5. POTENTIAL DUE TO A HORIZONTAL RING OF WAVE SOURCES
IN THE THEORY OF WATER WAVES

1. Introduction

Study of radiation or scattering problems involving surface waves in the presence of a vertical body of revolution partially immersed or submerged in a fluid region requires the consideration of potentials due to submerged circular horizontal rings of wave sources when the body and the fluid region have a common vertical axis of symmetry, because the problems can be formulated in terms of a suitable distribution of rings of wave sources around the body (cf. Fenton (1978), Gorgui and Kassem (1978)). Similarly the study of internal waves at the interface of two superposed fluids due to the presence of a vertical body of revolution which may intersect the interface, requires the consideration of potentials due to horizontal ring sources submerged in either of the two fluids.

If the sources around the ring are time harmonic with angular frequency $\omega$ and are all in the same phase, then the velocity potential in an incompressible, inviscid and irrotational fluid can be expressed as $\text{Re}\{\varphi \exp(-i\omega t)\}$ where $\varphi$ satisfies the Laplace's equation in the fluid region except
at points on the ring. In an unbounded fluid region the potential due to a circular ring of wave sources is

\[ \phi_o = 2\pi a \int_0^\infty \exp \left\{ -k|y-\eta| \right\} J_0(ka) \, J_0(kr) \, dk \]  

(1.1)

where \( a \) is the radius of the ring with centre at \((0,0,\eta)\), using a cylindrical co-ordinate system \((r,\theta,y)\), \(y\)-axis being taken as the axis of the ring (cf. Fenton (1978)). However, in a fluid with a free surface (mean FS being the plane \( y = 0 \)) or in a two-fluid medium with a mean horizontal interface, the FS or the interface conditions will contribute another term which can be regarded as the image of the potential \( \phi_o \) in the FS or the interface conditions.

In the present section we derive the velocity potentials due to a horizontal ring of wave sources submerged in a one-fluid medium (in sub-section 2) or in either of the two fluids of a two-fluid medium (in sub-section 3). The one-fluid medium may extend infinitely downwards or may be of uniform constant depth \( h \). For the two-fluid medium the upper fluid extends infinitely upwards while the lower fluid may extend infinitely downwards or may be of uniform constant depth. The effect of surface tension at the FS of the
one-fluid medium or interfacial tension at the interface of the two fluids is also considered.

2. One-fluid medium

We use a cylindrical co-ordinate system \((r, \theta, y)\) in which \(y\)-axis is taken vertically downwards passing through the centre of the ring situated at \((0,0,\eta)\) \((\eta > 0)\) and the plane \(y = 0\) is the mean FS of a fluid occupying \(y > 0\) if it extends infinitely downwards or \(0 < y < h\) if it is of uniform depth \(h\). The potential \(\psi\) then satisfies the Laplace's equation in the fluid region except at points on the ring together with the linearized FS condition

\[
K\psi + \frac{3\psi}{\partial y} = 0 \quad \text{on } y = 0
\]  

(2.1)

where \(K = \omega^2/g\), \(g\) being the gravity,

\[
\nabla \psi \to 0 \quad \text{as } y \to \infty
\]  

(2.2)

if the fluid is of infinite depth, otherwise
\[ \frac{\partial \psi}{\partial y} = 0 \quad \text{on} \quad y = h \]  

(2.3)

if the fluid is of uniform constant depth \( h \).

At points near the ring,

\[ \psi \to \psi_0 \text{ as } \left\{ (r-a)^2 + (y-\eta)^2 \right\}^{1/2} \to 0 \]  

(2.4)

where \( \psi_0 \) is given in (1.1). Also \( \psi \) is required to satisfy the condition that it should represent outgoing waves as \( r \to \infty \) which is the so called 'radiation condition' at infinity.

1) **Ring source submerged in fluid of infinite depth**

In this case \( \psi \) can be represented as (cf. Hulme (1981))

\[ \psi = \psi_0 + \int_0^\infty A(k) \exp(-ky) J_0(kr) \, dk, \]  

(2.5)

Using the FS condition (2.1) and the representation (1.1) for \( \psi_0 \), \( A(k) \) is uniquely determined as
\[ A(k) = 2\pi a \frac{k+K}{k-K} \exp(-k\eta) J_0(ka). \]

After rearrangement we can express \( \varphi \) in (2.5) as

\[
\varphi = 2\pi a \int_0^\infty \left[ \exp\left(-k(y-\eta)\right) + \exp\left(-k(y+\eta)\right) \right] j_0(ka) j_0(kr) \, dk
\]

\[
+ 4\pi a K \int_0^\infty \exp\{-k(y+\eta)\} j_0(ka) j_0(kr) \, dk \tag{2.6}
\]

where the contour from 0 to \( \infty \) in the second integral is indented below the pole at \( k = K \) to account for the radiation condition as \( r \to \infty \). This was already obtained by Hulme (1981).

By expressing \( j_0(z) \) as \( \frac{1}{2} \left\{ H_0^{(1)}(z) + H_0^{(2)}(z) \right\} \) and rotating the contour in each integral involving \( H_0^{(1)}(ka) H_0^{(j)}(kr) \) \((i,j = 1,2)\) in (2.6) appropriately in the first or fourth quadrants the following alternative representation of \( \varphi \) for \( r > a \) is obtained from which the outgoing nature of the waves becomes evident as \( r \to \infty \).
\[ \varphi = 16a \int_0^\infty \frac{(k \cos \eta - K \sin \eta)(k \cos \psi - K \sin \psi)}{k^2 + K^2} I_0(ka)K_0(kr) dk \]

\[ + 4\pi i a K \exp(-K(y+\eta)) J_0(Ka) H_0^1(Kr) \]

where \( I_0(z) \) and \( K_0(z) \) are modified Bessel functions. (2.7) can be identified with the corresponding result given by Rhodes-Robinson (1979).

If the effect of surface tension \( T' \) at the FS is included, then the FS condition (2.1) is modified as

\[ K \varphi + \frac{\partial \varphi}{\partial y} + T \frac{\partial^3 \varphi}{\partial y^3} = 0 \text{ on } y = 0 \]  

where \( T = T'/\rho g \), \( \rho \) being the density of the fluid. In this case we can similarly obtain

\[ \varphi = 2\pi a \int_0^\infty \left[ \exp\{-k|y-\eta|\} + \exp\{-k(y+\eta)\} \right] J_0(ka) J_0(kr) \, dk \]

\[ + 4\pi a K \int_0^\infty \frac{\exp(-K(y+\eta))}{k(l + Tk^2) - K} J_0(ka) J_0(kr) \, dk \]  

(2.9)
where now the contour in the second integral is indented below the simple pole at \( k = \alpha \), \( \alpha \) being the only real positive zero of \( k(1 + Tk^2) - K \). An alternative representation valid for \( r > \alpha \) is similarly given by

\[
\varphi = 16\alpha \int_0^\infty \frac{k(1-Tk^2)\cos kn - K \sin kn}{k^2(1 - Tk^2) + \alpha^2} I_0(ka)K_0(kr) \, dk
\]

\[
+ 4\pi^2 \frac{\exp\{-\alpha(y+n)\}}{3Ta^2 + 1} J_0(\alpha a) H_3^1(\alpha r). \tag{2.10}
\]

This form of \( \varphi \) was obtained by Rhodes-Robinson (1979) by employing a different technique. For \( T = 0 \) we note that \( \alpha \) coincides with \( K \) and (2.9), (2.10) reduce to (2.6), (2.7) respectively.

ii) Ring source submerged in fluid of finite depth

Here \( \varphi \) satisfies the Laplace's equation in \( 0 < y < h \) except at points on the ring \( r = \alpha, y = \eta \ (0 < \eta < h) \), the FS condition (2.1), the bottom condition (2.3), the condition (2.4)
and the radiation condition as \( r \to \infty \). We assume

\[
\phi = \varphi - 2\pi a \int_0^\infty \exp\left(-k(y+\eta)\right) J_0(ka) J_0(kr) \, dk \\
+ \int_0^\infty \begin{cases} A \cosh k(h-y) + B \sinh ky \end{cases} J_0(ka) J_0(kr) \, dk.
\]

The FS condition (2.1) and the bottom condition (2.3) determine \( A \) and \( B \) uniquely and the final form of \( \phi \) in this case is

\[
\phi = 2\pi a \int_0^\infty \left[ \exp\left(-k(y-\eta)\right) - \exp\left(k(y+\eta)\right) \right] J_0(ka) J_0(kr) \, dk \\
+ 4\pi a \int_0^\infty \frac{k \cosh k(h-\eta) \cosh k(h-y)}{k \sinh kh - k \cosh kh} \, dk \\
+ \exp(-kh) \sinh kh \int_0^\infty \frac{J_0(ka)}{\cosh kh} J_0(kr) \, dk \tag{2.11}
\]

where the contour in the second integral is indented below the
pole at \( k = k_0 \), say, to account for the outgoing nature of
the waves as \( r \to \infty \). Now \( k \sinh kh - K \cosh kh \) has only
two real zeros at \( \pm k_0 \), say, and a countably infinite
number of imaginary zeros at \( \pm ik_n \), \( n = 1, 2, \ldots \) where
\( k_n \tan k_nh + K = 0 \). By suitably changing the countour of
integration to the whole real axis with indentations above
the pole at \( k = -k_0 \) and below the pole at \( k = k_0 \) after
replacing the Bessel functions in terms of Hankel functions
and enclosing by semicircles of large radius in the upper or
lower half planes in the appropriate cases, we obtain the
following alternative representation of \( \varphi \) valid for \( r > a \)
which makes evident the outgoing nature of the waves as
\( r \to \infty \),

\[
\varphi = 32\pi a \sum_{l=1}^{\infty} \frac{k_n \cos k_n(h-\eta) \cos k_n(h-y)}{2k_0 h + \sinh 2k_0 h} I_0(k_n a) K_0(k_nr)
\]

\[
+ 8\pi i ak_0 \frac{\cosh k_0(h-\eta) \cosh k_0(h-y)}{2k_0 h + \sinh 2k_0 h} J_0(k_0 a) H_0^\prime(k_0 r). \quad (2.12)
\]

This form of \( \varphi \) coincide with the corresponding result given
by Hulme (1981) and Fenton (1978) except for an obvious
multiplying constant.
When the effect of surface tension at the FS is included, then in place of (2.11) we obtain

\[
\varphi = 2\pi a \int_0^\infty \left[ \exp \{ -k|y - \eta| \} - \exp \{ -k(y + \eta) \} \right] J_0(ka) J_0(kr) \, dk
\]

\[
+ 4\pi a \int_0^\infty \frac{k(l+Tk^2) \cosh k(h-\eta) \cosh k(h-y)}{k(l+Tk^2) \sinh kh - K \cosh kh} \, dk
\]

\[
+ \exp(-kh) \sinh kh \sinh ky \int \frac{J_0(ka)}{\cosh kh} J_0(kr) \, dk \tag{2.13}
\]

where the contour in the second integral is now indented below the simple pole at \( k = \alpha_0, \alpha_0 \) being the only positive zero of \( k(l+Tk^2) \sinh kh - K \cosh kh \). This has only two real zeros at \( \pm k_0 \) and an infinite number of zeros at \( k = \pm i\alpha_n, n = 1, 2, \ldots \)

where

\[
\alpha_n(1-T\alpha_n^2) \tan \alpha_n h + K = 0 .
\]

(2.13) has the alternative representation valid for \( r > a \).
\[ \varphi = \sum_{n=0}^{\infty} \frac{a_n (1-T_n^2) \cos a_n (h-\eta) \cos a_n (h-y)}{2 \alpha_n h (1-T_n^2) + (1-3T_n^2) \sin 2\alpha_n h} I_0(k_n a) K_0(k_n r) \]

\[ + \frac{\sigma_o (1+T_o^2) \cosh \sigma_o (h-\eta) \cosh \sigma_o (h-y)}{2 \sigma_o h (1+T_o^2) + (1+3T_o^2) \sinh 2\sigma_o h} J_0(\sigma_o a) H_0^1(\sigma_o r) \]

(2.14) can be identified with the corresponding result obtained by Rhodes-Robinson (1979).

3. Two-fluid medium

Let \( \rho_1, \rho_2 \) denote the densities of the lower and upper fluids respectively (\( \rho_2 < \rho_1 \)). The two fluids are inviscid, incompressible and we are concerned with irrotational motion of the two fluids under gravity due to a horizontal ring of wave sources all oscillating harmonically in the same phase with circular frequency \( \omega \). After suppressing the time dependent factor \( \exp(-i\omega t) \), the velocity potentials in the lower and upper fluids can be expressed by \( \varphi_1, \varphi_2 \) respectively. As before y-axis is taken to be the vertical line passing through the centre of the ring and points to the lower fluid,
the plane \( y = 0 \) being the mean SS of the two fluids. Then \( \varphi_1, \varphi_2 \) satisfy (cf. Gorgui and Kassem (1978) the Laplace's equation in the lower and upper fluid region respectively except at points on the ring in the appropriate situation, the linearized SS conditions

\[
\frac{\partial \varphi_1}{\partial y} = \frac{\partial \varphi_2}{\partial y} \quad \text{on} \quad y = 0, \quad (3.1)
\]

\[
K \varphi_1 + \frac{\partial \varphi_1}{\partial y} = s(K \varphi_2 + \frac{\partial \varphi_2}{\partial y}) \quad \text{on} \quad y = 0, \quad (3.2)
\]

where \( s = \rho_2/\rho_1 \),

\[
\nabla \varphi_1 \to 0 \quad \text{as} \quad y \to \infty, \quad (3.3)
\]

\[
\nabla \varphi_2 \to 0 \quad \text{as} \quad y \to -\infty \quad (3.4)
\]

when the upper fluid extends infinitely upwards and the lower fluid extends infinitely downwards, and \( \varphi_1, \varphi_2 \) represent outgoing waves as \( r \to \infty \). However, if the lower fluid is of uniform constant depth \( h \), then (3.3) is to be replaced by
\frac{\partial \varphi_1}{\partial y} = 0 \text{ on } y = h \quad (3.5)

1) **Ring source submerged in lower fluid of infinite depth**

Let the centre of the ring be at a distance \( \eta \) from the mean SS and the radius of the ring be \( a \). We represent \( \varphi_1, \varphi_2 \) by

\[
\varphi_1 = \varphi_0 + \int_0^\infty A(k) \exp(-ky) J_0(ka) J_0(kr) \, dk ,
\]

\[
\varphi_2 = \int_0^\infty B(k) \exp(ky) J_0(ka) J_0(kr) \, dk ,
\]

where \( \varphi_0 \) is given by (1.1). Using the SS conditions (3.1) and (3.2), \( A \) and \( B \) are obtained uniquely. After some rearrangements, \( \varphi_1 \) and \( \varphi_2 \) are given by

\[
\varphi_1 = 2\pi a \int_0^\infty \left[ \exp\{-k|y-\eta|\} + \exp\{-k(y+\eta)\}\right] J_0(ka) J_0(kr) \, dk
\]

\[
+ 4\pi a M(1+s) \int_0^\infty \frac{\exp\{-k(y+\eta)\}}{k-M} J_0(ka) J_0(kr) \, dk , \quad y > 0 \quad (3.6)
\]
\[ \varphi_2 = -4\pi aM(1+s)^{-1} \int_0^\infty \frac{\exp\{k(y-\eta)\}}{k-M} J_o(ka) J_o(kr) \, dk, \quad y < 0 \quad (3.7) \]

where \( M = (1+s)(1-s)^{-1} K \), \( \quad (3.8) \)

and the contours in the integral of (3.7) and the second integral of (3.6) are indented below the simple pole at \( k = M \) to ensure the satisfaction of the radiation condition as \( r \to \infty \). Alternative representations of \( \varphi_1 \) and \( \varphi_2 \) (valid for \( r > a \)) which make evident the outgoing waves as \( r \to \infty \) are

\[ \varphi_1 = 16a \int_0^\infty \frac{I_0(ka)}{X} \frac{K_0(kr)}{k^2 + M^2} \, dk \]

\[ + 4\pi i aM(1+s)^{-1} \exp\{-M(y+\eta)\} J_0(Ma) H_0^{(1)}(Mr) \quad (3.9) \]

where \( X = (k \cos k\theta - M \sin k\theta)(x \cos ky - M \sin ky) \)

\[ + sM(1+s)^{-1} \left\{ k \sin k|y+\eta| + M \cos k(y+\eta) \right\} , \]
\[ \varphi_2 = -16aM(1+s)^{-1} \int_0^\infty \frac{k \sin k(y-\eta) - M \cos k(y-\eta)}{k^2 + M^2} I_0(ka) K_0(kr) \, dk \]

\[ - 4\pi^2 aM(1+s)^{-1} \exp\{M(y-\eta)\} J_0(Ma) H_0^{(1)}(Mr). \]  

When the effect of interfacial tension \( T' \) at the SS is included then the SS condition (3.2) is to be replaced by (cf. Rhodes-Robinson (1980))

\[ K\varphi_1 + \frac{\partial \varphi_1}{\partial y} s(K\varphi_2 + \frac{\partial \varphi_2}{\partial y}) = -T \begin{bmatrix} \frac{\partial^3 \varphi_1}{\partial y^3} \\ \frac{\partial^3 \varphi_2}{\partial y^3} \end{bmatrix} \]  

on \( y = 0 \) \hfill (3.11)

where \( T = T'/(\rho_1 g) \).

In this case we obtain

\[ \varphi_1 = 2\pi a \int_0^\infty \left[ \exp\{-k|y-\eta|\} - \exp\{-k(y+\eta)\} \right] J_0(ka) J_0(kr) \, dk + \]
\[
\varphi_2 = -4\pi a M(l+s)^{-1} \int_0^\infty \frac{\exp\{-k(y-\eta)\}}{k(1+Sk^2) - M} J_0(ka) J_0(kr) \, dk,
\]

where \( S = T' \{g(\rho_1-\rho_2)^{-1}\}^{-1} \),

and the contour is indented below the pole at \( k = \beta \), \( \beta \) being the only positive zero of \( k(1+Sk^2) - M \). We note that when \( T' = 0, \beta = M \). For \( r > a \) we have the following alternative representations for \( \varphi_1, \varphi_2 \) which make evident the outgoing nature of the waves as \( r \to \infty \),

\[
\varphi_1 = 16a \int_0^\infty \frac{Y I_0(ka) K_0(kr)}{k^2(1-Sk^2)^2 + M^2} \, dk
\]

\[
+ 4\pi \frac{a M(l+s)^{-1}}{3S^2 + 1} \frac{\exp\{-2(y+\eta)\}}{k^2(1-Sk^2)^2 + M^2} J_0(\beta a) H_0^{(1)}(\beta r),
\]

where \( Y = \{ k(1-Sk^2) \cos \eta - M \sin \eta \} \{ k(1-Sk^2) \cos ky - M \sin ky \} + \).
\[ + sM(l+s)^{-1} \left\{ k(l-Sk^2) \sin k(y+\eta) + M \cos k(y+\eta) \right\} , \]

\[
\varphi_2 = -16aM(l+s)^{-1} \int_0^\infty \frac{k(l-Sk^2) \sin k(y-\eta) - M \cos k(y-\eta)}{k^2(l-Sk^2)^2 + M^2} I_0(ka)K_0(kr)dk 
\]

\[
-4\pi^2 aM(l-s)^{-1} \frac{\exp{i\beta(y-\eta)}}{3S^2 + 1} J_0(\beta \alpha) \frac{H_0^{(1)}(\beta r)}{J_0(\beta \alpha)} . \quad (3.17)
\]

ii) Ring source submerged in lower fluid of finite depth

When the lower fluid is of uniform constant depth \( h \), we can represent \( \varphi_1, \varphi_2 \) by

\[
\varphi_1 = 2\pi a \int_0^\infty \left[ \exp\{-k|y-\eta|\} - \exp\{-k(y+\eta)\} \right] J_0(ka)J_0(kr)dk 
\]

\[
+ \int_0^\infty \{ A \cosh k(h-y) + B \sinh ky \} J_0(ka)J_0(kr)dk ,
\]

\[
\varphi_2 = \int_0^\infty C \exp(ky) J_0(kr)dk ,
\]
where \( A, B, C \) are to be obtained by using the SS conditions (3.1), (3.2) and the bottom condition (3.5). The final result is

\[
\varphi_1 = 2\pi a \int_0^\infty \left[ \exp\left\{-k|y-\eta|\right\} - \exp\left\{-k(y+\eta)\right\} \right] J_0(ka) J_0(kr) \, dk
\]

\[+ 4\pi a \int_0^\infty \frac{k(1-s)-sK \cosh k(h-\eta) \cosh k(h-\eta)}{\Delta(k)} \cosh kh J_0(ka) \cosh kh J_0(kr) \, dk, \tag{3.18}\]

\[
\varphi_2 = -4\pi k \int_0^\infty \cosh k(h-\eta) \exp(ky) \frac{\Delta(k)}{\cosh kh} J_0(ka) J_0(kr) \, dk, \tag{3.19}\]

where \( \Delta(k) = \{k(1-s)-sK\} \cosh kh-K \cosh kh , \tag{3.20} \)

and the contour in (3.19) and the second integral of (3.18) is indented below the simple pole at \( k = \beta_o \) where \( \beta_o \) is the only real positive zero of \( \Delta(k) \).
As $r \to \infty$, we obtain

$$
\varphi_1 \sim 8\pi^2 a \frac{(1-s)\beta_0 - sk}{2(s)\beta_0 - sk} \cosh \beta_0 (h-\eta) \cosh \beta_0 (h-y) \frac{\exp(-k(y+\eta))}{k-M} \int_0^\infty J_0(ka) \frac{J_0(kr)}{k-M} \, dk, \quad (3.21)
$$

We note that as $h \to \infty$, (3.6) and (3.7) are recovered from (3.18) and (3.10).

These results can easily be extended to include the effect of interfacial tension at the SS.

iii) Ring source submerged in upper fluid, lower fluid of infinite depth

In this case a circular ring of radius $a$ and centre at $(0,0,-\eta)(\eta > 0)$ is present in the upper fluid. We obtain

$$
\varphi_2 = -4\pi a s M (1+s)^{-1} \int_0^\infty \frac{\exp(-k(y+\eta))}{k-M} \, dk. \quad (3.21)
$$
\[ \varphi_2 = 2\pi a \int_0^\infty \left[ \exp \left\{ -k|y-n|^{1/2} \right\} + \exp \left\{ k(y-n)^{1/2} \right\} \right] J_0(ka) J_0(kr) \, dk \]

\[ + 4\pi a M(1+s)^{-1} \int_0^\infty \frac{\exp \left\{ k(y-n)^{1/2} \right\}}{k-M} J_0(ka) J_0(kr) \, dk \quad (3.22) \]

for \( r > a \), these have the following alternative representations,

\[ \varphi_1 = 16\pi a M(1+s)^{-1} \int_0^\infty \frac{k \sin k(y+n)+k \cos k(y+n)}{k^2 + M^2} I_0(ka) K_0(kr) \, dk \]

\[ -4\pi^2 a M(1-s)^{-1} \exp \left\{ -M(y-n)^{1/2} \right\} J_0(Ma) H_0^{(1)}(Mr), \quad (3.23) \]

\[ 2k \cos k\eta(k \cos ky + M \sin ky) \]

\[ + 2M \cos k(y-n)-k \sin k(y-n)^{1/2} \]

\[ \varphi_2 = 2a \int_0^\infty + \frac{2M \cos k(y-n)-k \sin k(y+n)^{1/2}}{k^2 + M^2} I_0(ka) K_0(kr) \, dk \]

\[ + 4\pi^2 a M(1+s)^{-1} \exp \left\{ M(y-n)^{1/2} \right\} J_0(Ma) H_0^{(1)}(Mr). \quad (3.24) \]
When the effect of interfacial tension $T^i$ in the SS is included then in place of (3.21) and (3.22) we obtain

$$
\varphi_1 = -4\pi a M(1+s)^{-1} \int_0^\infty \frac{\exp\{-k(y+\eta)\}}{k(l+Sk^2)-M} J_0(ka) J_0(kr) \, dk, \quad (3.25)
$$

$$
\varphi_2 = 2\pi a \int_0^\infty \left[ \exp\{-k|y+\eta|\} + \exp\{k(y-\eta)\} \right] J_0(ka) J_0(kr) \, dk
+ 4\pi a M(1+s)^{-1} \int_0^\infty \frac{\exp\{k(y-\eta)\}}{k(l+Sk^2)-M} J_0(ka) J_0(kr) \, dk, \quad (3.26)
$$

where the contour in (3.25) and in the second integral of (3.26) are indented below the pole at $k = \beta$. For $r > a$, these have the alternative representation

$$
\varphi_1 = 16\pi a M(1+s)^{-1} \int_0^\infty \frac{k(l-Sk) \sin k(y+\eta)+M \cos k(y+\eta)}{k^2(l-Sk^2)+M^2} I_0(ka)K_0(kr) \, dk
$$

$$
-4\pi^2 a M(1+s)^{-1} \frac{\exp\{-2(y+\eta)\}}{3\beta^2 + 1} J_0(\beta a) H_0^{(1)}(\beta r), \quad (3.27)
$$
\[ \varphi_2 = 8a \int_0^{\infty} \frac{P I_0(ka) K_0(kr)}{k^2(1 + sk^2) + M^2} \, dk \]

\[ + 4\pi l \cos M(l + s) \frac{\exp(\beta(y-n))}{3\beta^2 + 1} J_0(\beta a) R_0(1) R_1(\beta r) \]  \hspace{1cm} (3.28)

where \[ P = 2k(1 - sk^2) \cos ky \{ (1 - sk^2) \cos ky + M \sin ky \} \]

\[ + M \{ M \cos k(y+n) - k(1 - sk^2) \sin k(y-n) \} \]

\[ + M \{ M \cos k(y+n) - k(1 - sk^2) \sin k(y+n) \}. \]

iv) Ring source submerged in upper fluid, lower fluid of finite depth

In this case

\[ \varphi_1 = -4\pi a K \int_0^{\infty} \frac{\exp(-ky)}{\Delta(k)} J_0(ka) J_0(kr) \, dk \]  \hspace{1cm} (3.29)

\[ \varphi_2 = 2\pi a \int_0^{\infty} \left[ \exp(-k|y+n|) + \exp(-k(y-n)) \right] J_0(ka) J_0(kr) \, dk + \]
These results can be extended to include the effect of interfacial tension $T'$ at the SS.

4. Discussion

Ring source potentials in a one-fluid or a two-fluid medium are obtained. The results for the one-fluid medium are already known to some extent, but the results for the two-fluid medium appear to be new. For the two-fluid medium considered here the upper fluid extends infinitely upwards while the lower fluid is either of infinite or finite depth. However, one can extend the results to include the case when the upper fluid is of finite height bounded by a FS or a rigid horizontal plane.