CHAPTER – III

ANALYSIS OF FUZZY ERLANG’S LOSS QUEUING MODEL
AND FUZZY TANDEM QUEUES

Section 1 of this chapter proposes a procedure for constructing the membership functions of the performance measures in a finite capacity loss queuing system with arrival rate and service rate being fuzzy numbers. To investigate the performance measures of the queuing system a pair of mathematical non linear programme is formulated to calculate the upper and lower bound of system characteristics. By extending finite capacity loss system queue in fuzzy environment, would have wider applications. Section 2 proposes a procedure for constructing the membership functions of the performance measures in a queuing network with instantaneous Bernoulli’s feedback in which the arrival rate and the service rate are fuzzy numbers. The $\alpha$-cut approach is used to transform a fuzzy queue into a family of conventional crisp queue. By means of membership functions of the system characteristics a set of non-linear programs is developed to describe the family of crisp queues with IBF in tandem with single server. A numerical example is solved to illustrate the validity of the proposed approach.

SECTION 1 : ANALYSIS OF FUZZY ERLANG’S LOSS QUEUING MODEL :
NON LINEAR PROGRAMMING APPROACH

3.1. INTRODUCTION

In teletraffic engineering the term turn is used to describe any entity that will carry one call. The number of trunks, to be provided obviously depends on

Section 1 of this chapter has been published in the International Journal of Fuzzy Mathematics and Systems, Vol.1, No.1 (2011), pp.1-10. Section 2 of this chapter has been presented in the International Conference ICMCM-11 and published in the proceedings organized by PSG College of Technology, Coimbatore during 19.12.11 to 21.12.11, pp.227-234.
the traffic to be carried. On a group of trunks, the average number of calls in progress depends on both the number of calls which arrive and their duration.

In order to obtain analytical solutions to teletraffic problems it is necessary to have a mathematical model of the traffic offered to telecommunication system. Erlang determined the grade of service is, the loss probability of a lost call system having N trunks, when offered traffic is A.

A system in which customers have to leave when the space is full because of limited waiting space is called loss system. The formulas hold irrespective of the form of service time distribution. This is known as robustness property.

We think of waiting lines as the circumstances we encounter at the grocery store but in the telecommunications world lines can also form for packets waiting for trunk to become available. Queuing theory define, a set of formulas that describe waiting line behaviours and can be applied to these and similar telecommunication situation. Telecommunication based service activities can be independent and random as well. For example the length of a telephone call or size of a packet of data will impact the service time of telephone switches and routes.

Efficient methods have been developed for analyzing the queuing system when its parameters such as arrival rate and service rate are known exactly. However, there are cases that these parameters may not be
presented precisely due to uncontrollable factors. Specifically in many practical applications, the statistical data may be obtained subjectively i.e., arrival rate and service rate are more suitably described by linguistic terms such as fast, moderate, or slow rather by probability distribution based on statistical theory. Imprecise information of this kind will determine the system performance measure accurately. To deal with imprecise information, Zadeh [82] introduced the concept of fuzziness. Fuzzy set theory is a well known concept for modeling imprecision or uncertainty arising from mental phenomenon. Specifically fuzzy queues have been discussed by several researchers.

Buckley [13] investigated multiple channel queuing system with finite or infinite waiting capacity and calling population. Negi and Lee [61] formulated the $\alpha$-cut and two variable simulation approaches for analyzing fuzzy queues on the basis of Zadeh extension principle. Li and Lee [51] proposed an approach for analyzing fuzzy queues.

Unfortunately their approach provided only crisp solutions. In other words the membership functions of the performance measures are not completely described. Kao et al [41] applied parametric programming to construct the membership functions of the performance measures for four simple fuzzy queues with one or two fuzzy variable namely M/F/1, F/M/1, F/F/1 and FM/FM/1 where F denotes fuzzy time and FM denotes fuzzified exponential time.
Clearly when the arrival rate or service rate are fuzzy numbers, the performance measure of state dependent service queues will be fuzzy as well. In this model, we develop a solution procedure that is capable of evaluating the fuzzy performance measure for state dependent service queues with fuzzified exponential arrival rate and service rates. The membership functions of performance measures are derived by applying $\alpha$-cut and Zadeh extension principle. A pair of mathematical program is formulated to calculate the lower and upper bounds of the $\alpha$-cut of the performance measure. Consequently the membership function of the performance measure is derived analytically or numerically by enumerating different values of $\alpha$.

3.1.2. $M/M/S/S$ : Loss Model System

This model envisages that a unit who finds, on arrival that all $s$ channels are busy leaves the system without waiting for service. This is called $S$ channel loss system and was investigated by Erlang.

For this birth death process we have

$$\lambda_n = l, \quad \mu_n = n\mu, \quad n = 0, 1, 2, \ldots, s-1.$$  
$$\lambda_n = 0, \quad \mu_n = n\mu, \quad n \geq s.$$  

We get the steady state prob. as

$$P_n = \frac{(\lambda/\mu)}{n!}P_0 \quad n = 1, 2, \ldots, s$$
\[
& \left[ 1 + \sum_{K=1}^{s} \frac{\left(\lambda/\mu\right)^K}{n!} \right] P_0 = 1
\]

\[
P_0 = \left[ \sum_{K=1}^{s} \frac{\left(\lambda/\mu\right)^K}{K!} \right]^{-1}
\]

\[
P_n = \left[ \sum_{K=0}^{s} \frac{\left(\lambda/\mu\right)^K}{K!} \right]^{-1}
\]

\[
= \left( \frac{\left(\lambda/\mu\right)^n}{n!} \sum_{K=0}^{s} \frac{\left(\lambda/\mu\right)^K}{K!} \right)^{-1}
\]

\[n = 0, 1, 2, \ldots s.\]

This is known as Erlang's Formula or Erlang's First Formula.

The probability that an arriving unit is loss to the system (which is the same as that all the channels are busy) is given by

\[
P_s = \left[ \frac{\left(\lambda/\mu\right)^s / s!}{\sum_{K=0}^{s} \frac{\left(\lambda/\mu\right)^K}{K!}} \right]
\]

This is known as Erlang's Loss Formula or blocking formula and is denoted by \(B(S, \lambda/\mu).\)

### 3.1.3. System with Limited Waiting Time

Apart from limitations of space, a second kind limitation which arises in problems dealing with practical situations need consideration. For eg. a long distance telephone call may be booked to be put through within a limited time (say, during office hours). While \(a = \lambda/\mu\) is the offered load. \(a' = a[1 - B(c, a)]\) is the carried load. The overflow rate of blocked (lost) customers is \(aB(c, a)\);
the sum of the carried load and the overflow rate equals the offered load ‘a’.
The throughput, defined as the rate at which customers (units) depart from the system after being served, is given by $a^*\mu = \lambda[1 - B(c, a)]$; this is the rate at which customers are accepted for service.

A system in which customers have to leave when the space is full because of limited waiting space is called a loss system, whereas a system where all the arriving customers can wait is called delay system.

3.1.4. Expected Number of Busy Channels

Let B be the RV denoting the no. of busy channels, we have

$$E[B] = \sum_{n=1}^{s} nP_n = \sum_{n=1}^{s} \frac{n(\lambda/\mu)}{n!} P_0$$

$$= \sum_{n=1}^{s} \frac{n(\lambda/\mu)^n}{(n-1)!} P_0 = \frac{\lambda}{\mu} P_0 \sum_{n=1}^{s} \frac{(\lambda/\mu)^n}{(n-1)!}$$

$$= \frac{\lambda}{\mu} P_0 \left[ \sum_{n=1}^{s} \frac{(\lambda/\mu)^n}{n!} - \frac{n(\lambda/\mu)^s}{s!} \right]$$

$$= \frac{\lambda}{\mu} \left[ 1 - P_c \right]$$

$$= \frac{\lambda}{\mu} \left[ 1 - B\left( C, \frac{\lambda}{\mu} \right) \right]$$
3.1.5. FM/FM/S/S : Loss Model System

Consider a queueing model in which customers arrive at the system according to Poisson process with fuzzy arrival rate $\tilde{\lambda}$. All customers are served according to exponential time fuzzy service rate $\tilde{\mu}$.

A system in which customers have to leave when the space is full because of limited waiting space is called a loss system. This model is denoted by FM/FM/S/S.

In this model the arrival rate $\tilde{\lambda}$ and service rate $\tilde{\mu}$ are approximately known and are represented by the following convex fuzzy sets.

$$\tilde{\lambda} = \left\{ x, \mu_{\tilde{\lambda}}(x) / x \in X \right\} \quad \ldots (3.1)$$

$$\tilde{\mu} = \left\{ y, \mu_{\tilde{\mu}}(y) / y \in Y \right\}$$

where $X$ and $Y$ are the crisp universal sets of the arrival rate and service rate $\mu_{\tilde{\lambda}}(x)$ and $\mu_{\tilde{\mu}}(y)$ are the corresponding membership functions. Let $P(x, y)$ denote the system performance measure of interest. When $\tilde{\lambda}$ and $\tilde{\mu}$ are fuzzy numbers $P(\tilde{\lambda}, \tilde{\mu})$ is also a fuzzy number. According to Zadeh’s extension principle, the membership function of the performance measure $P(\tilde{\lambda}, \tilde{\mu})$ is defined.

$$\mu_{P(\tilde{\lambda}, \tilde{\mu})} = \text{Sup}_{x \in X, y \in Y} \left\{ \mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y) / Z = \frac{\lambda}{\mu} \left( 1 - \mathbb{B} \left( \frac{\lambda}{\mu} \right) \right) \right\} \quad \ldots (3.2)$$
Without loss of generality, assume that the system performance measure of interest are $L$, $L_q$, $w$ and $w_q$.

### 3.1.6. Solution Procedure

To re-express the membership function system characteristic in an understandable and usual form we adopt Zadeh’s approach, which relies on $\alpha$-cuts of $\mu_{P(\tilde{\lambda}, \tilde{\mu})}$.

The $\alpha$ cuts or $\alpha$ level sets of $\tilde{\lambda}$ and $\tilde{\mu}$ are defined as

$$
\tilde{\lambda}(\alpha) = \left\{ x \in X / \mu_{\tilde{\lambda}}(x) \geq \alpha \right\} \\
\tilde{\mu}(\alpha) = \left\{ y \in Y / \mu_{\tilde{\mu}}(y) \geq \alpha \right\}
$$

(3) can be expressed in another form

$$
\lambda(\alpha) = \left[ \min_{x \in X} \{ x / \mu_{\tilde{\lambda}}(x) \geq \alpha \}, \max_{x \in X} \{ x / \mu_{\tilde{\lambda}}(x) \geq \alpha \} \right] \\
= [X^L_\alpha, X^U_\alpha] 
$$

$$
\mu(\alpha) = \left[ \min_{y \in Y} \{ y / \mu_{\tilde{\mu}}(y) \geq \alpha \}, \max_{y \in Y} \{ y / \mu_{\tilde{\mu}}(y) \geq \alpha \} \right] \\
= [y^L_\alpha, y^U_\alpha] 
$$

By the convexity of a fuzzy number, the bounds of these intervals are functions of $\alpha$ and can be obtained as

$$
x^L_\alpha = \min \mu_{\tilde{\lambda}}^{-1}(\alpha) \\
x^U_\alpha = \max \mu_{\tilde{\lambda}}^{-1}(\alpha)
$$
\[ y^l_x = \min \mu^{-1}_\lambda(\alpha) \]
\[ y^u_x = \max \mu^{-1}_\lambda(\alpha) \]

Clearly the membership function of \( P(\hat{\lambda}, \hat{\mu}) \) is defined in (3.2) is also parameterised by \( \alpha \). Consequently we can use it's \( \alpha \)-cuts to construct the membership function from the membership function stated in (3.4). \( \mu_\lambda(z) \) is the minimum of \( \mu_\lambda(x), \mu_\lambda(y) \). To tackle from the membership value, we need either \( \mu_\lambda(x) = \alpha \) and \( \mu_\lambda(y) \geq \alpha \) or \( \mu_\lambda(x) \geq \alpha \) and \( \mu_\lambda(y) = \alpha \). Such that
\[
z = \frac{\lambda^l}{\mu} \left[ 1 - B \left( C, \frac{\lambda^l}{\mu} \right) \right]
\]
to satisfy \( \mu_\lambda(z) = \alpha \). This can be accomplished via parametric NLP technique. For the former case the parametric non-linear programs for finding the parametric non-linear programs for finding the lower and upper bounds of \( \alpha \)-cut of \( \mu_\lambda \) are

\[
[E(B)]^L = \min_{x,y \in \mathbb{R}^n} \frac{x}{y} \left[ 1 - B \left( C, \frac{x}{y} \right) \right] ; \quad x^L_x \leq x \leq x^U_x \quad y \in \mu(\alpha)
\]

\[
[E(B)]^U = \max_{x,y \in \mathbb{R}^n} \frac{x}{y} \left[ 1 - B \left( C, \frac{x}{y} \right) \right] ; \quad x^L_x \leq x \leq x^U_x \quad y \in \mu(\alpha)
\]

and for the latter case

\[
[E(B)]^L = \min_{x,y \in \mathbb{R}^n} \frac{x}{y} \left[ 1 - B \left( C, \frac{x}{y} \right) \right] ; \quad y^L_y \leq y \leq y^U_y \quad x \in \lambda(\alpha)
\]

\[
[E(B)]^U = \min_{x,y \in \mathbb{R}^n} \frac{x}{y} \left[ 1 - B \left( C, \frac{x}{y} \right) \right] ; \quad y^L_y \leq y \leq y^U_y \quad x \in \lambda(\alpha)
\]
Using the above equations we can have

\[
\left[ E(B)^L \right] = \min_{x,y \in \mathbb{R}^+} \frac{x}{y} \left[ 1 - B \left( C, \frac{x}{y} \right) \right] ; \quad x^L_a \leq x \leq x^U_a, y^L_a \leq y \leq y^U_a
\]

\[
\left[ E(B)^U \right] = \max_{x,y \in \mathbb{R}^+} \frac{x}{y} \left[ 1 - B \left( C, \frac{x}{y} \right) \right] ; \quad x^L_a \leq x \leq x^U_a, y^L_a \leq y \leq y^U_a
\]

If both \( L^L_a \) and \( L^U_a \) are invertible with respect to \( \alpha \) then a left shape function \( L(z) = \left( L^L_a \right)^{-1} \) and a right shape function \( R(Z) = \left( L^U_a \right)^{-1} \) can be obtained from which the membership function \( \mu_L \) is constructed.

\[
\mu_L^L(z) = \begin{cases} L(z) & z_1 \leq z \leq z_2 \\ 1 & z_2 \leq z \leq z_3 \\ R(z) & z_3 \leq z \leq z_4 \end{cases}
\]

### 3.1.7. Numerical Example

In an automatic telegraph switching system with 3 equipments incoming messages are stored in a queue until the retransmitting equipment of an outgoing trunk can send them. Messages arrive at the rate that can be represented by a trapezoidal fuzzy number \( \tilde{\lambda} = [1, 2, 3, 4] \) per hr and the time taken to retransmit messages may be assumed to have an exponential distribution with a mean time that can be represented by a trapezoidal fuzzy number, \( \tilde{\mu} = [6, 7, 8, 9] \). The system wants to know about the carried load
(expected no. of busy channels), expected no. of idle channels, busy probability.

Expected number of Busy Channels

\[ E(B) = \frac{x}{y} [1 - B(3, x/y)] \]

Busy probability

\[ P(X = 1) = \frac{\frac{x}{y} [1 - B(3, x/y)]}{3} \]

Expected number of busy channels

\[ E(B) = \frac{6xy^2 + 6x^2y + 3x^3}{6y^3 + 6xy^2 + 3x^2y + x^3} \]

Let \( \lambda = [1 \ 2 \ 3 \ 4] \), \( \mu = [6 \ 7 \ 8 \ 9] \)

\[ \begin{bmatrix} x^L_a & x^U_a \end{bmatrix} = [\alpha + 1, 4 - \alpha] \]

\[ \begin{bmatrix} y^L_a & y^U_a \end{bmatrix} = [6 + \alpha, 9 - \alpha] \]

\[ E(B)^L = \frac{6(1 + \alpha)(6 + \alpha)^2 + 6(1 + \alpha)^2(6 + \alpha) + 3(1 + \alpha)^3}{6(6 + \alpha)^3 + 6(1 + \alpha)(6 + \alpha)^2 + 3(1 + \alpha)^2(6 + \alpha) + (1 + \alpha)} \]

\[ E(B)^U = \frac{6(4 - \alpha)(9 - \alpha)^2 + 6(4 - \alpha)^2(9 - \alpha) + 3(4 - \alpha)^3}{6(9 - \alpha)^3 + 6(4 - \alpha)(9 - \alpha)^2 + 3(4 - \alpha)^2(9 - \alpha) + \alpha} \]

Busy probability

\[ P[X = 1] = \frac{E(B)}{3} \]
\[
\{P[X = 1]\}_L = \frac{[E(B)]_L}{3}
\]

\[
\{P[X = 1]\}_U = \frac{[E(B)]_U}{3}
\]

With the help of MATLAB® 6.0, following table and graphs gives the system characteristics values with different possibility levels from 0 to 1.

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>([E(B)]_L)</th>
<th>([E(B)]_U)</th>
<th>(P[X=1]_L)</th>
<th>(P[X=1]_U)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.180180</td>
<td>0.434232</td>
<td>0.06006</td>
<td>0.144744</td>
</tr>
<tr>
<td>0.2</td>
<td>0.193356</td>
<td>0.428051</td>
<td>0.064452</td>
<td>0.142683</td>
</tr>
<tr>
<td>0.3</td>
<td>0.206103</td>
<td>0.421720</td>
<td>0.068701</td>
<td>0.140573</td>
</tr>
<tr>
<td>0.4</td>
<td>0.218443</td>
<td>0.415234</td>
<td>0.072814</td>
<td>0.138411</td>
</tr>
<tr>
<td>0.5</td>
<td>0.230393</td>
<td>0.408587</td>
<td>0.076796</td>
<td>0.136195</td>
</tr>
<tr>
<td>0.6</td>
<td>0.241972</td>
<td>0.401775</td>
<td>0.080657</td>
<td>0.133925</td>
</tr>
<tr>
<td>0.7</td>
<td>0.253195</td>
<td>0.394789</td>
<td>0.084398</td>
<td>0.13596</td>
</tr>
<tr>
<td>0.8</td>
<td>0.264077</td>
<td>0.387625</td>
<td>0.088025</td>
<td>0.129208</td>
</tr>
<tr>
<td>0.9</td>
<td>0.274634</td>
<td>0.380275</td>
<td>0.091544</td>
<td>0.126758</td>
</tr>
<tr>
<td>1.0</td>
<td>0.284879</td>
<td>0.372733</td>
<td>0.094959</td>
<td>0.124244</td>
</tr>
</tbody>
</table>

The minimal expected busy time lies between 0.180 and 0.434 and maximal expected busy time lies between 0.284 and 0.372.
SECTION 2 : FUZZY ANALYSIS OF MEAN AND VARIANCE OF SOJOURN TIME DISTRIBUTION IN TANDEM QUEUES

3.2. INTRODUCTION

We discuss queues in series (or tandem), which is a simple one with 2 service stations such that a unit arrives from outside to the first station,
receives service there and proceeds to second one and after receiving service there departs from the system. The model analysis for queuing network with overtaking has been studied by many researchers like Boxma, Daduna, Dallery, Gershwin [10] etc. An important practical problem is in the job shops where different products may follow different routes and some of the machines may perform operations on different types of products. However in many practical applications, the statistical information may be obtained subjectively i.e., the arrival pattern and service pattern are more suitably described by linguistic terms such as fast, slow and moderate rather than by probability distribution. In open networks, the response time or sojourn time of a customer is defined as the time from its entry into the network until it exits from the network.

Here we consider the two stage tandem network. The system consists of two nodes with respective service rates $\mu_0$ and $\mu_1$ which are fuzzy numbers. The external arrival rate is $\lambda$ which is also a fuzzy number. After an exponentially distributed service time departures are routed to node 2 with probability $1-q$ and fed back to node 1 with probability $q$, where $q$ is considered as a fuzzy number. In this model we try to fuzzify the mean and variance of sojourn time distribution in queuing network with overtaking.
3.2.1. Fuzzy Queuing Network with Overtaking

One important generalization of the birth death process that we consider is a network of queues. Such networks can model problems of contention that arise when a set of resources is shared. A node or a service centre represents each resource. Thus in a model for computer system performance analysis, we may have a service centre associated with one or more servers. After completion of service at one service centre, the job may move to another centre for further service, re-enter the same centre or leave the system.

Queuing networks have been successfully used in performance modeling of computer and communication systems. Most of the analysis techniques discussed have concentrated on the evaluation of averages of various performance measures such as throughput, utilization and response time. For real time situations however the knowledge of response time or sojourn time distribution is required in order to compute the probability of missing a deadline.

Here we develop a fuzzy queuing network model with overtaking, ie. an FM/FM/1-IBF queue in series with a second queue. We consider a fuzzy queuing network in which customers arrive at node according to Poisson process with fuzzy rate $\tilde{\lambda}$. After an exponentially distributed service
time at node 1 with service $\tilde{\mu}_0$, customers are routed to node 2 with a fuzzy probability $\tilde{q}$. The service time at node 2 is also fuzzy rate $\tilde{\mu}_1$.

In this model the arrival rate $\tilde{\lambda}$ and service rates at node 1 and node 2 $\tilde{\mu}_0$ and $\tilde{\mu}_1$ and also the fuzzy probability $\tilde{q}$ are known and are represented by the following fuzzy set

$$\tilde{\lambda} = \{x ; \mu_\lambda(x)/x \in X\}$$

$$\tilde{\mu} = \{y ; \mu_\mu(y)/y \in Y\}$$

$$\tilde{q} = \{q ; \mu_q(u)/u \in U\}$$

where $X$, $Y$ and $U$ are the crisp universal set of the arrival rate, service rate and probability. Let $P(x, y, u)$ denote the system performance measure of interest. Clearly when arrival rate are fuzzy the system performance measure is also fuzzy. In this model we approach the problem via mathematical program techniques. A pair of parametric non-linear programs is developed to find the $\alpha$-cuts of $P(\tilde{\lambda}, \tilde{\mu}, \tilde{q})$ based on Zadeh’s Extension Principle [8].

The membership function performance measure is

$$\mu_{P(\tilde{\lambda}, \tilde{\mu}, \tilde{q})}(z) = \sup_{x \in X, y \in Y, z \in Z} \left\{\mu_{P(x, y, z)}(z) = P(x, y, z)\right\}$$

The variance sojourn time of a queuing network for a crisp queuing system is

$$N = \frac{1}{(\mu_1(1 - q) - \lambda)\mu_1(q^2 - 1) + \lambda q} + \frac{1}{(\mu_2 - \lambda)^2}$$
The membership function for \( \tilde{N} \) is

\[
\mu_{\tilde{N}}(z) = \sup_{x \in X, y \in Y, z \in Z} \left\{ \mu_{\tilde{N}(x,y), \tilde{N}(y,z)} / z = \frac{1}{(y_1 - x_1)^2} - x_2 \right\}
\]

Membership function for the other performance measures is

(i) Mean sojourn

\[
\mu_{\tilde{L}}(z) = \sup_{x \in X, y \in Y, z \in Z} \left\{ \mu_{\tilde{L}(x,y), \tilde{L}(y,z)} / z = \frac{1}{(y_1 - x_1)^2} + \frac{1}{y_2 - x_2} \right\}
\]

(ii) Number of customers in the system

\[
\mu_{\tilde{q}}(z) = \sup_{x \in X, y \in Y, z \in Z} \left\{ \mu_{\tilde{q}(x,y), \tilde{q}(y,z)} / z = \frac{x}{y_1 - x} + \frac{x}{y_2 - x} \right\}
\]

(iii) Expected waiting time in the system

\[
\mu_{\tilde{w}}(z) = \sup_{x \in X, y \in Y, z \in Z} \left\{ \mu_{\tilde{w}(x,y), \tilde{w}(y,z)} / z = \frac{1}{R(y_1 - x)} + \frac{x}{y_2 - x} \right\}
\]

where \( R \) is the response time.

Although the membership functions are theoretically correct, they are not in the usual forms for practical use and they are very difficult to imagine their shapes.

### 3.2.2. Mathematical Programming Approach

One approach to construct the membership function \( \mu_{p(\tilde{\lambda}, \tilde{\mu}, \tilde{q})} \) is to derive the \( \alpha \)-cuts of \( \mu_{p(\tilde{\lambda}, \tilde{\mu}, \tilde{q})} \). The \( \alpha \)-cuts are defined as
\[
\lambda(\alpha) = \{ x \in X / \mu_X(x) \geq \alpha \}
\]
\[
\mu(\alpha) = \{ y \in Y / \mu_Y(y) \geq \alpha \}
\]
\[
q(\alpha) = \{ u \in U / \mu_U(u) \geq \alpha \}
\]

Note that \( \lambda(\alpha), \mu(\alpha) \) and \( q(\alpha) \) are crisp sets rather than fuzzy sets. Therefore the \( \alpha \)-level sets of \( \lambda, \mu, q \) are crisp intervals which can be expressed in the following forms.

\[
\lambda(\alpha) = \left[ x_a^l, x_a^u \right] = \left[ \min_{x \in X} \{ x / \mu_X(x) \geq \alpha \}, \max_{x \in X} \{ x / \mu_X(x) \geq \alpha \} \right]
\]
\[
\mu(\alpha) = \left[ y_a^l, y_a^u \right] = \left[ \min_{y \in Y} \{ y / \mu_Y(y) \geq \alpha \}, \max_{y \in Y} \{ y / \mu_Y(y) \geq \alpha \} \right]
\]
\[
q(\alpha) = \left[ u_a^l, u_a^u \right] = \left[ \min_{u \in U} \{ u / \mu_U(u) \geq \alpha \}, \max_{u \in U} \{ u / \mu_U(u) \geq \alpha \} \right]
\]

These intervals indicate where the constant arrival rate, service rate and probability respectively lie at possibility level \( \alpha \). By the concept of \( \alpha \)-cuts the imbedded fuzzy Markov chain in FM/FM/1-IBF can be decomposed into a family of ordinary Markov chains with different transition probability matrices, which are also parameterized by \( \alpha \). The arrival rate, service rate and probability level can also be represented by different levels of confidence interval. Consequently the FM/FM/1-IBF with different \( \alpha \) level sets \( \{ \lambda(\alpha)/0 \leq \alpha \leq 1 \}, \{ \mu(\alpha)/0 \leq \alpha \leq 1 \} \) and \( \{ q(\alpha)/0 \leq \alpha \leq 1 \} \). These sets represent sets of movable boundaries forming nested structures for expressing the relationship between ordinary sets and fuzzy sets. By the convexity of a fuzzy number the bounds of these intervals are functions of \( \alpha \) and can be obtained
as \( x_a^l = \min \mu^{-1}_x(\alpha), \ x_a^u = \max \mu^{-1}_x(\alpha), \ y_a^l = \min \mu^{-1}_y(\alpha), \ y_a^u = \max \mu^{-1}_y(\alpha) \) and \\
\( q_a^l = \min \mu^{-1}_q(\alpha), \ q_a^u = \max \mu^{-1}_q(\alpha) \) respectively.

Assume that the performance measure of interest is \( N \), ie. \( P(x, y, u) = N \).

From the membership functions stated above which is not in the usual form and is very difficult to imagine its shape \( \mu_N(z) \) is the minimum of \( \mu_x(x), \mu_y(y) \) and \( \mu_q(u) \). To take from the membership value we need at least one of the following cases held such that \( z \) satisfies \( \mu_N(z) = \alpha \).

Case (i) : \( \{ \mu_x(x) = \alpha, \mu_y(y) = \alpha, \mu_q(u) = \alpha \} \)

Case (ii) : \( \{ \mu_x(x) = \alpha, \mu_y(y) = \alpha, \mu_q(u) = \alpha \} \)

Case (i) : \( \{ \mu_x(x) = \alpha, \mu_y(y) = \alpha, \mu_q(u) = \alpha \} \)

This can be accomplished via parametric non-linear programming techniques. For the former case the corresponding parametric non-linear programs for finding the lower and upper bounds of the \( \alpha \) cuts of \( \mu_N \) are

\[
N_a^l = \min \left\{ \frac{1}{(y_2 - x)^2} \frac{y_1(q^2 - 1) - xq}{(y_1 - q - x)^2} + \frac{1}{y_1(q^2 - 1) + xq} \right\}
\]

such that \( x_a^l \leq x \leq x_a^u ; y \in \mu(\alpha), u \in q(\alpha) \).

\[
N_a^u = \max \left\{ \frac{1}{(y_2 - x)^2} \frac{y_1(q^2 - 1) - xq}{(y_1 - q - x)^2} + \frac{1}{y_1(q^2 - 1) + xq} \right\}
\]
such that $x_a^l \leq x \leq x_a^u$ ; $y \in \mu(\alpha), u \in q(\alpha)$.

For the second case

$$N_a^l = \min \left\{ \frac{1}{y_1(q^2 - 1) - xq} \frac{y_1(q^2 - 1) - xq}{(y_1(1 - q) - x)^2} + \frac{1}{(y_2 - x)^2} \right\}$$

such that $y_a^l \leq y \leq y_a^u$ ; $x \in \lambda(\alpha), u \in q(\alpha)$.

$$N_a^u = \max \left\{ \frac{1}{y_1(q^2 - 1) - xq} \frac{y_1(q^2 - 1) - xq}{(y_1(1 - q) - x)^2} + \frac{1}{(y_2 - x)^2} \right\}$$

such that $y_a^l \leq y \leq y_a^u$ ; $y \in \lambda(\alpha), u \in q(\alpha)$.

For the third case

$$N_a^l = \min \left\{ \frac{1}{y_1(q^2 - 1) - xq} \frac{y_1(q^2 - 1) - xq}{(y_1(1 - q) - x)^2} + \frac{1}{(y_2 - x)^2} \right\}$$

such that $q_a^l \leq q \leq q_a^u$ ; $x \in \lambda(\alpha), y \in \mu(\alpha)$.

$$N_a^u = \max \left\{ \frac{1}{y_1(q^2 - 1) - xq} \frac{y_1(q^2 - 1) - xq}{(y_1(1 - q) - x)^2} + \frac{1}{(y_2 - x)^2} \right\}$$

such that $q_a^l \leq q \leq q_a^u$ ; $x \in \lambda(\alpha), y \in \mu(\alpha)$.

According to the definition of $\lambda(\alpha), \mu(\alpha) \& q(\alpha) ; x \in \lambda(\alpha), y \in \mu(\alpha)$ and $u \in q(\alpha)$ can be replaced by $x \in [x_a^l, x_a^u]$ ; $y \in [y_a^l, y_a^u]$ ; $q \in [q_a^l, q_a^u]$ respectively. All $\alpha$-cuts form a nested structure with respect to $\alpha$ ie. given $0 < \alpha_2 < \alpha_1 \leq 1$ we have
\[ \left[ x^l_{a_i}, x^u_{a_i} \right] \subseteq \left[ x^l_{a_2}, x^u_{a_2} \right]; \left[ y^l_{a_i}, y^u_{a_i} \right] \subseteq \left[ y^l_{a_2}, y^u_{a_2} \right] \quad \text{and} \quad \left[ u^l_{a_i}, u^u_{a_i} \right] \subseteq \left[ u^l_{a_2}, u^u_{a_2} \right] \]

ie., \( N^l_a \) is non decreasing with respect to \( \alpha \) and \( N^u_a \) is non increasing with respect to \( \alpha \). This property assumes that \( \tilde{N} \) is a fuzzy number that posses convexity

\[
N^l_a = \min \left\{ \frac{1}{(y_1(1 - q) - x)^2} \frac{y_1(q^2 - 1) - x_q}{y_1(q^2 - 1) + x_q} + \frac{1}{(y_2 - x)^2} \right\}
\]

such that \( x^l_a \leq x \leq x^u_a \); \( y^l_a \leq y \leq y^u_a \); \( q^l_a \leq q \leq q^u_a \)

\[
N^u_a = \max \left\{ \frac{1}{(y_1(1 - q) - x)^2} \frac{y_1(q^2 - 1) - x_q}{y_1(q^2 - 1) + x_q} + \frac{1}{(y_2 - x)^2} \right\}
\]

such that \( x^l_a \leq x \leq x^u_a \); \( y^l_a \leq y \leq y^u_a \); \( q^l_a \leq q \leq q^u_a \)

This pair of mathematical programs involves the systematic study of how the optimal solution change when \( x^l_a, x^u_a, y^l_a, y^u_a, q^l_a \) and \( q^u_a \) vary over the interval \( \alpha \in (0, 1] \) they fall into the category of parametric NLP. As \( \alpha \) increases since the objective function becomes more and more complicated, the difficulty in solving the pair of mathematical programs also increases. If both \( N^l_a \) and \( N^u_a \) are invertible with respect to \( \alpha \) then left shape function \( L(Z) = \left( N^l_a \right)^{-1} \) and a right shape function \( R(Z) = \left( N^u_a \right)^{-1} \) can be obtained from which the membership function \( \mu_{\tilde{N}} \) is constructed.

\[
\mu_{\tilde{N}}(Z) = \begin{cases} 
L(z) & ; z_1 \leq z \leq z_2 \\
1 & ; z_2 \leq z \leq z_3 \\
R(z) & ; z_3 \leq z \leq z_4 
\end{cases}
\]
Otherwise the set of intervals \( \{ (N^a_\alpha, N^u_\alpha) / \alpha \in (0, 1) \} \) still reveals the shape of \( \mu_N \) although exact function is not known explicitly.

### 3.2.3. Numerical Example

Consider the open central server queuing model with two channels with service rate at channel 1 following a fuzzy rate \( y_1 = [5, 6, 7, 8] \) and at channel 2 following a fuzzy rate \( y_2 = [9, 10, 11, 12] \) where both service rates are exponentially distributed. The arrival rate follow Poisson distribution following a fuzzy rate \( x = [1, 2, 3, 4] \) with a feedback probability \( q = [0.1, 0.2, 0.3, 0.4] \). We are interested in finding the mean and variance of sojourn time, the number of customers in the system and the waiting time of a customer in the system.

\[
N = \frac{1}{y_1(1 - q) - x} \left( \frac{y_1(q^2 - 1) - xq}{y_1(q^2 - 1) + xq} + \frac{1}{y_2 - x} \right)
\]

\[
L = \frac{1}{y_1(1 - q) - x} + \frac{1}{y_2 - x}
\]

\[
L_S = \frac{x}{y_1 - x} + \frac{1}{y_2 - x}
\]

\[
W_q = \frac{1}{R} L_S
\]

Let the arrival rate, service rates and probability be \( x = [1, 2, 3, 4] \); \( y_1 = [5, 6, 7, 8] \); \( y_2 = [9, 10, 11, 12] \); \( q = [0.1, 0.2, 0.3, 0.4] \) respectively.
Then 

\[ [x_a^l, x_a^u] = [1 + \alpha, 4 - \alpha] \]

\[ [y_{1a}^l, y_{1a}^u] = [5 + \alpha, 8 - \alpha] \]

\[ [y_{2a}^l, y_{2a}^u] = [9 + \alpha, 12 - \alpha] \]

\[ [q_a^l, q_a^u] = [0.1 + 0.1\alpha, 0.4 - 0.1\alpha] \]

\[ N_a^l = \frac{1}{(5+\alpha)((0.1+0.1\alpha)^2-1)-(1+\alpha)(0.1+0.1\alpha)} \]

\[ + \frac{1}{((9+\alpha)(1+\alpha)^2)} \]

\[ N_a^u = \frac{1}{(8-\alpha)((0.4-0.1\alpha)^2-1)-(4+\alpha)(0.4+0.1\alpha)} \]

\[ + \frac{1}{((12-\alpha)(4-\alpha)^2)} \]

\[ L_a^l = \frac{1}{((5+\alpha)(1-(0.1+0.1\alpha))-(1+\alpha)^2)} + \frac{1}{(9+\alpha)-(1+\alpha)} \]

\[ L_a^u = \frac{1}{((8-\alpha)(1-(0.4-0.1\alpha))-(4-\alpha)^2)} + \frac{1}{(12+\alpha)-(4-\alpha)} \]

\[ (LS)_a^l = \frac{(1+\alpha)}{(5+\alpha)-(1+\alpha)} + \frac{(1+\alpha)}{(9+\alpha)-(1+\alpha)} \]

\[ (LS)_a^u = \frac{(4-\alpha)}{(8-\alpha)-(4-\alpha)} + \frac{(4-\alpha)}{(12-\alpha)-(4-\alpha)} \]

Average response time is 

\[ R = \frac{1}{x} \left[ \frac{y_1}{1 - y_1} + \frac{y_2}{1 - y_2} \right] \]

\[ R = 0.375; \ W_q = \frac{1}{R} L_s \]
MATLAB® 6.0 is used to depict the following table and figures.

<table>
<thead>
<tr>
<th>α</th>
<th>N^l</th>
<th>N^u</th>
<th>L^l</th>
<th>L^u</th>
<th>L^s</th>
<th>L^s^u</th>
<th>W^s</th>
<th>W^s^u</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.100382</td>
<td>2.55471</td>
<td>0.410714</td>
<td>1.375</td>
<td>0.375</td>
<td>1.5</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>0.1</td>
<td>0.104099</td>
<td>1.895407</td>
<td>0.415782</td>
<td>1.213139</td>
<td>0.4125</td>
<td>1.4625</td>
<td>1.1</td>
<td>3.9</td>
</tr>
<tr>
<td>0.2</td>
<td>0.108497</td>
<td>1.46198</td>
<td>0.421208</td>
<td>1.090250</td>
<td>0.45</td>
<td>1.425</td>
<td>1.2</td>
<td>3.8</td>
</tr>
<tr>
<td>0.3</td>
<td>0.112958</td>
<td>1.16211</td>
<td>0.427023</td>
<td>0.993809</td>
<td>0.4875</td>
<td>1.3875</td>
<td>1.3</td>
<td>3.7</td>
</tr>
<tr>
<td>0.4</td>
<td>0.117956</td>
<td>0.94658</td>
<td>0.433261</td>
<td>0.916139</td>
<td>0.525</td>
<td>1.35</td>
<td>1.4</td>
<td>3.6</td>
</tr>
<tr>
<td>0.5</td>
<td>0.123516</td>
<td>0.78649</td>
<td>0.439960</td>
<td>0.852272</td>
<td>0.5625</td>
<td>1.3125</td>
<td>1.5</td>
<td>3.5</td>
</tr>
<tr>
<td>0.6</td>
<td>0.129630</td>
<td>0.66453</td>
<td>0.447164</td>
<td>0.798854</td>
<td>0.6</td>
<td>10275</td>
<td>1.6</td>
<td>3.4</td>
</tr>
<tr>
<td>0.7</td>
<td>0.136467</td>
<td>0.56954</td>
<td>0.454924</td>
<td>0.753535</td>
<td>0.6375</td>
<td>1.2375</td>
<td>1.7</td>
<td>3.3</td>
</tr>
<tr>
<td>0.8</td>
<td>0.144092</td>
<td>0.49419</td>
<td>0.463294</td>
<td>0.651315</td>
<td>0.675</td>
<td>1.2</td>
<td>1.8</td>
<td>3.2</td>
</tr>
<tr>
<td>0.9</td>
<td>0.152629</td>
<td>0.43344</td>
<td>0.472342</td>
<td>0.680864</td>
<td>0.725</td>
<td>1.1625</td>
<td>1.9</td>
<td>3.1</td>
</tr>
<tr>
<td>1</td>
<td>0.162213</td>
<td>0.383787</td>
<td>0.482142</td>
<td>0.651315</td>
<td>0.75</td>
<td>1.125</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

![Fig.3.3. Variance Sojourn Time of Tandem Queue](image)

![Fig.3.4. Mean Sojourn Time of Tandem Queue](image)