Chapter 3

Amphicheiral links and their HOMFLY polynomials

Link polynomials are powerful tools and they have a great importance in the study of knots and links because they have the property that if one oriented link is ambient isotopic to another then the two will have the same polynomial. The concept of HOMFLY polynomials was introduced in [34]. In this Chapter we show how certain bounds on the possible diagrams presenting a given oriented reduced alternating amphicheiral knot or link can be found from HOMFLY polynomials. Finally we give some applications of the HOMFLY polynomials on amphicheiral knots and links.

3.1 Introduction

Whether or not a molecule has mirror image symmetry is quite important chemically. A molecule that is distinct from its mirror image is said to be cheiral, whereas one that can chemically change itself into its mirror image is said to be amphicheiral. The word cheiral comes from the ancient Greek word cheir, which means 'hand'. A pair of hands is a prototypical example of a pair of cheiral structures, because a left hand can never change itself into a right hand. A pair cheiral molecules that are mirror images of one another are called enantiomers.

One important technique to detect the amphicheirality is to make use of a link polynomial. In general, these polynomials are associated with oriented knots and links, that is, links with an arrow assigned to each component to indicate a particular direction. Such polynomials are powerful tools and they play important role in the study of knots and links because they have the property that if one oriented link is ambient isotopic to another then
the two will have the same polynomial. If in particular link polynomial is a topological
invariant and if $K$ is an oriented link that is ambient isotopic to its mirror image $\bar{K}$, then
the polynomial of $K$ and the polynomial of its mirror image $\bar{K}$ will be the same. If these
polynomials are not the same, then right away we will know that $K$ is not amphicheiral.
Conway was the first to show that skein relations [24] can be used axiomatically for defining
a knot invariant. His discovery stimulated further beautiful work presenting polynomials
based on skein relations. By using these polynomials, some odd problems solved, e.g., Tait's
problem [61].

There are several useful link polynomials, including the Jones polynomial (Jones, 1985),
the Kauffman polynomial (Kauffman, 1987), and the two variable HOMFLY polynomial
(Freyd et al, 1985). These polynomial are somewhat similar in flavor, so we present only
one. The polynomial that we will present is the HOMFLY polynomial [34], named after
five of the authors who discovered it, Hoste, Ocneanu, Millett, Freyd, Lickorish and Yetter.
This polynomial was also independently discovered by Przytycki and Traczyk [54]. All of
the link polynomials are actually Laurent polynomials, which means that the variables in
the polynomials can be raised to negative as well as positive powers. Recently, Y. Altun has
studied the HOMFLY polynomial in his paper "On the HOMFLY polynomial" [8] where he
has calculated the HOMFLY polynomial for twist knots, based on the Jones polynomial.

In this Chapter we show how certain bounds on the possible diagrams presentating a given
oriented reduced alternating amphicheiral link can be found from its two variable polynomials
$P_K(\alpha, z)$, called HOMFLY polynomials [34] and finally we present some applications of the
HOMFLY polynomials to the reduced alternating amphicheiral links.

3.2 Some basic concepts and notations

In this section we recall some notations which will be used in the next sections.

Knot (link) polynomials are powerful tools because they have the property that if one
oriented knot(link) is ambient isotopic to another then the two will have the same polynomial.
If a particular link polynomial is a topological invariant and if $L$ is an oriented link that is
ambient isotopic to its mirror image $\bar{L}$, then the polynomial of $L$ and the polynomial of its
mirror image $\bar{L}$ will be the same.

In 1923, the famous American mathematician James Alexander [6, 7] derived a polynomial
invariant of knots and links from the fundamental group. This invariant is, certainly, weaker
than the fundamental group itself, but the invariant polynomial is much easier to recognize:
one can easily compare two polynomials (unlike groups given by their presentation). The
next stage of development of knot theory was the discovery of the Conway polynomial [24].
This discovery is based on so-called skein relations. These relations are purely combinatorial
and based on the notion of the planar diagram. The Alexander polynomial can also be interpreted in terms of skein relations. Moreover, Alexander knew about this. However, Conway was the first to show that skein relations can be used axiomatically for defining a knot invariant. This discovery stimulated further beautiful work presenting polynomials based on skein relations. By using these polynomials, some old problems were solved, e.g., Tait's problem [61]. The most powerful of the skein polynomials is the Jones polynomial (by V. F. R. Jones in 1985) of two variables. Beside these, there are several useful link polynomials, including the Kauffman polynomial (by L. H. Kauffman in 1987) and the two variable polynomial HOMFLY polynomial (by Hoste, Ocneanu, Millett, Freyd, Lickorish and Yetter in 1985). Though the HOMFLY polynomial was also independently discovered by Przytycki and Traczyk in 1987 [54]. All of the skein polynomials are actually Laurent polynomials, which means that the variables in the polynomials can be raised to negative as well as positive powers. Here, among the skein polynomials, we mainly concerned with the HOMFLY polynomial and the Kauffman polynomial.

**Alexander polynomial:** In the beginning Alexander probably discovered this polynomial by thinking about covering spaces, but his paper was strictly combinatorial, using linear algebra, determinants, and the Reidemeister moves. He showed that if two oriented knots or links \( K, K' \) are ambient isotopic, then \( \Delta_K(t) = \Delta_{K'}(t) \) where \( \equiv \) means equal up to a multiple of \( \pm t^n \) for some integer \( n \). The polynomial was seen to be quite good at distinguishing knots and links, although it did not distinguish a knot or link from its mirror image.

**Conway polynomial:** In 1970 John Horton Conway published a remarkable paper [24] in which he showed that the Alexander polynomial could be sharpened to an invariant with a simple recursive definition. The Conway polynomial, \( \nabla_K(z) \), determined by the conditions:

i) \( \nabla \begin{array}{c} \searrow \\ \downarrow \\ \nearrow \end{array} - \nabla \begin{array}{c} \nearrow \\ \downarrow \\ \searrow \end{array} = z \begin{array}{c} \leftarrow \\ \rightarrow \end{array} \)

ii) \( \nabla \begin{array}{c} \rightarrow \\ \downarrow \\ \leftarrow \end{array} = 1 \)

iii) \( \nabla_K(z) = \nabla_{K'}(z) \) whenever \( K \) and \( K' \) are ambient isotopic.

Conway explained that his polynomial was related to the Alexander polynomial by the formula \( \Delta_K(t) = \nabla_K(\sqrt{t} - 1/\sqrt{t}) \).

**Jones polynomial:** In 1984 Vaughan Jones lectured on his new invariant derived from a representation of the Artin braid group into a von Neumann algebra [39]. And Jones
proved (among other things) that his (Laurent) polynomial satisfied an identity

\[ t^{-1}V(t) - tV(t) = (\sqrt{t} - 1/\sqrt{t})V(\sqrt{t}) \]

**HOMFLY polynomial:** With the Jones formula standing in juxtaposition to the Conway formula, a number of people leapt at once to the generalization

\[ \alpha^{-1}P - \alpha P = zP \]

giving a two variable polynomial \( P(\alpha, z) \) specializing both to the Conway \((\alpha = 1)\) and the Jones \((\alpha = t, z = \sqrt{t} - 1/\sqrt{t})\) polynomials. This is the HOMFLY polynomial [34].

**Kauffman polynomial:** This polynomial has defined by Louis H. Kauffman in 1987 and on an unoriented link diagram \( K \) [42]. The Kauffman polynomial of \( K \) is the Laurent polynomial \(< K > = < K > (A) \in \mathbb{Z}[A, A^{-1}] \) defined by the following rules:

i) \(< O > = 1,\)

ii) \(< O K > = (-A^2 - A^{-2}) < K >,\)

iii) \(< \big\triangleup \big\triangleup > = A \big\rangle \big\langle + A^{-1} \big\langle \big\rangle \big\rangle >\)

\[ \big\langle \big\rangle \big\rangle > = A^{-1} \big\rangle \big\langle + A \big\langle \big\rangle \big\rangle >\]

In order to define the HOMFLY polynomial we first orient a \( K \); then we fix a particular projection \( K \). The HOMFLY polynomial of \( K \) will be defined in terms of the crossings of this oriented projection. We want to distinguish to different types of oriented crossings, which we will call positive crossing and negative crossing. A positive crossing corresponds to a right-handed twist and a negative crossing corresponds to a left-handed twist. These two types of crossings are illustrated in the following way:

+crossing \quad -crossing
Let $D$ be a knot (link) diagram of an oriented knot $K$ with total number of positive crossings $n^+$ and total number of negative crossings $n^-$ as shown in the above way. Then the number of crossings in the diagram $D$ is given by $n = n^+ + n^-$.  

**Definition 3.2.1** [34] The HOMFLY polynomial $P_K(a, z)$ (or simply by $P(K)$) corresponding to the oriented knot (link) $K$ is a two variable polynomial with a function $P_K : \{\text{Oriented} S^3\} \rightarrow \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ such that $P_K$ takes the value 1 on the unknot and, if $K_+, K_-$ and $K_0$ are links that have diagrams $D_+, D_-$ and $D_0$ that are the same except near a single point where they are as in the following way,

$$K_+ \quad \quad \quad K_- \quad \quad \quad K_0$$

then $\alpha^{-1}P_{K+} - \alpha P_{K-} = zP_{K_0}$.

HOMFLY polynomial is a two-variable polynomial $P_K(a, z)$ giving both to the Conway ($\alpha = 1$) and the Jones polynomial ($\alpha = t, z = \sqrt{t} - 1/\sqrt{t}$). The validity of the above skein relation for diagrams related as in the above way is obvious [34]. Note that the above equation determines uniquely any one of $P_{K+}, P_{K-}$ and $P_{K_0}$ from knowledge of the other two and a solution of the equation is of the form $[P_{K+}, P_{K-}, P_{K_0}] = (x, x, \lambda x)$ where $x$ is arbitrary and $\lambda = z^{-1}(\alpha^{-1} - \alpha)$.

In the course of the proof of the existence of the HOMFLY polynomial, the idea of an ascending link diagram will be used. The idea is as follows: A diagram $D$ of an oriented link is ordered if an ordering is chosen for the link components and based if a base point is selected in $D$ on each link component. If $D$ is so ordered and based, the associated ascending diagram $\beta D$ is formed from $D$ by changing the crossings so that on a journey around all the components in the given order, always beginning at the base point of each component, each crossing is first encountered as an under-pass. That means that the link represented by $\beta D$ can be thought of as lying in $\mathbb{R}^3$ above the diagram, with each component entirely below those following it in the given order, an with each component ascending as we moves around it away from its base point, but eventually dropping vertically back to that base point. Thus $\beta D$ represent a trivial link. It is important to remember, given $D$, that $\beta D$ depends on those two choices, component order and base points.

Now we recall the definition of the Seifert circles of a diagram $D$ of an oriented knot (link) $K$. 

45
Definition 3.2.2 [48] Let $K$ be an oriented link. A Seifert surface for a link $K$ is a closed compact orientable two-dimensional surface in $S^3$, whose boundary is the link $K$ and the orientation of the link $K$ is induced by the orientation of the surface. See the Figure 3.1.

![Figure 3.1: Seifert surface of a knot](image)

Definition 3.2.3 [48] Let $D$ be a planar diagram of the link $K$. If we smooth link crossings as shown in the following way, then we obtain a set of closed non-intersecting simple curves on the plane. These curves are called Seifert circles. The number of Seifert circles in a diagram $D$ is denoted by $s(D)$.

![Smoothing diagram crossings](image)

Let $D_n$ be the set of all oriented link diagram in the plane with at most $n$-crossings. Let inductively that $P_D : D_{n-1} \rightarrow \mathbb{Z}[x^{\pm 1}, z^{\pm 1}]$ has been defined such that on $D_{n-1}$,
i) the above skein relation holds for any three diagrams in $D_{n-1}$ related in the usual way,

ii) if $D$ is any ascending diagram of a link in $D_{n-1}$ with $|D|$ components then $P_D = \lambda^{|D|-1}$ for $D$.

**Theorem 3.2.1** [48] Any oriented link $K$ in $S^3$ has a Seifert surface.

The above theorem ensures us about the existing of Seifert circles in a diagram $D$ for any oriented link $K$ in $S^3$.

**Definition 3.2.4** [48] A knot diagram $K$ is called ascending (starting from a point $A$ on it different from any vertex) if while walking along $K$ from $A$ (in some direction) each crossing is first passed under and then over.

**Theorem 3.2.2** [48] Each ascending diagram represent an unknot.

### 3.2.1 Determination of HOMFLY polynomial

Here we show (by giving an example) how to determine the HOMFLY polynomial of a given oriented link.

From the definition of the HOMFLY polynomial we get that:

1. $P(O) = 1$
2. $\alpha^{-1}P_{K_+} - \alpha P_{K_-} = zP_{K_0}$
3. $P(K)$ is not changed by an ambient isotopy of $K$.

From the Figure 3.2 (upper portion), $K_+$ and $K_-$ are ambient isotopic, therefore, $P(K_+) = P(K_-) = 1$. Thus,

$$\alpha^{-1} - \alpha = zP(K) \text{ i.e., } P(K) = z^{-1}(\alpha^{-1} - \alpha).$$

To find the HOMFLY polynomial of the Hopf link, we suppose that $K_0$ (in the Figure 3.2 (lower portion)) is ambient isotopic to the unknot and $K_+$ is ambient ambient isotopic to the link whose polynomial we computed above. Now by the condition 3, we can substitute this polynomial into the equation given in the condition 2 to get the equation:

$$\alpha^{-1}(z^{-1}(\alpha^{-1} - \alpha)) - \alpha P(K_-) = z,$$

i.e.,

$$\alpha^{-2}z^{-1} - z^{-1} - \alpha P(K_-) = z,$$

i.e.,

$$P(K_-) = \alpha^{-3}z^{-1} - \alpha^{-1}z^{-1} - \alpha^{-1}z. \text{ This is illustrated in the Figure 3.2.}$$

Similarly, for the Hopf link,
the HOMFLY polynomial is given by \( \alpha^3 z^1 - \alpha z^{-1} - \alpha z \), which is illustrated in the Figure 3.3. It can be seen from these examples that computing the HOMFLY polynomial of any complicated knot or link will be quite cumbersome. However, there are a number of excellent computer programs that will compute all of the link polynomials for any link drawn with up to approximately fifty crossings (see e.g., the programs *Knot Theory By Computer*, Ochiai & Yamada, 1992 and *Knotscape*, Hoste & Thistlethwaite, 1998). Peter Suber has prepared a list of current resources on knot theory including software programs for computing knot polynomials (see http://www.earlham.edu/ peters/knotlink.htm).

**Remark 1** It is to be noted that the orientation of the components of a link may affect its HOMFLY polynomial. The HOMFLY polynomial of the Hopf link that is oriented as in Figure 3.3 is different than the HOMFLY polynomial that we computed above for the oriented Hopf link in Figure 3.2. Specifically, the roles of \( \alpha \) and \( \alpha^{-1} \) have been reversed in the HOMFLY polynomial of the link in Figure 3.3 relative to the HOMFLY polynomial of...
The Hopf link illustrated as $L_-$ in Figure 3.2.

The subsequent sections (Amphicheirality of oriented links by using HOMFLY polynomial, Some applications of HOMFLY polynomials to amphicheiral links) of this Chapter contain my results.

3.3 Amphicheirality of oriented links by using HOMFLY polynomial

In contrast with links, we can see as follows that the orientation of a knot has no effect on its HOMFLY polynomial. Let $K$ be a knot (a link with one component). Then reversing the orientation of $K$ has the effect of reversing all of the arrows in the projection. In particular, it reverses the direction of both arrows occurring at any crossing. As can be seen from the positive crossing illustrated in the following Figure, a positive crossing remains positive after both arrows reversed; it is simply rotated by $180^\circ$. Negative crossings and null crossing also remain unchanged.

Thus the HOMFLY polynomial of a knot $K$ is independent of the orientation of the knot. If we reversed the orientation of every component of a link, the HOMFLY polynomial also would not change. However, each component of a link can be oriented in two different ways. Thus we can change the orientation of one component without changing the orientation of every component, and this may affect the HOMFLY polynomial, as it did for the Hopf link.

Link polynomials can be quite useful for distinguishing knot and oriented links, because two knots or oriented links with different polynomials cannot be ambient isotopic. However, the link polynomials do not distinguish every pair of knots or links. In fact, Kanenobu [40] has shown that there are infinitely many knots with the same HOMFLY polynomial.

**Theorem 3.3.1** Let $K$ be an oriented link with HOMFLY polynomial $P(K)$. Let $\overline{P}(K)$ be denote the polynomial obtained from $P(K)$ by interchanging $\alpha$ and $\alpha^{-1}$. If $P(K) \neq \overline{P}(K)$ then $K$ is not amphicheiral as an oriented link.

**Proof** : Let $\overline{K}$ be denote the mirror image of the oriented link $K$. Then $\overline{K}$ is obtained by reversing all of the crossings of $K$. That is, $\overline{K}$ has positive crossings where $K$ has negative crossings and $\overline{K}$ has negative crossings where $K$ has positive crossings. So, by progressively changing the crossings of $\overline{K}$ as we compute its polynomial, the roles of $\overline{K}_+$ and $\overline{K}_-$ are the reverse of those of $K_+$ and $K_-$ in the computation of the polynomial for $K$. It follows
from the definition of HOMFLY polynomial that, for $K$ and all of the simpler links obtained from $K$, the roles $\alpha$ and $\alpha^{-1}$ are the opposite of what they were for $K$ in the equation

$$\alpha^{-1}P_{K+} - \alpha P_{K-} = zP_{K_0}.$$ 

Hence in $P(K)$, the roles of $\alpha$ and $\alpha^{-1}$ have been interchanged relative to what they were in $P(K)$. Thus $P(K) = P(K)$. Now we know that the HOMFLY polynomial is a topological invariant, so if there is an ambient isotopy from the oriented link $K$ to the oriented link $\overline{K}$, then $P(K) = P(\overline{K}) = \overline{P}(K)$. Thus if $P(K) \neq \overline{P}(K)$ then $K$ must be topologically not amphicheiral as an oriented link.

Remark 1 If $K$ is a knot and $P(K) \neq \overline{P}(K)$ then $K$ is not amphicheiral, independent of the orientation of $K$.

The proof follows from the definition of a knot. Since $K$ is a knot, then the orientation has no effect on the polynomial of the knot, and $P(K) \neq \overline{P}(K)$ implies that the knot $K$ is not amphicheiral, independent of its orientation.

3.4 Some applications of HOMFLY polynomials to amphicheiral links

In this Section we determine the upper and lower exponents of the HOMFLY polynomial for reduced alternating amphicheiral finks. But before that we begin with the following: Consider $P_K(\alpha, z)$ as a Laurent polynomial in $\alpha$ with defining equation:

$$P_K(\alpha, z) = c_\alpha(z)\alpha^{a_\alpha} + \cdots + c_{\alpha}(z)\alpha^a$$

with $c_\alpha(z) \neq 0 \neq c_{\alpha}(z)$, with its range $[\alpha,\alpha]$ in $\alpha$, where $\alpha$ and $\alpha$ denoting the minimum and maximum exponents of $\alpha$ that appear in the HOMFLY polynomial $P_K(\alpha, z)$ of an oriented fink $K$.

In the similar way we can consider $P_#(\alpha, z)$ as a Laurent polynomial in $z$ with the defining equation:

$$P_#(\alpha, z) = c_\beta(z)z^{b_\alpha} + \cdots + c_{\beta}(z)z^b$$

with $c_\beta(z) \neq 0 \neq c_{\beta}(z)$, with its range $[z,z]$ in $z$, where $z$ and $z$ denoting the minimum and maximum exponents of $z$ that appear in the HOMFLY polynomial $P_K(\alpha, z)$ of an oriented link $K$.

Theorem 3.4.1 Let $D$ be a diagram of an oriented reduced alternating amphicheiral link $K$, having $s(D)$ Seifert circles and $n$-crossings. Then, $\overline{z} \leq n - s(D) + 1$.

Proof: We prove the theorem by induction on the crossing number $n$ of the diagram $D$. If $n = 0$ i.e., if the diagram correspond to an unknot, then the number of Seifert circles is equal to $|D|$ where $D$ is defined earlier. Also, $P_D = \lambda^{[D]-1}$, where $\lambda = z^{-1}(\alpha^{-1} - \alpha)$ and hence the result follows.

Now the skein relation for $P_D$ is given by,

$$\alpha^{-1}P_D^+ - \alpha P_D^- = zP_D$$

where $D^+, D^-$ and $D^0$ are the diagrams related in the usual way
with the same number of Seifert circles. The inequality by induction true on $D^0$. Now to get an ascending diagram we use induction again on $n$ and then our task will be to prove the result for the ascending diagrams. Now it is required to prove for an ascending diagram $D$ that,

$$1 - [D] \leq n - s(D) + 1 \text{ i.e., } s(D) \leq n + [D].$$

Suppose that $n \neq 0$. Let $D'$ be the result of annulling a crossing of $D$. Thus the inequality is true for $D'$ and also $s(D) = s(D')$, $[D] = [D'] \pm 1$ and $n = n' + 1$. Hence it follows the inequality.

**Theorem 3.4.2** For any diagram $D$ of an oriented reduced alternating amphicheiral link $K$ with $n$-crossings, $\alpha \leq s(D) - 1$.

**Proof:** Let $D$ be an oriented link diagram whose Laurent polynomial is given by, $\alpha^{w(D)} P_D(\alpha, z)$ i.e., $P_D(\alpha, z)$, from Lemma 2.2.1. We prove the theorem by induction on the crossing number $n$. If $n = 0$, then the inequality follows. The skein relation for $P_D(\alpha, z)$ is given by $\alpha^{-1}P_{D^+} - \alpha P_{D^-} = zP_{D^0}$. Again, $D^+, D^-$ and $D^0$ all have the same number of Seifert circles and, as the required inequality is true by induction for $D^0$, it is sufficient to prove it for ascending diagrams. Let $D$ be an ascending diagram so that $P_D(\alpha, z) = \lambda^{[D]-1}$. Now we have to show that $[D] \leq s(D)$.

Suppose that in $D$, some crossings of a component with itself is deleted to give another ascending diagram $D'$. Now by induction on the number of crossings, $w(D) + [D'] \leq s(D') = s(D')$ i.e., $[D'] \leq s(D') = s(D)$, by Lemma 2.2.1. Although, $[D'] = [D] + 1$ and $w(D') = w(D) \pm 1$ i.e., $w(D') = \pm 1$, by Lemma 2.2.1, and so the inequality is true for $D$. If there is no crossing at which a component crosses itself, let $D''$ be obtained from $D$ by deleting a negative crossing where one component crosses another. This can be done as all linking numbers are zero. Let us choose the two components as close as possible, in the ordering of the components of $D$ as an ascending diagram and then $D''$ is also ascending. By induction on the number of crossings, $w(D'') + [D''] \leq s(D'') = s(D)$. Now $w(D'') = w(D) + 1$ i.e., $w(D'') = 1$, by Lemma 2.2.1. Also, $[D''] = [D] - 1$. Thus the inequality holds for $D$.

**Theorem 3.4.3** For any diagram $D$ of an oriented reduced alternating amphicheiral link $K$ with $n$-crossings, $\alpha \geq -s(D) + 1$.

The proof is immediate from the above theorem and the fact that, for an amphicheiral link $\alpha = -\overline{\alpha}$.

**Proposition 3.4.1** For any diagram $D$ of an oriented reduced alternating amphicheiral link $K$, $-s(D) + 1 \leq \alpha < 0 < \overline{\alpha} \leq s(D) - 1$.

The proof follows from the above two Theorems.
Proposition 3.4.2 For any diagram $D$ of $K$, $s(D) \geq 1 + (\overline{a} - a)/2$.

Proof: Let $D$ be an oriented diagram of a reduced alternating amphicheiral link $K$ with $n$-crossings. Now from the above two Theorems we have,

\begin{align*}
\overline{a} & \leq s(D) - 1 \quad (1) \\
-a & \leq s(D) - 1 \quad (2)
\end{align*}

Adding (1) and (2) we get the result.

Remark 1 For an oriented link $K$ the above Proposition gives the lower bound on the number of Seifert circles in any diagram that might represent $K$.

Remark 2 The bounds of $\overline{a}$ and $a$ in $P_K(\alpha, z)$ depend only on the number of Seifert circles of the given diagram and last Proposition gives the braid index of the link $K$.

3.5 Relation between HOMFLY polynomial and Kauffman polynomial

The HOMFLY polynomial and the Kauffman polynomial are independent invariants in the sense that they distinguish different pair of knots. Thus neither polynomial can be seen to be trivially contained in the other by means of some subtle change of variables. Examples are shown in the Figure.

![Knots](image)

The knots $8_8$ and $10_{129}$ have the same HOMFLY polynomial but distinct Kauffman polynomials (even when taking the variable $a = 1$). Knots $11_{255}$ and $11_{257}$ have the same Kauffman polynomial but distinct HOMFLY polynomials (and even have distinct Alexander polynomials).