Chapter 6

A study of knotted 4-valent graphs

Growing out of the knot and link polynomials, several polynomial invariants have been developed for embedded graphs. Some of those are restricted to graphs, all of whose vertices have a valence of three or all of whose vertices have a valence of four. In most cases, the polynomial for embedded graphs with a valence of four are actually only useful for the so-called rigid vertex graphs. The aim of this Chapter is to prove an invariant of knotted 4-valent graphs by using a generalized form of Reidemeister moves and $t_k$-moves for knots and links. Moreover, we have study the Kauffman bracket polynomial [42] and Jones polynomial [39] for knotted graphs.

6.1 Introduction

We have already familiar with the notion of a four valent graph, see Chapter 2. Now by a rigid vertex graph, we mean a graph in which each vertex is replaced by a two-dimensional disk with the edges attached to the boundary of the disk. The order of the edges around the boundary of the disk is fixed by any ambient isotopy of the rigid vertex graph because each edge must remain attached to the boundary of the disk throughout the ambient isotopy. For example, consider the rigid vertex embedded graphs in the following Figure.
We imagine the circle as small disks. If the circle represented vertices, then the two ordinary graphs would be ambient isotopic, by interchanging the positions of the inner and outer diagonals of the second picture. However, this ambient isotopy changes the order of the edges around the vertices, and there is no corresponding ambient isotopy of the rigid vertex graphs.

Here we use a set of diagrammatic local moves that generalize the Reidemeister moves for diagrams of classical knots (links) introduced by Kauffman [42], see Figure 6.1. But before that we need the notion of rigid vertex of a knotted graph. Here we consider a knotted graph as an embedding of a 4-valent graph into $\mathbb{R}^3$ [42, 3, 68].

Figure 6.1: Rigid vertex moves for knotted graphs

A rigid vertex (RV4) 4-valent graph is a 4-valent graph [42] whose vertices are replaced by rigid 2-disks. Each disk has four strands attached to it. A knotted RV4 graph means an embedding of a RV4 graph into $\mathbb{R}^3$. In [42], Kauffman associated a collection $L(G)$ of links to each knotted RV4 graph $G$ and showed that the ambient isotopy class of $L(G)$ is an invariant of the RV equivalence of the graph $G$. He also defines [43] a simpler invariant that he asserts is just as effective at detecting amphicheirality for rigid vertex embedded graphs as any of the polynomials. An element of $L(G)$ is obtained by making a connection at each vertex, replacing the vertex locally by a configuration that connects the four edges in pairs. There are four ways to do this as shown in Figure 6.2.

In practice, the ambient isotopy class of $L(G)$ is very useful to distinguish knotted RV4 graphs in 3-space. In the case of a plane graph $G$, however, the ambient isotopy class of $L(G)$ is not an invariant of the (plane vertex) equivalence of the graph $G$ because the move (IV) may change the ambient isotopy type of a link in $L(G)$. In this Chapter we have showed that if we take the t3-equivalence class of $L(G)$, then it is an invariant of the knotted graph $G$. Moreover, we study on t3-moves and knotted graphs. Finally, we have
discussed and investigated some invariants of knotted graphs with the help of Kauffman bracket polynomial \[42\] and Jones polynomial \[39\].

6.2 Preliminary discussions and notations

In this section we give some definitions followed by some results.

**Definition 6.2.1** \[42\] Let \( G \) be an RV4 graph. Let \( L(G) \) be the collection of knots and links obtained from \( G \) by choosing one replacement of each of the types, given in the Figure 6.2, at each vertex of \( G \).

**Definition 6.2.2** \[42\] Let \( X \) be a collection of knots and links. Two such collections will be said to be ambient isotopic \((X \equiv X')\) if every member of the first collection is ambient isotopic to some member of the second collection and vice versa.

The symbol \( \equiv \) will be used both for RV4 graphs and for ambient isotopy of knots and links. In general, if \( G \) has \( n \) rigid verities, then \( L(G) \) will contain \( 4^n \) diagrams, some trivial and some ambient isotopic.

**Theorem 6.2.1** \[42\] Let \( G \) and \( G' \) be equivalent RV4-graphs in three dimensional space. Then their associated link collection are ambient isotopic, i.e., \( L(G) \equiv L(G') \).

The subsequent sections (\( t_3 \) moves and knotted graphs, \( t_3 \) move and Kauffman polynomial, Jones polynomial and knotted graphs) of this Chapter contain my results.

6.3 \( t_3 \) moves and knotted graphs

In this section we have introduced another local moves, called \( t_3 \) moves, due to \[53\]. Let \( G \) be a knotted graph with the vertex set \( V(G) = \{v_i : I = 1,2,...,n\}, n \geq 0 \) and \( D \) be a diagram of \( G \).
Let $R$ be the collection of four tangle diagrams as shown in Figure 6.2, if we denote them by the symbol $R^1, R^2, R^3, R^4$, then $R = \{R^1, R^2, R^3, R^4\}$. Let $h : V(G) \to R$ be an assignment of a member $h(v_j)$ in $R$ for each vertex $v_j$ of $G$. Now considering all the assignments of $G$, we get the set $S$ as $S = \{h_1, h_2, ..., h_{4^n}\}$. Now for each assignment $h_j \in S$, let $[D, h_j]$ denote the knot or link diagram obtained from $D$ by replacing all the vertices of $G$ as shown in Figure 6.2 with the assignment $h$. Let $L(D)$ be denote the collection of all $4^n$ link diagrams $[D, h_j]$ associated to $D$. Now if $|V(G)| = 0$, then we define $L(D) = \{D\}$. Let $K$ be a link diagram. Then we have the following definition,

**Definition 6.3.1** Two links $K_1$ and $K_2$ are said to be $t_3$-equivalent if their diagrams can be transformed to each other by a finite sequence of Redemeister moves and the two $t_3$ moves as shown in Figure 6.3.

**Lemma 6.3.1** Consider the four link diagrams that are identical except a small neighborhood where they are as shown in the Figure,

(A) \hspace{1cm} (B) \hspace{1cm} (C) \hspace{1cm} (D)

then (A) is $t_3$ equivalent with (B) and (C) is $t_3$ equivalent with (D).

Now from the construction of the above moves we have the following proposition,

**Proposition 6.3.2** Let $G$ be a knotted graph and let $D$ and $D'$ be any two diagrams of $G$. Then there exists a permutation $P$ on the set $I = \{1, 2, ..., 4^n\}$ such that the link $[D, h_j]$ is $t_3$ equivalent to the link $[D', h_{P(j)}]$ for each $j = 1, 2, ..., 4^n$.

**Proof**: Let $D = D_0, D_1, ..., D_s = D'$ be a sequence of knotted graph diagrams connecting $D$ and $D'$, where $D_i$ is obtained from $D_{i-1}$ by applying exactly one of the moves (I), (II), (III), (IV), and (V). For each $j = 1, 2, ..., 4^n$, there exists a sequence $D_{j1}, D_{j2}, ..., D_{jn}$ of diagrams such that $D_{j1} = D$ and $D_{jn} = D'$, and each $D_{ji}$ is obtained from $D_{ji-1}$ by applying a permutation of the moves (I), (II), (III), (IV), and (V). Therefore, for each $j = 1, 2, ..., 4^n$, there exists a sequence $h_1, h_2, ..., h_n$ of assignments such that $h_1(v_j) = h_1$ and $h_n(v_j) = h_n$, and the diagram $[D, h_j]$ is $t_3$ equivalent to the diagram $[D', h_{P(j)}]$, where $P$ is the permutation on the set $I = \{1, 2, ..., 4^n\}$.
(III), (IIIA) and (IV). Let $v_1, v_2, ..., v_n$ be the vertices of $G$ and $S$ is defined earlier, then, for each pair $(i, j), 1 \leq i \leq s, 1 \leq j \leq 4^n$, we denote $D_{ij}$ the knot or link $(D_i, h_j)$. For each $i = 1, 2, ..., s$, define a permutation $P_i$ on $S$ such that the links $D_{i-1j}$ and $D_{iP_i(j)}$ are ambient isotopic for each $j = 1, 2, ..., 4^n$ as follows:

First Part: $D_i$ is obtained from $D_{i-1}$ by applying the Reidemeister move (I), (II), (III), or (IIIA). Then it is clear that the moves (I)-(III) do not affect vertex connection replacements. So $D_{i-1j}$ and $D_{ij}$ are ambient isotopic for each $j$.

Now it is clear from the above figure that a vertex connection replacement at a vertex by a tangle $R \in R$ and the affect of the move (IIIA). This shows that the links $D_{i-1j}$ and $D_{ij}$ are ambient isotopic for each $j$. Here, we define $P_i$ to be identity permutation.

Second Part: $D_i$ is obtained from $D_{i-1}$ by applying the Reidemeister move (IV). We assumed that the move (IV) is accomplished at the vertex $v_1$ without loss of generality, from the above figure, it is clear that all possible vertex connection replacements in the diagram $D_{i-1}$ and the corresponding replacements in the diagram $D_i$ at the vertex $v_1$. For the type
(P) of Reidemeister move (IV) in the above figure, we see that $R^1$ and $(P_3)$ are ambient isotopic by Reidemeister move (II), $R^2$ and $(P_2)$ are ambient isotopic by Reidemeister move (I), $R^4$ and $(P_1)$ are plain isotopic, and $R^3$ is $t_3$-equivalent to $(P_4)$, by the above Lemma 6.3.1.

Similarly, for the type (Q) we can show the ambient isotopics between the diagrams. Now, for each $h_j \in F(G)$, let $h_j : V(G) \rightarrow R$ be an assignment of $G$ defined by $h_j(v_k) = h_j(v_k)$ for $2 \leq k \leq n$ and $h_j(v_1) = R^1$ if $h_j(v_1) = R^1$,

$h_j(v_1) = R^2$ if $h_j(v_1) = R^2$,

$h_j(v_1) = R^3$ if $h_j(v_1) = R^3$,

$h_j(v_1) = R^4$ if $h_j(v_1) = R^4$. Thus, $h_j \in F(G)$ and it follows from the above discussion and observation that the mapping $f : F(G) \rightarrow F(G)$ defined by $f(h_j) = h_j \forall h_j \in F(G)$ is bijective and so it induces the desired permutation $P_i$ on $S$. Similarly, we can obtain a permutation for the type (Q).

Thus we get $P = P_1 P_2 \cdots P_4$. Then $(D, h_j) = (D_0, h_j)$ is $t_3$ equivalent to $(D_0, h_{P(I)}) = (D', h_{P(I)})$ for each $j = 1, 2, \ldots, 4^n$.

Now we have the following definition follows from the above discussion,

**Definition 6.3.2** Let $U_1$ and $U_2$ be two collections of links. Now $U_1$ and $U_2$ are said to be $t_k$ equivalent if every member of $U_1$ is $t_k$ equivalent to some member of $U_2$ and vice versa.

**Remark** 1. Let $G$ be a knotted graph and let $D$ be a diagram of $G$. Then the $t_3$ equivalent class $L_3(G)$ of the collection $L(D)$ is an invariant of $G.

Consider the following example,
Let $G_1, G_2, G_3$ and $G_4$ be four knotted graphs as shown below. Then the $t_3$ equivalent class of the knotted graphs are given by $L_3(G_1) = \{0_1, 0_2\}$, $L_3(G_2) = \{0_2, 0_3\}$, $L_3(G_3) = \{0_1, 0_2\}$, $L_3(G_4) = \{0_1, 0_2, 0_3\}$, where $0_i$ denotes the unlink with $i$ trivial components. Since $L_3(G_1)$ and $L_3(G_2)$ are not $t_3$ equivalent, $G_1$ and $G_2$ are not equivalent and hence $G_2$ is knotted. Similarly, $G_3$ and $G_4$ are not equivalent.

### 6.4 $t_3$ move and Kauffman polynomial

In this section we construct a new $t_3$ move invariants of links by using Kauffman bracket polynomial [42] and consequently give topological invariants of knotted graphs. Let $K$ be a link and let $D$ be a diagram of $K$. Then the Kauffman-polynomial [42] of $K$ is the Laurent polynomial $< D > = < D > (A) \in \mathbb{Z}[A, A^{-1}]$ defined by the following rules:

i) $< O > = 1$,

ii) $< O D > = (-A^2 - A^{-2}) < D >$,

iii) $< X > = A < | > + A^{-1} < \bigcirc >$

It must be noted that the Kauffman polynomial is a regular isotopy invariant and

$< \bigcirc > = -A^3 < | >$, $< \bigcirc > = -A^{-3} < | >$

So it is not an ambient isotopy invariant. Also, it is not invariant under the $t_3$ moves since,

$< \bigcirc \bigcirc \bigcirc \bigcirc > = A^{-3} < \bigcirc \bigcirc > + (A^7 - A^3 + A^{-1}) < | >$

$< \bigcirc \bigcirc \bigcirc \bigcirc > = A^3 < \bigcirc \bigcirc > + (A - A^{-3} + A^7) < | >$

Let $Z_n = e^{i\pi n/12}$, where $n = 1, 5, 7, 11, 13, 17, 19, 23$ and $i = \sqrt{-1}$. Then each $Z_n$ is a nonzero common root of the two equations $A - A^{-3} + A^{-7} = 0$ and $A^7 - A^3 + A^{-1} = 0$. Substituting $Z_n$ in the Kauffman polynomial $< D >$, we get a regular isotopy invariant $< D >_n$ of $D$ as, $< D >_n = [ < D > ]_{A=Z_n}$. Thus from the above discussion we have constructed a real number, denoted by $d_n$, defined by $d_n = < D >_n < D >'_n$, where $< D >' = [ < D > ]$ at $A = A^{-1}$ is a polynomial obtained from $< D > (A)$ by interchanging $A$ and $A^{-1}$. Now we have the following proposition,

**Proposition 6.4.1** Let $D$ be a link diagram of a link $K$. Then for each $n = 1, 5, 7, 11, 13, 17, 19$ and 23, the real number $d_n$ is a $t_3$ move invariant of knots and links.
Proof: The real number $d_n$ is a regular isotopy invariant, follows from the construction of it. Now consider,

$$d\left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n = -z_n^3 \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n \circ (-z_n^{-3}) \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n'$$

$$= \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n \circ \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n'$$

$$= d\left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n$$

Similarly,

$$d\left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n = -z_n^3 \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n \circ (-z_n^{-3}) \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n'$$

$$= \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n \circ \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n'$$

$$= d\left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n$$

Thus we get that $d_n$ is an ambient isotopy invariant. Since $Z_n - Z_n^{-3} + Z_n^{-7} = 0$ and $Z_n^2 - Z_n^3 + Z_n^{-1} = 0$, it follows from the previous discussion that,

$$\left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n = z_n^{-3} \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n$$

$$\left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n = z_n^3 \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n$$

Now we have,

$$d\left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n = z_n^3 \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n \circ z_n^{-3} \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n'$$

$$= \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n \circ \left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n'$$

$$= d\left< \begin{array}{c} \hline \\ \hline \end{array} \right>_n$$
In the similar way we get,

\[ d\langle \begin{array}{c}
\otimes
\end{array}\rangle_n = d\langle\begin{array}{c}
\circ
\end{array}\rangle_n \]

Hence, \( d_n \) is an invariant under the \( t_3 \) moves. This completes the proof.

**Remark 2** If we consider the \( t_4 \) moves, then we have the following:

\[ A^3\langle\begin{array}{c}
\otimes
\end{array}\rangle + (-A^{-10} + A^{-6} - A^{-2} + A^2)\langle\begin{array}{c}
\circ
\end{array}\rangle \]

\[ A^3\langle\begin{array}{c}
\otimes
\end{array}\rangle + (-A^{-10} + A^{-6} - A^{-2} + A^2)\langle\begin{array}{c}
\circ
\end{array}\rangle \]

Thus from the above two equations we may derive the equations as,

\[ -A^{10} + A^6 - A^{-2} + A^2 = 0 \& -A^{-10} + A^{-6} - A^{-2} + A^2 = 0 \]

This is equivalent to,

\[ A^{12} - A^8 + A^4 - 1 = 0. \]  

Thus we may generalize the results stated in this Chapter for \( t_4, t_5, \ldots, t_n \) moves also for any finite \( n \).

Now from the Proposition 6.3.2 and the Proposition 6.4.1, we define the following:

**Definition 6.4.1** Let \( G \) be a knotted graph with \( n \) vertices and \( D \) be a diagram of \( G \). Then for each \( n = 1, 5, 7, 11, 17, 19, \) and \( 23 \), we get a real number \( (G)_n \), defined by \( (G)_n = \sum d([D, h_j])_n, j = 1, 2, \ldots, 4^n \), where each \([D, h_j]\) as the usual meaning given earlier.

Now from the above definition and our discussions we have the following result:

**Theorem 6.4.1** Let \( G \) be a knotted graph with \( n \) vertices and \( D \) be a diagram of \( G \). Then \( (G)_n \) is a topological invariant of \( G \) for each \( n \) as given in the Definition 6.4.1.

The above theorem can verify by the following illustration: Let \( G_1, G_2, G_3 \) and \( G_4 \) be four knotted graphs as given in the figure-8. Now if we put \( n = 1 \) in \( Z_n = e^{\text{mev}/12} \), then we obtain \((G_1)_1 = 6, (G_2)_1 = 27, (G_3)_1 = 24 \) and \((G_4)_1 = 36 \). This shows that the invariant \((G)_1\) distinguish the graphs from each other. In the similar way we get the other invariants for different \( n \).
6.5 Jones polynomial and knotted graphs

The Jones polynomial [39] assigns an invariant to each oriented link \( L \) in \( \mathbb{R}^3 \) which is a Laurent polynomial \( V_L(t) \) in the variable \( \sqrt{t} \). The product \( V_L(t)V_L(t^{-1}) \) is another Laurent polynomial in \( t \) and can be expressed in terms of \( \mu = t + t^{-1} \) to give an ordinary polynomial, we denote it by \( A_L(\mu) \) and defined by, \( A_L(\mu) = A_L(t + t^{-1}) = V_L(t)V_L(t^{-1}) \). Now the polynomial \( J_L(\mu) \) is defined by, \( J_L(\mu) = (\mu + 2)A_L(\mu) \), where the factor \( (\mu + 2) \) is included so that the unknot has invariant equal to \( (\mu + 2) \) and the empty link has polynomial equal to 1. The invariant \( J_L \) does not depend on the orientation of the link \( L \) and cannot distinguish a link from its mirror image.

The advantage of \( J_L \) is that the definition extends to an invariant of embedded graphs in \( \mathbb{R}^3 \), a generalization of the idea of a link in which vertices where a number of edges meet are allowed. But we restrict ourselves to the case for which every vertex is 4-valent i.e. for knotted graphs in \( \mathbb{R}^3 \). The invariant for the knotted vertex is defined by,

\[
X = P(\mu)(\mu R_1 + \mu R_2 + R_3 + R_4),
\]

where \( R_1, R_2, R_3, R_4 \) are as defined earlier, in Section 6.3 and \( P(\mu) \) is the normalizing factor can be chosen arbitrarily. Now the graph invariant \( J_G \) for an embedded graph \( G \) is defined by applying this relation to every vertex in a diagram for \( G \), then evaluating the resulting link diagrams \( L \) using the previously defined \( J_L \). One must note that this does not depend on the projection and gives an invariant of ambient isotopy of the embedding.

Choosing \( P = 1 \) would give a polynomial for each embedded graph. The definition used here is \( P(\mu) = 1/(\mu + 1)(\mu + 2) \). This choice has the disadvantage that the invariant for graphs is no longer always a polynomial but a rational function.

Table 1 gives some examples of the evaluation of the invariant for 4-valent graphs and links. In the table, the examples with the same letter, e.g. \( A \) and \( A_1 \), are the same graph but with different embeddings. Two edges can be removed from \( A_1 \) to give a trefoil knot \( F \) and so it is perhaps not surprising that the invariant can distinguish \( A \) and \( A_1 \). The examples \( B, B_1, B_2 \) again share the same graph. \( B_1 \) is linked in the sense that removing the two outer edges gives the Hopf link \( E \). However \( B_2 \) is an unlinked knotted graph in this sense: any way of removing edges from \( B_2 \) to make a link results only in a number of unlinked unknots. Yet \( B \) and \( B_2 \) differ; they have different \( J_L \) invariants.
Now we have the following lemma,

**Lemma 6.5.1** The invariant satisfies the following relation:

\[
(1 - \mu) + (\mu - 1) = -\mu^2 + 2\mu^2 + 2
\]

**Proof**: The invariant \( J_L \) for a link \( L \) is related to the (suitably normalized) Kauffman bracket polynomial for the same diagram \( C(A) \), by \( J_L(A^4 + A^{-4}) = C(A)C(A^{-1}) \). The braid generator \( b \) for the Kauffman bracket satisfies the quadratic relation \( A^{-1}b^2 + (A^2 - A^{-2})b - A = 0 \).

The braid generator for the invariant \( J_L \) can be represented by

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**Figure 6.4: Table-1**

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( A_1 )</th>
<th>( B )</th>
<th>( B_1 )</th>
<th>( B_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-( \mu^2 + 2\mu^2 + 2 )</td>
<td>( \mu + 2 )</td>
<td>( \mu^2 + 2 )</td>
<td>-( \mu^4 + \mu^4 + 3\mu + 2 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>10</td>
<td>3</td>
<td>4</td>
<td>10</td>
</tr>
</tbody>
</table>

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Thus the lemma follows from using the quadratic relation for $b$.

**Theorem 6.5.1** The definition of $J$ for knotted 4-valent graphs in $\mathbb{R}^3$ is independent of the diagram and is an invariant of ambient isotopy.

To prove invariance under ambient isotopy of the knotted graph it suffices to check a set of extended Reidemeister moves [70]. The symmetry relation

![Diagram]

follows immediately from the definition, while the permutation property of the vertex,

![Diagram]

follows from a calculation using the above Lemma 6.5.1. The invariant is an example of the more general 'rigid vertex' invariants described earlier and in [44], which do not necessarily have the invariance of type-2 under the permutations of edges meeting at the vertex.

**Remark 3** A further property of the invariant is.

![Diagram]

This can be taken as the way of fixing the normalizing factor $P$. 