3. A NOTE ON THE TWO-DIMENSIONAL SOURCE POTENTIALS IN A TWO FLUID MEDIUM*

1. Introduction

When a line singularity is present in either of the fluids of a two-fluid medium, in earlier method of derivation expressions of velocity potentials were initially assumed in such a way that these contained a number of unknown constants and a number of unknown functions. The total number of these unknowns exceeded the number of given boundary conditions. Extra conditions were then found by matching the logarithmic terms in the boundary conditions and from a convergence criterion of the integrals. In the present section it is shown that these matching and convergence conditions are in fact not necessary to derive the potentials. Gorgui and Kassem (1978) obtained potential functions due to different types of singularities present in either of the fluids of a two-fluid medium. To get acquaintance with their method of deriving the potential functions due to a line singularity, let us consider a two-fluid medium with lower fluid of finite depth 'h' say, and the upper fluid extending into...

infinity upwards and let there be a line singularity present in the lower fluid. We choose a co-ordinate system with \(xz\) plane as the mean surface of separation (SS) of the two fluids, \(y\)-axis vertically downwards and passing through the singularity at \((0,\eta)\). Let \(\phi_1\) and \(\phi_2\) denote the velocity potentials in the lower and upper fluids respectively. Then \(\phi_i\)'s \((i = 1,2)\) are functions of \(x, y\) only and these satisfy

\[
\nabla^2 \phi_1 = 0, \quad y > 0 \quad \text{expect at } (0, \eta) \tag{1.1}
\]

\[
\nabla^2 \phi_2 = 0, \quad y < 0 ,
\]

\[
K\phi_1 + \frac{\partial \phi_1}{\partial y} = s(K\phi_2 + \frac{\partial \phi_2}{\partial y}) \quad \text{on } y = 0 ,
\]

\[
\frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} \quad \text{on } y = 0 , \tag{1.2}
\]

\[
\frac{\partial \phi_1}{\partial y} = 0 \quad \text{on } y = h
\]

\[
\phi_1 \to \log r \quad \text{as } r = \left\{x^2 + (y-\eta)^2 \right\}^{1/2} \to 0 \tag{1.3}
\]
\( \text{grad} \varphi_2 \to 0 \text{ as } y \to -\infty \) \hspace{1cm} (1.4)

and \( \varphi_1, \varphi_2 \) behave as diverging waves as \( |x| \to \infty \) \hspace{1cm} (1.5)

which is the so called 'radiation condition' at infinity.

Conditions in (1.2) are the linearised SS condition, the condition of the continuity of the vertical component of velocity, and the condition of no flow at the bottom perpendicular to it respectively. Here \( K = \omega^2/g \) is the wave number, \( \omega \) being the circular frequency, \( g \) being the gravity, and \( s = \rho_2/\rho_1 \) \( (\rho_2 < \rho_1) \) where \( \rho_1, \rho_2 \) are the densities of the lower and upper fluids respectively.

To solve the boundary value problem described by (1.1) to (1.5), Gorgui and Kassem (1978) assumed the solution to be

\[
\begin{align*}
\varphi_1 &= \log r + c_1 \log r' + \int_0^\infty A(k) \cosh k(h-y) \\
&\quad + B(k) \sinh ky \cos kx \, dy, \quad y < 0, \\
\varphi_2 &= c_2 \log r + \int_0^\infty C(k) c^{ky} \cos kx \, dk, \quad y < 0
\end{align*}
\] (1.6)
where $r' = \{x^2 + (y + \eta)^2\}^{1/2}$, $c_1, c_2$ are unknown constants and $A(k), \beta(k), C(k)$ are unknown functions of $k$. The conditions (1.3), (1.4) are automatically satisfied, and the contour of integration is to be chosen in such a way that the 'radiation condition' (1.5) will also be satisfied automatically. Thus the two unknown constants and the three unknown functions are to be found from the three boundary conditions in (1.2). Jorgui and Kassem (1978) obtained two more conditions involving $c_1$ and $c_2$, the first by matching the logarithmic terms involved in the SS condition, and the second from a convergence criterion of the integrals in (1.6). Although this first condition is plausible, the second conditions appears to be somewhat artificial.

Here it is demonstrated that by suitably decomposing the given boundary value problem into two component problems we can obtain the potential functions without using the above mentioned 'matching' and 'convergence' conditions when there exists a line singularity in either of the two fluids.

2. Reduction to component problems

For simplicity, we consider only the case of a two-fluid medium as mentioned in sub-section 1. Let us write
\[ \varphi_i = \psi_i + X_i, \quad i = 1, 2 \]  

(2.1)

where \( \psi_i \)'s (\( i = 1, 2 \)) satisfy

\[ \begin{align*}
\nabla^2 \psi_1 &= 0, \quad 0 < y < h \text{ except at } (0, \eta) \\
\nabla^2 \psi_2 &= 0, \quad y < 0,
\end{align*} \]  

(2.2)

with \( \psi_1 = s \psi_2 \) on \( y = 0 \),

\[ \frac{\partial \psi_1}{\partial y} = \frac{\partial \psi_2}{\partial y} \]  

on \( y = 0 \),

\[ \begin{align*}
\psi_1 &\rightarrow \log r \quad \text{as } r \to 0, \\
|\nabla \psi_2| &\to 0 \quad \text{as } y \to -\infty.
\end{align*} \]  

(2.3)

(2.4)

(2.5)

Then obviously \( \chi_i \)'s (\( i = 1, 2 \)) satisfy
\( \nabla^2 \chi_1 = 0, \quad 0 < y < h \)  
(2.6)  

\( \nabla^2 \chi_2 = 0, \quad y < \sigma \)  

with  

\[
K \chi_1 + \frac{3}{2} (\psi_1 + \chi_1) = K \chi_2 + \frac{3}{2} (\psi_2 + \chi_2) \text{ on } y = 0 ,
\]

\[
\frac{\partial \chi_1}{\partial y} = \frac{\partial \chi_2}{\partial y} \quad \text{on } y = 0 ,
\]

\[
\frac{\partial \chi_1}{\partial y} = -\frac{\partial \psi_1}{\partial y} \quad \text{on } y = h ,
\]

\[
|\text{grad } \chi_2| \to 0 \text{ as } y \to -\infty ,
\]

(2.8)  

and

both \( \chi_1 \) and \( \chi_2 \) satisfy the radiation condition as \( |x| \to \infty \)  
(2.9).
It may be noted that in the first and third condition of (2.7) \( \psi_1 \) and \( \psi_2 \) are supposed to be known. Thus the boundary value problem described by (1.1) to (1.5) is decomposed into two component problems described by (2.2) to (2.5) and (2.6) to (2.9).

3. Solution of the component problems

a) Derivation of \( \psi_i \) (i = 1,2)

Solutions for \( \psi_i \) (i = 1,2) satisfying (2.2), (2.4) and (2.5) may be represented by

\[
\psi_1 = \log r + c_1 \log r', \\
\psi_2 = c_2 \log r
\]

where \( c_1 \) and \( c_2 \) are unknown constants to be determined. The two conditions in (2.3) are sufficient to determine these constants. These conditions give

\[
c_1 = -\frac{1-s}{1+s}, \quad c_2 = \frac{2}{1+s}
\]
b) Derivation of $X_i (i = 1, 2)$

Solutions for $X_i (i = 1, 2)$ satisfying (2.6) and (2.8) may be represented by

$$
X_1 = \int_0^\infty \left\{ A \cosh k(h-y) + B \sinh k(y) \right\} \cos kx \, dk
$$

$$
X_2 = \int_0^\infty C e^{ky} \cos kx \, dk
$$

where $A, B, C$ are unknown functions of $k$ to be determined from the three conditions in (2.7). The radiation condition (2.9) will be satisfied automatically if the contour of integration in the integrals in (3.3) is chosen properly.

Making use of the integral representations

$$
\frac{\partial}{\partial y} (\log r) = \int_0^\infty e^{-k(y-\eta)} \cos kx \, dk, \; y > \eta
$$

$$
= \int_0^\infty e^{-k(y-\eta)} \cos kx \, dk, \; y < \eta
$$
and \[ \frac{2}{\log r} = \int_{0}^{\infty} e^{-k(y+\eta)} \cos kx \, dk, \quad y + \eta > 0, \]

the three boundary conditions in (2.7) give respectively

\[ A(K \cosh kh - k \sinh kh) + BK - sG (K+k) = 2 \frac{1-s}{1+s} e^{-k\eta} \]

\[ -A \sinh kh + B = C \] \hspace{1cm} (3.5)

\[ KB \cosh kh = -\frac{2}{l+s} (\sinh k\eta + s \cosh k\eta) e^{-kh}. \]

The three conditions in (3.5) determine the functions \( A, B, C \), so that \( X_i \)'s (i = 1, 2) are determined. The contour of the integrals in \( X_i \) (i = 1, 2) is deformed below \( k_o \) to ensure the satisfaction of the radiation condition as \( |x| \to \infty \), where \( k_o \) is the real positive zero of \( A(k) \) defined by

\[ A(k) = K \cosh kh + \{ s(K+k) - k \} \sinh kh, \] \hspace{1cm} (3.6)
4. Solution of the problem

By (2.1) we can now write the velocity potentials as

\[
\varphi_1 = \log r - \frac{1-s}{1+s} \log r',
\]

\[
(1-s) \cosh k(h-\eta) +
\]

\[
\frac{2}{1+s} \int_0^\infty \frac{se^{-kh}}{k} \left\{ (k-s(K+k)) \cosh k\eta - k \sinh k\eta \right\} \cosh k(h-y) \, dk
\]

\[
- \frac{\sinh k\eta + s \cosh k\eta}{k} e^{-kh} \sinh ky \left[ \frac{\cos kx}{\cosh kh} \right] \, dk
\]

\[
\varphi_2 = \frac{2}{1+s} \log r - \frac{2}{1+s} \int_0^\infty \frac{(1-s)e^{-k\eta}}{k \cosh kh} \left\{ \frac{k-s(K+k)y e^{k\eta}}{k \cosh kh} \sinh ky e^{ky} \cos kx \right\} \, dk
\]

\[
+ \int_0^\infty \frac{\sinh k\eta + \cosh k\eta}{k \cosh kh} e^{ky} \cos kx \, dk
\]

where \( \Delta(k) \) is given by (3.6).
These results coincide with those given by Gorgui and Kassem (1978) except for some additive constants (integrals in k). It may be noted that by putting $s = 0$ we recover the potential for a line singularity present in water of finite depth with a free surface as given by Thorne (1953).

5. Conclusion

In sub-section 2 we have considered only the case when a line singularity is present in the lower fluid. However, the same type of decomposition can be used for obtaining the potential functions when a line singularity exists in the upper fluid, or if the surface tension effect at the SS is included (cf. Mandal (1981) and Chapter 2 section 1), or if the upper fluid is bounded by a free surface while the lower fluid extends to infinity downwards (cf chapter 1, section 1).