Chapter-1

$S^*_g$-Open Sets In Topological Spaces

1.1 Introduction

In this chapter, analogous to P.Battacharya and Lahiri’s semi generalized closed sets, we define a new class of sets namely $S^*_g$-open sets using semi generalized closure operator instead of closure operator in the semi open set. We further show that the concept of $S^*_g$-open set is a weaker form of open sets but stronger than the concept of semi-open set and semi-generalized open set and it is independent with semi*-open set. We discussed the characterization of $S^*_g$-open sets and also we investigate some fundamental properties of $S^*_g$-open sets. We also define $S^*_g$-interior and $S^*_g$-closure of a subset and study some of its basic properties. Also we established that the class of $S^*_g$-closed sets are placed between the class of closed sets and the class of semi-closed sets.

1.2 Preliminaries

**Definition 1.2.1:** Let A be a subset of a topological space $(X, \tau)$.

(i) A is called a **semi-open set** [36] if there is an open set U in X such that $U \subseteq A \subseteq Cl(U)$ or equivalently if $A \subseteq Cl(Int(A))$.

(ii) A is called a **semi-closed set** [37] if its complement $(X-A)$ is a semi-open set.

(iii) The family of all semi-open sets(resp. semi-closed set) in $(X, \tau)$ is denoted by $SO(X, \tau)$(resp. $SC(X, \tau)$)

(iv) The **semi-interior** [8](briefly $sInt(A)$) of A is defined as the union of all semi-open sets of X contained in A.
(v) The semi-closure \([v]\)(briefly sCl(A)) of \(A\) is defined as the intersection of all semi-closed sets of \(X\) containing \(A\).

**Definition 1.2.2:** A subset \(A\) of a topological space \((X, \tau)\) is called

(i) a **generalized closed set** \([35]\)(briefly \(g\)-closed) if \(Cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).

(ii) a **generalized open set** \([35]\)(briefly \(g\)-open) if its complement is a generalized closed.

(iii) The family of all generalized closed sets(resp. generalized open sets) in \((X, \tau)\) is denoted by \(gO(X, \tau)\)(resp. \(gC(X, \tau)\))

(iv) The **generalized interior** \([23]\)(briefly \(Int^*(A)\)) of \(A\) is defined as the union of all \(g\) -open sets of \(X\) contained in \(A\).

(v) The **generalized closure** \([23]\)(briefly \(Cl^*(A)\)) of \(A\) is defined as the intersection of all \(g\)-closed sets in \(X\) containing \(A\).

**Definition 1.2.3:** A subset \(A\) of a topological space \((X, \tau)\) is called

(i) a **semi-generalized closed set** \([7]\)(briefly \(sg\)-closed) if \(sCl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \(X\).

(ii) a **semi-star generalized closed set** \([12]\)(briefly \(S^*g\)-closed)) if \(Cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \(X\).

(ii) The complement of a semi-generalized closed set(resp. semi-star generalized closed set) is called a **semi-generalized open set** \([7]\)(resp. semi star-generalized open set[12])). It is denoted by \(Sg\)-open(resp. \(S^*g\)-open).

(iii) The family of all semi-generalized closed sets(resp. semi-generalized open sets) in \((X, \tau)\) is denoted by \(SgO(X, \tau)\)(resp. \(SgC(X, \tau)\))

(iv) The family of all generalized closed sets(resp. generalized open sets) in \((X, \tau)\) is denoted by \(S^*gO(X, \tau)\)(resp. \(S^*gC(X, \tau)\))
(v) The **semi-generalized interior** [64](briefly $s\text{Int}^*(A)$) of $A$ is defined as the union of all $Sg$-open sets of $X$ contained in $A$.

(vi) The **semi-generalized closure**[68](briefly $s\text{Cl}^*(A)$) of $A$ is defined as the intersection of all $Sg$-closed sets in $X$ containing $A$.

**Definition 1.2.4:** A subset $A$ of a topological space $(X, \tau)$ is called

(i) a **semi*-open set** [62](briefly $S^*$-open) if there is an open set $U$ in $X$ such that $U \subseteq A \subseteq Cl^*(U)$ or equivalently if $A \subseteq Cl^*(\text{Int}(A))$.

(ii) a **semi*-closed set** [61](briefly $S^*$-closed) if its complement is a semi*-open set.

(iii) The class of all semi*-open sets(resp. semi*-closed sets) in $(X, \tau)$ is denoted by $S^*O(X, \tau)$(resp. $S^*C(X, \tau)$)

**Results 1.2.5:**[68] Let $A$ and $B$ be two subsets of a space $(X, \tau)$

(i) $A \subseteq s\text{Cl}^*(A) \subseteq s\text{Cl}(A) \subseteq Cl(A)$

(ii) $s\text{Cl}^*(\emptyset) = \emptyset$ and $s\text{Cl}^*(X) = X$

(iii) $s\text{Cl}^*(A \cup B) = s\text{Cl}^*(A) \cup s\text{Cl}^*(B)$

(iv) $s\text{Cl}^*(s\text{Cl}^*(A)) = s\text{Cl}^*(A)$

(v) $s\text{Cl}^*(A \cap B) \subseteq s\text{Cl}^*(A) \cap s\text{Cl}^*(B)$[63]

**Results 1.2.6:**[64] Let $A$ and $B$ be two subsets of a space $(X, \tau)$

(i) $s\text{Int}^*(\emptyset) = \emptyset$ and $s\text{Int}^*(X) = X$

(ii) $s\text{Int}^*(A \cup B) \supseteq s\text{Int}^*(A) \cup s\text{Int}^*(B)$

(iii) $s\text{Int}^*(A) \subseteq A$

(iv) If $A \subseteq B$, then $s\text{Int}^*(A) \subseteq s\text{Int}^*(B)$

(v) $s\text{Int}^*(A \cap B) = s\text{Int}^*(A) \cap s\text{Int}^*(B)$

(vi) $s\text{Int}^*(s\text{Int}^*(A)) = s\text{Int}^*(A)$ and $(s\text{Int}^*(A))^c = s\text{Cl}^*(A^c)$
1.3 $S^*_g$-Open Sets and their basic properties

In this section we introduced a new class of sets called $S^*_g$-open sets in topological spaces and investigated certain basic properties of these sets.

**Definition 1.3.1:** A subset $A$ of a topological space $(X, \tau)$ is called a $S^*_g$-open set if there is an open set $U$ in $X$ such that $U \subseteq A \subseteq sCl^*(U)$. The collection of all $S^*_g$-open sets in $(X, \tau)$ is denoted by $S^*O(X, \tau)$.

**Theorem 1.3.2:** A subset $A$ of a topological space $(X, \tau)$ is $S^*_g$-open set iff $A \subseteq sCl^*(Int(A))$.

**Proof:** **Necessity.** If $A$ is $S^*_g$-open, then there exists an open set $U$ such that $U \subseteq A \subseteq sCl^*(U)$. Since $U = Int(U) \subseteq Int(A)$, $sCl^*(U) = sCl^*(Int(U)) \subseteq sCl^*(Int(A))$. Hence $A \subseteq sCl^*(Int(A))$.

**Sufficiency.** Let $A \subseteq sCl^*(Int(A))$. Let $U = Int(A)$, then $U$ is an open set in $(X, \tau)$ such that $U \subseteq A \subseteq sCl^*(U)$.

**Theorem 1.3.3:** Every open set is $S^*_g$-open.

**Proof:** Let $A$ be an open set in $X$. Then $Int(A) = A$. Since $A \subseteq sCl^*(A)$, we have $A \subseteq sCl^*(Int(A))$. Therefore by the necessary and sufficient condition for $S^*_g$-open, $A$ is $S^*_g$-open.

**Remark 1.3.4:** The converse of theorem 1.3.3 need not be true as can be seen from the following example.

**Example 1.3.5:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Then $S^*O(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Here the subsets $\{a, b\}$ and $\{a, c\}$ are $S^*_g$-open but not open.
Theorem 1.3.6: Let \( \{A_\alpha\} \) be a collection of \( S^*_g \)-open sets in a topological space \((X, \tau)\). Then \( \bigcup A_\alpha \) is \( S^*_g \)-open.

Proof: Since \( A_\alpha \) is \( S^*_g \)-open, for each \( \alpha \) there is an open set \( U_\alpha \) in \((X, \tau)\) such that \( U_\alpha \subseteq A_\alpha \subseteq sCl^*(U_\alpha) \). Then \( \bigcup U_\alpha \subseteq \bigcup A_\alpha \subseteq \bigcup sCl^*(U_\alpha) \subseteq sCl^*(\bigcup U_\alpha) \). Since \( \bigcup U_\alpha \) is open, \( \bigcup A_\alpha \) is \( S^*_g \)-open.

Theorem 1.3.7: If \( A \) is \( S^*_g \)-open and \( B \) is open in a topological space \((X, \tau)\), then \( A \cup B \) is \( S^*_g \)-open.

Proof: Follows from Theorem 1.3.3 and Theorem 1.3.6.

Theorem 1.3.8: If \( A \) is \( S^*_g \)-open and \( B \) is open in a topological space \((X, \tau)\), then \( A \cap B \) is \( S^*_g \)-open.

Proof: Since \( A \) is \( S^*_g \)-open in \( X \), there exists an open set \( U \) in \( X \) such that \( U \subseteq A \subseteq sCl^*(U) \). Since \( B \) is open, \( U \cap B \) is open and we have \( U \cap B \subseteq A \cap B \subseteq sCl^*(U) \cap B \subseteq sCl^*(U \cap B) \). Hence \( A \cap B \) is \( S^*_g \)-open.

Theorem 1.3.9: Let \( A \subseteq B \subseteq sCl^*(A) \). If \( A \) is \( S^*_g \)-open in the topological space \((X, \tau)\), then \( B \) is \( S^*_g \)-open.

Proof: Suppose \( A \) is \( S^*_g \)-open in \( X \), then there exists an open set \( U \) in \( X \) such that \( U \subseteq A \subseteq sCl^*(U) \). Therefore \( U \subseteq B \) which implies that \( sCl^*(U) \subseteq sCl^*(B) \). Hence \( U \subseteq B \subseteq sCl^*(U) \) and \( B \) is \( S^*_g \)-open.

Theorem 1.3.10: A subset \( A \) of a topological space \( X \) is \( S^*_g \)-open if and only if it contains a \( S^*_g \)-open set about each of its points.


Sufficiency: Let \( x \in A \). Suppose there is a \( S^*_g \)-open set \( U_x \) containing \( x \) such that \( U_x \subseteq A \). Then we have \( \bigcup \{U_x: x \in A\} = A \). By Theorem 1.3.6, \( A \) is \( S^*_g \)-open.
Theorem 1.3.11: Every \( S_g^* \)-open set is semi-open.

Proof: Let \( A \) be a \( S_g^* \)-open set. Then there exists an open set \( U \) in \( X \) such that \( U \subseteq A \subseteq sCl^*(U) \). But \( sCl^*(U) \subseteq sCl(U) \subseteq Cl(U) \). Therefore \( U \subseteq A \subseteq Cl(U) \). Hence \( A \) is semi-open.

Remark 1.3.12: Converse of the above theorem is not true as can be seen from the following example.

Example 1.3.13: Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\} \). Then \( SO(X, \tau) = \{X, \phi, \{a\}, \{c\}, \{a, c\}\} \) and \( S_g^*O(X, \tau) = \{X, \phi, \{a\}, \{c\}, \{a, c\}\} \). Here the subsets \( \{a, b\} \) and \( \{b, c\} \) are semi-open but not \( S_g^* \)-open.

Theorem 1.3.14: Every \( S_g^* \)-open set is semi-generalized open.

Proof: It follows from the facts that every \( S_g^* \)-open set is semi-open and every semi open set is semi-generalized open.

Remark 1.3.15: Converse of the above theorem is not true as can be seen from the following example.

Example 1.3.16: Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a,b\}\} \). In this topological space \( (X, \tau) \), \( SgO(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( S_g^*O(X, \tau) = \{X, \phi, \{a, b\}\} \). Here the subsets \( \{a\} \) and \( \{b\} \) are semi-generalized open but not \( S_g^* \)-open.

Theorem 1.3.17: For any topological space \( (X, \tau) \), \( \tau \subseteq S_g^*O(X, \tau) \subseteq SO(X, \tau) \subseteq SgO(X, \tau) \).

Proof: It follows from the Theorems 1.3.3, 1.3.11 and 1.3.14.

Remark 1.3.18: \( S_g^* \)-open sets and \( S^*g \)-open sets are independent as shown by the following examples.
Example 1.3.19: Let $X=\{a,b,c,d,e\}$ and $\tau=\{X, \phi, \{a\}\}$. Then $S^* g O(X, \tau) = \{X, \phi, \{a\}\}$. Then $S^*_g O(X, \tau) = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,b,c\}, \{a,b,d\}, \{a,b,e\}, \{a,c,d\}, \{a,c,e\}, \{a,d,e\}, \{a,b,c,d\}, \{a,b,d,e\}, \{a,c,d,e\}, \{a,b,c,e\}\}$. Here the subsets $\{a\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,e\}, \{a,c,d\}, \{a,c,e\}, \{a,d,e\}, \{a,b,c,d\}, \{a,b,d,e\}, \{a,c,d,e\}$ and $\{a,b,c,e\}$ are $S^*_g$-open but not $S^* g$-open.

Example 1.3.20: Let $X=\{a,b,c\}$ and $\tau=\{X, \phi, \{a,b\}\}$. Then $S^* g O(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$. Then $S^*_g O(X, \tau) = \{X, \phi, \{a\}\}$. Here the subsets $\{a\}$ and $\{b\}$ are $S^* g$-open but not $S^*_g$-open.

Remark 1.3.21: $S^*_g$-open sets and $S^*$-open sets are independent as shown by the following examples.

Example 1.3.22: Let $X=\{a,b\}$ and $\tau=\{X, \phi, \{a\}\}=S^* O(X, \tau)$. Then $S^*_g O(X, \tau) = \{X, \phi, \{a\}, \{a,b\}\}$. The subsets $\{a\}$ and $\{a,b\}$ are $S^*_g$-open but not $S^*$-open.

Example 1.3.23: Let $X=\{a,b,c\}$ and $\tau=\{X, \phi, \{a\}, \{c\}, \{a,c\}\}$. Then $S^*_g O(X, \tau) = \tau$, $S^* O(X, \tau) = \{X, \phi, \{a\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}$. The subsets $\{a,b\}$ and $\{b,c\}$ are $S^*$-open but not $S^*_g$-open.

Remark 1.3.24: From the above theorems and remarks, we have the following diagram.
1.4 $S_g^*$-interior

In this section, we define $S_g^*$-interior and study its properties.

**Definition 1.4.1:** Let A be a subset of a topological space $(X, \tau)$. The $S_g^*$-interior of A is defined as the union of all $S_g^*$-open sets of $(X, \tau)$ contained in A and is denoted by $s_g^*Int(A)$.

**Definition 1.4.2:** Let A be a subset of X. A point $x \in X$ is called $S_g^*$-interior point of A if A contains a $S_g^*$-open set containing $x$.

**Theorem 1.4.3:** If A is any subset of X, $S_g^*Int(A)$ is $S_g^*$-open. In fact $s_g^*Int(A)$ is the largest $S_g^*$-open set contained in A.

**Proof:** Follows from Definition 1.4.1 and Theorem 1.3.6.

**Theorem 1.4.4:** A subset A of X is $S_g^*$-open if and only if $s_g^*Int(A)=A$.

**Proof:** If A is $S_g^*$-open then $s_g^*Int(A)=A$ is obvious. Conversely, let $s_g^*Int(A)=A$. By Theorem 1.4.3, $s_g^*Int(A)$ is $S_g^*$-open and hence A is $S_g^*$-open.

**Theorem 1.4.5:** If A is a subset of X, then $s_g^*Int(A)$ is the set of all $S_g^*$-interior points of A.

**Proof:** If $x \in S_g^*Int(A)$, then x belongs to some $S_g^*$-open subset U of A. That is, x is a $S_g^*$-interior point of A.

**Corollary 1.4.6:** A subset A of X is $S_g^*$-open if and only if every point of A is a $S_g^*$-interior point of A.

**Proof:** It follows from theorem 1.4.4 and theorem 1.4.5.
**Theorem 1.4.7:** If $A$ and $B$ are subsets of a topological space $(X,\tau)$, then the following results hold:

(i) $s_g^*\text{Int}(\emptyset)=\emptyset$.

(ii) $s_g^*\text{Int}(X)=X$.

(iii) $s_g^*\text{Int}(A)\subseteq A$.

(iv) $A\subseteq B \implies s_g^*\text{Int}(A)\subseteq s_g^*\text{Int}(B)$.

(v) $\text{Int}(A)\subseteq s_g^*\text{Int}(A)\subseteq s\text{Int}(A)\subseteq s\text{Int}^*(A) \subseteq A$.

(vi) $s_g^*\text{Int}(A \cup B)\supseteq s_g^*\text{Int}(A) \cup s_g^*\text{Int}(B)$.

(vii) $s_g^*\text{Int}(A \cap B)\subseteq s_g^*\text{Int}(A) \cap s_g^*\text{Int}(B)$.

(viii) $s_g^*\text{Int}(s_g^*\text{Int}(A))=s_g^*\text{Int}(A)$.

(ix) $\text{Int}(s_g^*\text{Int}(A))=\text{Int}(A)$.

(x) $s_g^*\text{Int}(\text{Int}(A))=\text{Int}(A)$.

**Proof:** (i), (ii), (iii) and (iv) follows from definition 1.4.1

(v) follows from theorem 1.3.17.

(vi) and (vii) follow from (iv) above.

(viii) and (ix) follows from theorem 1.4.3 and theorem 1.4.4.

(x) Since $\text{Int}(A)$ is open, we have $\text{Int}(A)$ is $S_g^*$-open. Now by theorem 1.4.4, $s_g^*\text{Int}(\text{Int}(A))=\text{Int}(A)$.
1.5 $S_g^*$-Closed Sets in topological spaces

In this section first we define $S_g^*$-Closed Sets in topological spaces and obtain some characterizations of these sets.

**Definition 1.5.1:** A subset $A$ of a topological space $(X, \tau)$ is called a $S_g^*$-*closed set* if $X \setminus A$ is $S_g^*$-*open*. The collection of all $S_g^*$-*closed sets* in $(X, \tau)$ is denoted by $S_g^*C(X, \tau)$.

**Theorem 1.5.2:** A subset $A$ of a topological space $(X, \tau)$ is $S_g^*$-*closed set* if and only if there exists a closed set $F$ in $X$ such that $sInt^*(F) \subseteq A \subseteq F$.

**Proof:** *Necessity.* Suppose $A$ is $S_g^*$-*closed*, then $X \setminus A$ is $S_g^*$-*open*. By definition 1.3.1, there exists an open set $U$ in $X$ such that $U \subseteq (X \setminus A) \subseteq sCl^*(U)$. This implies $X \setminus U \supseteq A \supseteq X \setminus sCl^*(U)$. Since $X \setminus sCl^*(U) = sInt^*(X \setminus U)$, $sInt^*(X \setminus U) \subseteq A \subseteq X \setminus U$ where $X \setminus U$ is a closed set in $X$.

**Sufficiency.** Suppose there exists a closed set $F$ in $X$ such that $sInt^*(F) \subseteq A \subseteq F$. Then $X \setminus sInt^*(F) \supseteq X \setminus A \supseteq X \setminus F$. Note that $X \setminus sInt^*(F) = sCl^*(X \setminus F)$. Hence $X \setminus F \subseteq X \setminus A \subseteq sCl^*(X \setminus F)$ where $X \setminus F$ is an open set in $X$. This implies $X \setminus A$ is a $S_g^*$-*open set* and $A$ is a $S_g^*$-*closed set*.

**Theorem 1.5.3:** A subset $A$ of a topological space $(X, \tau)$ is $S_g^*$-*closed set* if and only if $sInt^*(Cl(A)) \subseteq A$.

**Proof:** *Necessity.* If $A$ is a $S_g^*$-*closed set*, then by Theorem 1.5.2 there exists a closed set $F$ such that $sInt^*(F) \subseteq A \subseteq F$. Take $F = Cl(F)$, then $F$ is closed. Now $Cl(A) \subseteq Cl(F)$ implies that $sInt^*(Cl(A)) \subseteq sInt^*(Cl(F)) = sInt^*(F)$. Hence $sInt^*(Cl(A)) \subseteq A$.

**Sufficiency.** Let $sInt^*(Cl(A)) \subseteq A$. Then for $F = Cl(A)$, we get $F$ is a closed set in $X$ such that $sInt^*(F) \subseteq A \subseteq F$. Hence by Theorem 1.5.2, $A$ is $S_g^*$-*closed in X.*
**Theorem 1.5.4:** If \( \{A_\alpha\} \) is a collection of \( S^*_g \)-closed sets in a topological space \( X \), then \( \bigcap A_\alpha \) is \( S^*_g \)-closed.

**Proof:** Suppose \( A_\alpha \) is \( S^*_g \)-closed in \( X \), then \( X \setminus A_\alpha \) is \( S^*_g \)-open in \( X \). By theorem 1.4.3, \( \bigcup (X \setminus A_\alpha) \) is \( S^*_g \)-open in \( X \) that is \( X \setminus \bigcap A_\alpha \) is \( S^*_g \)-open in \( X \). Hence \( \bigcap A_\alpha \) is \( S^*_g \)-closed.

**Theorem 1.5.5:** Suppose \( A \) is \( S^*_g \)-closed in \( X \) and \( s\text{Int}^*(A) \subseteq B \subseteq A \). Then \( B \) is \( S^*_g \)-closed in \( X \).

**Proof:** Suppose \( A \) is \( S^*_g \)-closed in \( X \), then \( (X \setminus A) \) is \( S^*_g \)-open in \( X \). Now \( s\text{Int}^*(A) \subseteq B \subseteq A \) implies that \( X \setminus s\text{Int}^*(A) \supseteq X \setminus B \supseteq X \setminus A \), that is, \( X \setminus A \subseteq X \setminus B \subseteq s\text{Cl}^*(X \setminus A) \). Now by Theorem 1.3.9, \( X \setminus B \) is \( S^*_g \)-open in \( X \). Hence \( B \) is \( S^*_g \)-closed in \( X \).

**Theorem 1.5.6:** Every closed set is \( S^*_g \)-closed.

**Proof:** Let \( A \) be a closed set in \( X \). Then \( X \setminus A \) is open in \( X \). By Theorem 1.3.3, \( X \setminus A \) is \( S^*_g \)-open. Hence \( A \) is \( S^*_g \)-closed.

**Remark 1.5.7:** The converse of Theorem 1.5.6 need not be true as shown in the following example.

**Example 1.5.8:** Let \( X=\{a,b,c,d,e\} \) and \( \tau=\{X,\phi,\{c,d,e\}\} \). Then \( S^*_g \mathcal{C}(X,\tau)=\{X,\phi,\{c\},\{d\},\{e\},\{c,d\},\{c,e\},\{d,e\},\{c,d,e\}\} \). Here the subsets \( \{c\},\{d\},\{e\},\{c,d\},\{c,e\},\{d,e\} \) and \( \{d,e\} \) are \( S^*_g \)-closed set but not closed.

**Theorem 1.5.9:** If a subset \( A \) is \( S^*_g \)-closed and \( B \) is closed, then \( A \cap B \) is \( S^*_g \)-closed.

**Proof:** This follows from theorem 1.5.6 and theorem 1.5.4.

**Theorem 1.5.10:** If \( A \) is \( S^*_g \)-closed in \( X \) and \( B \) is closed in \( X \), then \( A \cup B \) is \( S^*_g \)-closed.

**Proof:** Suppose \( A \) is \( S^*_g \)-closed in \( X \), then \( (X \setminus A) \) is \( S^*_g \)-open in \( X \). Also \( B \) is closed in \( X \) implies that \( (X \setminus B) \) is open in \( X \). Hence by theorem 1.3.8, \( (X \setminus A) \cap (X \setminus B) \) is \( S^*_g \)-open in \( X \). Therefore \( X \setminus (A \cup B) \) is \( S^*_g \)-open and so \( A \cup B \) is \( S^*_g \)-closed.
**Theorem 1.5.11:** Every $S^*_g$-closed set is semi-closed.

**Proof:** Let $A$ be a $S^*_g$-closed set. Then $(X \setminus A)$ is a $S^*_g$-open set. Now by theorem 1.3.11, $(X \setminus A)$ is semi-open which implies $A$ is semi-closed.

**Remark 1.5.12:** Converse of the above theorem is not true as can be seen from the following example.

**Example 1.5.13:** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. In this topological space $(X, \tau)$, $SC(X, \tau) = P(X) - \{a, b, c\}$ and $S^*_g C(X, \tau) = \{X, \phi, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Here the subsets $\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}$ and $\{b, c\}$ are semi-closed but not $S^*_g$-closed.

**Theorem 1.5.14:** Every $S^*_g$-closed set is semi-generalized closed.

**Proof:** It follows from the facts that every $S^*_g$-closed set is semi-closed and every semi-closed set is semi-generalized closed.

**Remark 1.5.15:** Converse of the above theorem is false as shown in the following example.

**Example 1.5.16:** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. In this space, $Sg C(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}\}$ and $S^*_g C(X, \tau) = \{X, \phi, \{a\}, \{d\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$. It shows that the subsets $\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}$ and $\{c, d\}$ are semi-generalized closed but not $S^*_g$-closed.

**Remark 1.5.17:** The concept of $S^*_g$-closed sets and $S^*g$-closed sets are independent as shown by the following examples.

**Example 1.5.18:** Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c, d, e\}\}$. $S^*g C(X, \tau) = \{X, \phi, \{a\}, \{c, d, e\}, \{b, c, d, e\}, \{a, c, d, e\}\}$. $S^*_g C(X, \tau) = \{X, \phi, \{c\}, \{d\}, \{e\}, \{a, c\}, \{a, d\}, \{a, e\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \{c, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}$. Here the
subsets \{c\}, \{d\}, \{e\}, \{a,c\}, \{a,d\}, \{a,e\}, \{c,d\}, \{c,e\}, \{d,e\}, \{a,c,d\}, \{a,c,e\} and \{a,d,e\} are $S^*_g$-closed but not $S^*$-closed.

**Example 1.5.19:** Let $X=\{a,b,c\}$ and $\tau=\{X, \phi, \{a,b\}\}$. $S^*gC(X, \tau)=\{X, \phi, \{c\}, \{b,c\}, \{a,c\}\}$. $S^*_gO(X, \tau) = \{X, \phi, \{c\}\}$. Here the subsets \{b,c\} and \{a,c\} are $S^*g$-closed but not $S^*_g$-closed.

**Remark 1.5.20:** $S^*_g$-closed sets and $S^*$-closed sets are independent as shown by the following examples.

**Example 1.5.21:** Let $X=\{a,b,c,d,e\}$ and $\tau=\{X, \phi, \{a\}\}$. $S^*C(X, \tau)=\{X, \phi, \{b,c,d,e\}\}$. $S^*_gO(X, \tau) = \{X, \phi, \{b\}, \{c\}\}$. Here the subsets \{b\}, \{c\}, \{d\}, \{e\}, \{b,c\}, \{b,d\}, \{b,e\}, \{c,d\}, \{b,c,d\}, \{b,c,e\}, \{b,d,e\}, \{c,d,e\}, \{b,c,d,e\}\} are $S^*_g$-closed but not $S^*$-closed.

**Example 1.5.22:** Let $X=\{a,b,c,d\}$ and $\tau=\{X, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$. In this space, $S^*C(X, \tau)=\{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$. Here the subsets \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\} and \{b,c\} are $S^*$-closed but not $S^*_g$-closed.

**Theorem 1.5.23:** Let $(X, \tau)$ be any topological space and $\mathcal{F}$ be the family of all closed sets, then $\mathcal{F} \subseteq S^*_gC(X, \tau) \subseteq SC(X, \tau) \subseteq SgC(X, \tau)$

**Proof:** It follows from Theorem 1.5.6, Theorem 1.5.11 and Theorem 1.5.14.
Remark 1.5.24: From the above discussions we have the following figure.
1.6 $S^*_g$-closure

In this section, we define $S^*_g$-closure and discuss its properties.

**Definition 1.6.1:** Let $A$ be a subset of a topological space $(X, \tau)$. The $S^*_g$-closure of $A$ is defined as the intersection of all $S^*_g$-closed sets of $X$ containing $A$. It is denoted by $s^*_g\text{Cl}(A)$.

**Theorem 1.6.2:** If $A$ is any subset of $X$, $s^*_g\text{Cl}(A)$ is $S^*_g$-closed. In fact $s^*_g\text{Cl}(A)$ is the smallest $S^*_g$-closed set in $X$ containing $A$.

**Proof:** Follows from Definition 1.6.1 and Theorem 1.5.4.

**Theorem 1.6.3:** A subset $A$ of $X$ is $S^*_g$-closed iff $s^*_g\text{Cl}(A) = A$.

**Proof:** If $A$ is $S^*_g$-closed then $s^*_g\text{Cl}(A) = A$ is obvious. Conversely, suppose $s^*_g\text{Cl}(A) = A$. By Theorem 1.6.2, $s^*_g\text{Cl}(A)$ is $S^*_g$-closed and hence $A$ is $S^*_g$-closed.

**Theorem 1.6.4:** If $A$ and $B$ be subsets of a topological space $(X, \tau)$, then the following results hold:

(i) $s^*_g\text{Cl}(\emptyset) = \emptyset$.

(ii) $s^*_g\text{Cl}(X) = X$.

(iii) $A \subseteq s^*_g\text{Cl}(A)$.

(iv) $A \subseteq B \Rightarrow s^*_g\text{Cl}(A) \subseteq s^*_g\text{Cl}(B)$.

(v) $A \subseteq s\text{Cl}^*(A) \subseteq s^*_g\text{Cl}(A) \subseteq s^*_g\text{Cl}(A) \subseteq \text{Cl}(A)$.

(vi) $s^*_g\text{Cl}(A \cup B) \supseteq s^*_g\text{Cl}(A) \cup s^*_g\text{Cl}(B)$.

(vii) $s^*_g\text{Cl}(A \cap B) \subseteq s^*_g\text{Cl}(A) \cap s^*_g\text{Cl}(B)$.

(viii) $s^*_g\text{Cl}(s^*_g\text{Cl}(A)) = s^*_g\text{Cl}(A)$.

(ix) $\text{Cl}(s^*_g\text{Cl}(A)) = \text{Cl}(A)$. 
(x)\( s^*_g Cl(Cl(A)) = Cl(A) \).

**Proof:** (i), (ii), (iii) and (iv) Follows from Definition 1.6.1.

(v) Follows from Result 1.2.5, Theorems 1.5.6, 1.5.11 and 1.5.14.

(vi) and (vii) Follows from (iv) above.

(viii) and (ix) Follows from Theorems 1.6.2 and 1.6.3.

(x) Since \( Cl(A) \) being closed, \( Cl(A) \) is \( s^*_g \)-closed. Hence by Theorem 1.6.3, \( s^*_g Cl(Cl(A)) = Cl(A) \).