \) is the index value in period \( t \) for the \( j \)th country/industry/speculative grade, \( \beta_j = (\beta_{j,0} ; \beta_{j,1} ; \beta_{j,2} ; \ldots ; \beta_{j,m}) \) are coefficients to be estimated for the \( j \)th country/industry/speculative grade, \( X_{j,t} = (X_{j,t,1} ; X_{j,t,2} ; \ldots ; X_{j,t,m}) \) are period \( t \) values of the macroeconomics variables for the \( j \)th country/industry, \( \nu_{j,t} \) is the error term assumed to be independent of \( X_{j,t} \) and identically normally distributed, i.e.

\[ \nu_t \sim N(0, \sigma^2) \text{ and } \nu_t \sim N(0, \Sigma_v). \]

Where \( \nu_t \) denotes \( \nu_{j,t} \sim N(0, \sigma^2) \text{ and } \nu_{j,t} \sim N(0, \Sigma_v) \), the vector of stacked index innovations \( \nu_{j,t} \) and \( \Sigma_v \) is the \( j \times j \) covariance matrix of the index innovations.

The macroeconomics variables are specified for each country. When sufficient data is available the model can be calibrated at the country/industry level. Both the probability of default \( P_{j,t} \) and the index, \( Y_{j,t} \), are then defined at the country/industry level, and the coefficient \( \beta_j \) are calibrated accordingly.

In the proposed implementation, each macroeconomics variable is assumed to follow a univariate, auto-regressive model of order 2 (AR2):

\[ X_{j,t,d} = \gamma_{j,d,0} + \gamma_{j,d,1} X_{j,t,d-1} + \gamma_{j,d,2} X_{j,t,d-2} + \epsilon_{j,d,t}, \]  

(97)
Where \( X_{j,i,t-1}, X_{j,i,t-2} \) denote the lagged values of the macroeconomic variable \( X_{j,i,t} \), \( y_j = (y_{j,i,0}, y_{j,i,1}, y_{j,i,2}) \) are coefficients to be estimated, \( e_{j,i,t} \) is the error term assumed to be independent and identically distributed, i.e.,

\[
e_{j,i,t} \sim N(0, \sigma_{e_{j,i,t}}) \text{ and } e_t \sim N(0, \Sigma_e).
\]

Where \( e_t \) denotes the vector of stacked error terms \( e_{j,i,t} \) of the \( j \times i \) AR(2) equations \( \Sigma_e \) is the \( (j \times i) \times (j \times i) \) covariance matrix of the error terms \( e_t \).

To calibrate the default probability model defined by (95)-(97), one has to solve the system

\[
\begin{align*}
& Y_{j,i,t} = \beta_{j,0} + \beta_{j,1} X_{j,i,t} + \ldots + \beta_{j,m} X_{j,m,j} + e_{j,i,t},
\end{align*}
\]

(98)

Where the vector of innovations \( E_t \) is

\[
E_t = \begin{bmatrix} V_{t}^j \\ e_t \end{bmatrix} \sim N(0, \Sigma)
\]

With \( \Sigma = \begin{bmatrix} \Sigma_e & \Sigma_{e,r} \\ \Sigma_{e,r}^T & \Sigma_r \end{bmatrix} \)

Where \( \Sigma_{r,e} \) and \( \Sigma_{e,m} \) denote the cross correlation matrices.

Once the system (98) has been calibrated, then one can use the Cholesky decomposition of \( \Sigma \), i.e.,

\[
\Sigma = AA'
\]

(99)

To simulate the distribution of speculative default probabilities, first, draw a vector of random variables \( Z_t \sim N(0, I) \) where each component is normally distributed \( N(0, 1) \).

Then, calculate

\[
E_t = A'Z_t
\]

Which is the stacked vector of error terms \( v_{j,t} \) and \( e_{j,i,t} \). Using these realizations of the error terms, one can derive the corresponding values for \( Y_{j,t} \) and \( P_{j,t} \).
II. Conditional transition matrix

The starting point is the unconditional Markov transition matrix based on Moody's or Standard & Poor's historical data, which we denote by $\Phi M$.

Transition probabilities are unconditional in the sense that they are historical averages based on more than 20 years of data covering several business cycles, across many different industries.

As we discussed earlier, default probabilities for non-investment grade obligors is higher than average during a period of recession. Also downgrade migrations increase, while upward migrations decrease. It is the opposite during a period of economic expansion:

\[
\frac{SDP}{\Phi SDP} > 1 \quad \text{In economic recession; (100)}
\]

\[
\frac{SDP}{\Phi SDP} < 1 \quad \text{In economic expansion;}
\]

Where $SDP_t$ is the simulated default probability for a speculative grade obligor, $\Phi SDP_t$ the unconditional (historical average) probability of default for a speculative grade obligor.

CreditPortfolioView proposes to use these ratios (100) to adjust the migration probabilities in $\Phi M$ in order to produce a transition matrix, $M$, conditional on the state of the economy:

\[
M_t = M \left( P_{j,t} / \Phi SDP \right),
\]

Where the adjustment consists of shifting the probability mass into downgraded and defaulted states when the ratio $P_{j,t} / \Phi SDP$ is greater than 1, and vice versa if the ratio is less than 1. Since one can simulate $P_{j,t}$ over any time horizon $t = 1, \ldots, T$, this approach can generate multi-period transition matrices:

\[
M_T = \prod_{t=1}^{T} M \left( P_{j,t} / \Phi SDP \right). \quad (101)
\]
One can simulate many times the transition matrix (101) to generate the distribution of the cumulative conditional default probability for any rating, over any time period.

The same Monte Carlo methodology can be used to produce the conditional cumulative distributions of migration probabilities over any time horizon.