Chapter-6

$S_g^*$-Compact and $S_g^*$-Connectedness in Topological Spaces

6.1 Introduction

In 1974, Das[14] defined the concept of semi-connectedness in topology and investigated its properties. Compactness is one of the most important, useful and fundamental concepts in topology. In 1981, Dorsett[21] introduced and studied the concept of semi-compact spaces. Since then, Hanna and Dorsett[22], Ganster[26] and Mohammad S. Sarsak[49] investigated the properties of semi-compact spaces.

In this chapter, first we introduce a new cover called $S_g^*$-open cover. Using this open cover we define a new space called $S_g^*$-compact space. Also we study that every $S_g^*$-compact space is compact and every semi-compact space is $S_g^*$-compact. Additionally we introduce another space called $S_g^*$-connected space and establish its relationship between connected space. Further we connect $S_g^*$-connected with various functions and obtain some more results.

6.2 Preliminaries

Definition 6.2.1: A topological space $(X, \tau)$ is said to be compact[83](resp. semi-compact[21]) if every open(resp. semi-open) cover of $(X, \tau)$ has a finite subcover.

Definition 6.2.2: A topological space $(X, \tau)$ is said to be connected[83](resp. semi-connected[21]) if $X$ cannot be expressed as the union of two non-empty open (resp. semi-open) sets in $X$. 
6.3 $S^*_g$-Compactness

Under this section, a new space is introduced in topological spaces which is named as $S^*_g$-Compact space. Also its properties are studied.

**Definition 6.3.1:** A collection $\{A_i : i \in \Lambda\}$ of $S^*_g$-open sets in a topological space $(X, \tau)$ is called a $S^*_g$-open cover of a subset $A$ in $(X, \tau)$ if $A \subseteq \bigcup_{i \in \Lambda} A_i$.

**Definition 6.3.2:** A topological space $(X, \tau)$ is called $S^*_g$-compact if every $S^*_g$-open cover of $(X, \tau)$ has a finite subcover.

**Definition 6.3.3:** A subset $A$ of a topological space $(X, \tau)$ is called $S^*_g$-compact relative to $X$ if for every collection $\{U_i : i \in \Lambda\}$ of a $S^*_g$-open subsets of $X$ such that $A \subseteq \bigcup \{U_i : i \in \Lambda\}$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $A \subseteq \bigcup \{U_i : i \in \Lambda_0\}$.

**Definition 6.3.4:** A subset $B$ of a topological space $X$ is said to be $S^*_g$-compact if $B$ is $S^*_g$-compact as a subspace of $X$.

**Theorem 6.3.5:**

(i) Every $S^*_g$-compact space is compact.

(ii) Every semi-compact space is $S^*_g$-compact.

**Proof:** (i) and (ii) Follows from the Definitions 6.2.1 and 6.3.2.

**Theorem 6.3.6:** Every $S^*_g$-closed subset of a $S^*_g$-compact space $(X, \tau)$ is $S^*_g$-compact relative to $(X, \tau)$.

**Proof:** Let $A$ be a $S^*_g$-closed subset of a $S^*_g$-compact space $(X, \tau)$. Then $A^c$ is $S^*_g$-open in $(X, \tau)$. Let $\{U_i : i \in \Lambda\}$ be a cover of $A$ by $S^*_g$-open subsets of $X$ such that $A \subseteq \bigcup \{U_i : i \in \Lambda\}$. So $A^c \cup \{U_i : i \in \Lambda\} = X$. Since $(X, \tau)$ is $S^*_g$-there exists a finite subset...
\( \Lambda_0 \) of \( \Lambda \) such that \( A \subset A^c \cup \{U_i : i \in \Lambda\} = X \). Then \( A \subset \cup \{U_i : i \in \Lambda\} \) and hence \( A \) is \( S^*_g \)-compact relative to \( X \).

**Theorem 6.3.7:** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a surjective \( S^*_g \)-continuous map. If \( (X, \tau) \) is \( S^*_g \)-compact, then \( (Y, \sigma) \) is compact.

**Proof:** Let \( \{A_i : i \in \Lambda\} \) be an open cover of \( Y \). Since \( f \) is \( S^*_g \)-continuous, \( \{f^{-1}(A_i) : i \in \Lambda\} \) is a \( S^*_g \)-open cover of \( X \). Also, since \( X \) is \( S^*_g \)-compact, it has a finite subcover, say \( \{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\} \). The surjectiveness of \( f \) implies \( \{A_1, A_2, \ldots, A_n\} \) is a finite subcover of \( Y \) and hence \( Y \) is compact.

**Theorem 6.3.8:** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( S^*_g \)-irresolute surjective map. If \( (X, \tau) \) is \( S^*_g \)-compact, then \( (Y, \sigma) \) is \( S^*_g \)-compact.

**Proof:** Let \( \{A_i : i \in \Lambda\} \) be a \( S^*_g \)-open cover of \( Y \). Since \( f \) is \( S^*_g \)-irresolute, \( \{f^{-1}(A_i) : i \in \Lambda\} \) is a \( S^*_g \)-open cover of \( X \). Also, since \( X \) is \( S^*_g \)-compact, it has a finite subcover, say \( \{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\} \). Now \( f \) is onto implies \( \{A_1, A_2, \ldots, A_n\} \) is a finite subcover of \( Y \) and hence \( Y \) is \( S^*_g \)-compact.

**Theorem 6.3.9:** If a map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( S^*_g \)-irresolute and a subset \( B \) of \( (X, \tau) \) is \( S^*_g \)-compact relative to \( X \), then the image \( f(B) \) is \( S^*_g \)-compact relative to \( Y \).

**Proof:** Let \( \{A_i : i \in \Lambda\} \) be any collection of \( S^*_g \)-open subsets of \( Y \) such that \( f(B) \subset \cup \{A_i : i \in \Lambda\} \). Then \( B \subset \cup \{f^{-1}(A_i) : i \in \Lambda\} \) holds. Since by hypothesis \( B \) is \( S^*_g \)-compact relative to \( X \), there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( B \subset \cup \{f^{-1}(A_i) : i \in \Lambda_0\} \). Therefore we have \( f(B) \subset \cup \{A_i : i \in \Lambda_0\} \) which shows that \( f(B) \) is \( S^*_g \)-compact relative to \( Y \).
**Theorem 6.3.10:** If a surjective map \( f: (X, \tau) \to (Y, \sigma) \) is strongly \( S^*_g \)-continuous and \((X, \tau)\) is a compact space, then \((Y, \sigma)\) is \( S^*_g \)-compact.

**Proof:** Let \( \{A_i : i \in \Lambda \} \) be a \( S^*_g \)-open cover of \( Y \). Since \( f \) is strongly \( S^*_g \)-continuous, \( \{f^{-1}(A_i) : i \in \Lambda \} \) is a open cover of \( X \). Thus the open cover has a finite subcover, say \( \{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\} \) as \( X \) is compact. The surjectiveness of \( f \) implies \( \{A_1, A_2, \ldots, A_n\} \) is a finite subcover of \( Y \) and hence \( Y \) is \( S^*_g \)-compact.

**Corollary 6.3.11:** If a surjective map \( f: (X, \tau) \to (Y, \sigma) \) is perfectly \( S^*_g \)-continuous and \((X, \tau)\) is a compact space, then \((Y, \sigma)\) is \( S^*_g \)-compact.

**Proof:** Since every perfectly \( S^*_g \)-continuous function is strongly \( S^*_g \)-continuous, the result follows from Theorem 6.3.10.
6.4 $S^*_g$-Connectedness

In this section, another space named as $S^*_g$-connected space is introduced and certain characterizations are proved.

**Definition 6.4.1:** A topological space $(X, \tau)$ is called a $S^*_g$-connected space if $X$ cannot be written as a disjoint union of two nonempty $S^*_g$-open sets.

**Example 6.4.2:** Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a\}\}$. Then $S^*_gO(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and it is $S^*_g$-connected.

**Theorem 6.4.3:** Every $S^*_g$-connected space is connected.

**Proof:** Let $(X, \tau)$ be a $S^*_g$-connected space. Suppose that $(X, \tau)$ is not connected, then $X = A \cup B$ where $A$ and $B$ are disjoint nonempty open sets in $(X, \tau)$. Since every open set is a $S^*_g$-open set, $(X, \tau)$ is not a $S^*_g$-connected space and so $(X, \tau)$ is connected.

**Remark 6.4.4:** The converse of Theorem 6.4.3 is not true as can be seen from the following example.

**Example 6.4.5:** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$. Clearly $(X, \tau)$ is connected. The $S^*_g$-open sets of $X$ are $\{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$. Here $(X, \tau)$ is not $S^*_g$-connected because $X = \{a\} \cup \{b, c, d\}$ where $\{a\}$ and $\{b, c, d\}$ are non-empty $S^*_g$-open sets.

**Theorem 6.4.6:** A contra $S^*_g$-continuous image of a $S^*_g$-connected space is connected.

**Proof:** Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a contra $S^*_g$-continuous map of a $S^*_g$-connected space $(X, \tau)$ onto a topological space $(Y, \sigma)$. Suppose $(Y, \sigma)$ is not connected. Let $A$ and $B$ form a disconnection of $Y$. Then $A$ and $B$ are clopen and $Y = A \cup B$ where $A \cap B = \emptyset$. Since $f$ is contra $S^*_g$-continuous, $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. Thus
\(X\) is the union of disjoint nonempty \(S^*_g\)-open sets in \((X, \tau)\). Also \(f^{-1}(A) \cap f^{-1}(B) = \emptyset\).

Hence \(X\) is not \(S^*_g\)-connected which is a contradiction. Therefore \(Y\) is connected.

**Theorem 6.4.7:** For a subset \(A\) of a topological space \((X, \tau)\), the following are equivalent.

(i) \((X, \tau)\) is \(S^*_g\)-connected.

(ii) The only subsets of \((X, \tau)\) which are both \(S^*_g\)-open and \(S^*_g\)-closed are the empty set \(\emptyset\) and \(X\).

(iii) Each \(S^*_g\)-continuous map of \((X, \tau)\) into a discrete space \((Y, \sigma)\) with at least two points is a constant map.

**Proof:** (i) \(\Rightarrow\)(ii): Suppose that \(S \subset X\) is a proper subset, which is both \(S^*_g\)-open and \(S^*_g\)-closed. Then \(S^c\) is also \(S^*_g\)-open and \(S^*_g\)-closed. Therefore \(X = S \cup S^c\) is a disjoint union of two nonempty \(S^*_g\)-open sets which contradicts the fact that \(X\) is \(S^*_g\)-connected. Hence \(S = \emptyset\) or \(S = X\).

(ii) \(\Rightarrow\)(i): Suppose that \(X = A \cup B\) where \(A\) and \(B\) are disjoint nonempty \(S^*_g\)-open sets in \((X, \tau)\). Since \(A = B^c\), \(A\) is \(S^*_g\)-closed. But by assumption \(A = \emptyset\), which is a contradiction. Hence (i) holds.

(ii) \(\Rightarrow\)(iii): Let \(f: (X, \tau) \to (Y, \sigma)\) be a \(S^*_g\)-continuous map where \((Y, \sigma)\) is a discrete space with at least two points. Then \(f^{-1}(\{y\})\) is \(S^*_g\)-closed and \(S^*_g\)-open for each \(y \in Y\) and \(X = \cup \{f^{-1}(\{y\}) : y \in Y\}\). By assumption, \(f^{-1}(\{y\}) = \emptyset\) or \(f^{-1}(\{y\}) = X\).

If \(f^{-1}(\{y\}) = \emptyset\) for all \(y \in Y\), then \(f\) will not be a map. Hence, there exists only one point say \(y_1 \in Y\) such that \(f^{-1}(\{y\}) \neq \emptyset\) and \(f^{-1}(\{y_1\}) = X\) which shows that \(f\) is a constant map.
(iii) $\Rightarrow$ (ii): Let $U$ be both $S_g^*$-open and $S_g^*$-closed in $(X, \tau)$. Suppose that $U \neq \emptyset$. Define $f: (X, \tau) \to (Y, \sigma)$ by $f(U) = \{y_1\}$ and $f(U^c) = \{y_2\}$ for some distinct points $y_1$ and $y_2$ in $(Y, \sigma)$, then $f$ is $S_g^*$-continuous. By assumption, $f$ is a constant map. Therefore $y_1 = y_2$ and so $U = X$.

**Theorem 6.4.8:** Let $(X, \tau)$ be $S_g^*$-connected. Then each contra $S_g^*$-continuous map of $(X, \tau)$ into a discrete space $(Y, \sigma)$ with at least two points is a constant map.

**Proof:** Let $f: (X, \tau) \to (Y, \sigma)$ be a contra $S_g^*$-continuous map where $(Y, \sigma)$ is a discrete space with at least two points. Then $X$ is covered by $S_g^*$-open and $S_g^*$-closed covering $\{f^{-1}(\{y\}): y \in Y\}$. Since $(X, \tau)$ is $S_g^*$-connected, the only subsets of $(X, \tau)$ which are both $S_g^*$-open and $S_g^*$-closed are the empty set $\emptyset$ and $X$. Therefore $f^{-1}(\{y\}) = \emptyset$ or $f^{-1}(\{y\}) = X$. If $f^{-1}(\{y\}) = \emptyset$ for all $y \in Y$, then $f$ fails to be a map. Then, there exists only one point say $y \in Y$ such that $f^{-1}(\{y\}) \neq \emptyset$ and $f^{-1}(\{y\}) = X$ which shows that $f$ is a constant map.

**Theorem 6.4.9:** If $f: (X, \tau) \to (Y, \sigma)$ is a $S_g^*$-continuous surjection and $(X, \tau)$ is $S_g^*$-connected, then $(Y, \sigma)$ is connected.

**Proof:** Suppose $(Y, \sigma)$ is not connected. Then $Y = A \cup B$ where $A$ and $B$ are disjoint nonempty open subsets of $(Y, \sigma)$. Since $f$ is $S_g^*$-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty $S_g^*$-open sets in $(X, \tau)$. This contradicts the fact that $(X, \tau)$ is $S_g^*$-connected and hence $(Y, \sigma)$ is connected.

**Theorem 6.4.10:** If a surjective map $f: (X, \tau) \to (Y, \sigma)$ is $S_g^*$-irresolute and $(X, \tau)$ is $S_g^*$-connected, then $(Y, \sigma)$ is $S_g^*$-connected.

**Proof:** If possible assume that $Y$ is not $S_g^*$-connected. Then $Y = A \cup B$ where $A$ and $B$ are nonempty disjoint $S_g^*$-open sets of $(Y, \sigma)$. Since $f$ is $S_g^*$-irresolute, $f^{-1}(A)$ and $f^{-1}(B)$
are $S^*_g$-open sets in $(X, \tau)$. Since $f$ is onto, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty.

Now $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. Thus $X$ is the union of disjoint nonempty $S^*_g$-open sets in $(X, \tau)$. This contradicts the fact that $(X, \tau)$ is $S^*_g$-connected and hence $(Y, \sigma)$ is $S^*_g$-connected.

**Theorem 6.4.11:** If a surjective map $f: (X, \tau) \rightarrow (Y, \sigma)$ is strongly $S^*_g$-continuous and $(X, \tau)$ is a connected space, then $(Y, \sigma)$ is $S^*_g$-connected.

**Proof:** Similar to the proof of the theorem 6.4.10.

**Theorem 6.4.12:** Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a perfectly $S^*_g$-continuous map, $(X, \tau)$ a connected space, then $(Y, \sigma)$ has an indiscrete topology.

**Proof:** Suppose that there exists a proper open set $U$ of $(Y, \sigma)$, then $U$ is $S^*_g$-open in $(Y, \sigma)$. Since $f$ is perfectly $S^*_g$-continuous, $f^{-1}(U)$ is a proper open and closed subset of $(X, \tau)$. This implies $(X, \tau)$ is not connected which is a contradiction. Therefore $(Y, \sigma)$ has an indiscrete topology.

**Theorem 6.4.13:** Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a totally $S^*_g$-continuous map, from a $S^*_g$-connected space $(X, \tau)$ onto any space $(Y, \sigma)$, then $(Y, \sigma)$ is an indiscrete space.

**Proof:** Suppose $(Y, \sigma)$ is not indiscrete. Let $A$ be a proper nonempty open subset of $(Y, \sigma)$. Then $f^{-1}(A)$ is a proper nonempty $S^*_g$-open and $S^*_g$-closed subset of $(X, \tau)$, which is a contradiction to the fact that $(X, \tau)$ is $S^*_g$-connected. Then $(Y, \sigma)$ must be indiscrete.

**Theorem 6.4.14:** If $f$ is a contra $S^*_g$-continuous map from a $S^*_g$-connected space $(X, \tau)$ onto any space $(Y, \sigma)$, then $(Y, \sigma)$ is not a discrete space.

**Proof:** Suppose $(Y, \sigma)$ is discrete. Let $A$ be any proper nonempty open and closed subset of $(Y, \sigma)$. Then $f^{-1}(A)$ is a proper nonempty $S^*_g$-open and $S^*_g$-closed subset of $(X, \tau)$,
which is a contradiction to the fact that \((X, \tau)\) is \(S_g^*\)-connected. Hence \((Y, \sigma)\) is not a discrete space.

**Theorem 6.4.15:** Suppose that \(X\) is a \(S_g^*-T_{1/2}\) space then \(X\) is connected if and only if it is \(S_g^*\)-connected.

**Proof:** Suppose that \(X\) is connected. Then \(X\) cannot be written as a union of two non-empty disjoint proper subsets of \(X\). Suppose \(X\) is not \(S_g^*\)-connected. Let \(A\) and \(B\) be any two \(S_g^*\)-open sets subsets of \(X\) such that \(X = A \cup B\), where \(A \cap B = \emptyset\). Since \(X\) is a \(S_g^*-T_{1/2}\) space, every \(S_g^*\)-open sets are open. Hence \(A\) and \(B\) are open sets which contradicts the fact that \(X\) is not connected. Then \(X\) is \(S_g^*\)-connected. The converse part follows from the theorem that every \(S_g^*\)-connected space is connected.

**Theorem 6.4.16:** If \(f: (X, \tau) \rightarrow (Y, \sigma)\) is slightly \(S_g^*\)-continuous surjective function and \(X\) is \(S_g^*\)-connected then \(Y\) is connected.

**Proof:** Suppose \(Y\) is not connected. Then there exists non-empty disjoint open set \(A\) and \(B\) such that \(Y = A \cup B\). Therefore \(A\) and \(B\) are clopen sets in \(Y\). Since \(f\) is slightly \(S_g^*\)-continuous and surjective, \(f^{-1}(A)\) and \(f^{-1}(B)\) are non-empty disjoint \(S_g^*\)-opensets in \(X\). Also \(f^{-1}(Y) = X = f^{-1}(B)\). This shows that \(X\) is not \(S_g^*\)-connected, a contradiction. Hence \(Y\) is connected.

**Theorem 6.4.17:** If \(f\) is slightly \(S_g^*\)-continuous function from a connected space \((X, \tau)\) onto a space \((Y, \sigma)\) then \(Y\) is not a discrete space.

**Proof:** Suppose that \(Y\) is a discrete space. Let \(A\) be a proper nonempty open subset of \(Y\). Then \(f^{-1}(A)\) is nonempty \(S_g^*\)-clopen subset of \(X\), which is a contradiction to the fact that \(X\) is \(S_g^*\)-connected. Hence \(Y\) is not a discrete space.
**Theorem 6.4.18:** A space $X$ is $S^*_g$-connected if every slightly $S^*_g$-continuous from $X$ into any $T_0$ space $Y$ is constant.

**Proof:** Let every slightly $S^*_g$-continuous function from a space $X$ into $Y$ be constant. If $X$ is not $S^*_g$-connected then there exists a proper nonempty $S^*_g$-clopen subset $A$ of $X$. Let $(Y, \sigma)$ be such that $Y=\{a, b\}$ and $\sigma=\{\emptyset, Y, \{a\}, \{b\}\}$ be a topology. Let $f:X \to Y$ be any function such that $f(A) = \{a\}$ and $f(X - A) = \{b\}$. Then $f$ is a non-constant and slightly $S^*_g$-continuous function such that $Y$ is $T_0$ which is a contradiction. Hence $Y$ is $S^*_g$-connected.

**Theorem 6.4.19:** A space $(X, \tau)$ is $S^*_g$-connected if and only if every totally $S^*_g$-continuous function from a space $(X, \tau)$ into any $T_0$ space $(Y, \sigma)$ is a constant map.

**Proof:** Suppose $f:(X, \tau) \to (Y, \sigma)$ is a totally $S^*_g$-continuous function where $(Y, \sigma)$ is a $T_0$-space. Suppose that $f$ is not a constant map, then we can select two points $x$ and $y$ such that $f(x) \neq f(y)$. Since $(Y, \sigma)$ is a $T_0$-space and $f(x)$ and $f(y)$ are distinct points of $Y$, there exists an open set $G$ in $(Y, \sigma)$ containing $f(x)$ but not $f(y)$. Since $f$ is a totally $S^*_g$-continuous function, $f^{-1}(G)$ is a $S^*_g$-clopen subset of $(X, \tau)$. Clearly $x \in f^{-1}(G)$ and $y \notin f^{-1}(G)$. Now $X = f^{-1}(G) \cup (f^{-1}(G))^C$ which is the union of non-empty $S^*_g$-open subsets of $X$. Thus $X$ is not $S^*_g$-connected space, which contradicts the fact that $X$ is $S^*_g$-connected. Hence $f$ is a constant map.

Conversely, suppose $(X, \tau)$ is not a $S^*_g$-connected space there exists a proper non-empty $S^*_g$-clopen subset $A$ of $X$. Let $Y = \{a, b\}$ and $\tau = \{Y, \emptyset, \{a\}, \{b\}\}$ be a topology for $Y$. Let $f:(X, \tau) \to (Y, \sigma)$ be a function such that $f(A) = \{a\}$ and $f(Y \setminus A) = \{b\}$. Then $f$ is non-constant and totally $S^*_g$-continuous such that $Y$ is $T_0$, which is a contradiction. Hence $X$ must be $S^*_g$-connected.
**Theorem 6.4.20:** Let \( f: (X, \tau) \to (Y, \sigma) \) be a totally \( S^*_g \)-continuous function and \( Y \) is a \( T_1 \)-space. If \( A \) is a non-empty \( S^*_g \)-connected subset of \( X \). Then \( f(A) \) is singleton.

**Proof:** Suppose that \( f(A) \) is not a singleton. Let \( f(x_1) = y_1 \in A \) and \( f(x_2) = y_2 \in A \).

Since \( y_1, y_2 \in Y \) and \( Y \) is a \( T_1 \)-space, there exists an open set \( G \) in \( (Y, \sigma) \) containing \( y \), but not \( y_2 \). Since \( f \) is totally \( S^*_g \)-continuous, \( f^{-1}(G) \) is \( S^*_g \)-continuous, \( f^{-1}(G) \) is \( S^*_g \)-clopen set containing \( x_1 \) but not \( x_2 \). Now \( X = f^{-1}(G) \cup (f^{-1}(G))^c \). Thus we have expressed \( X \) as a union of two non-empty \( S^*_g \)-open sets. This contradicts the fact that \( X \) is \( S^*_g \)-connected. Therefore \( f(A) \) is singleton.

**Theorem 6.4.21:** Every semi-connected space is \( S^*_g \)-connected.

**Proof:** Let \( (X, \tau) \) be a semi-connected space. Suppose that \( (X, \tau) \) is not \( S^*_g \)-connected, then \( X = A \cup B \) where \( A \) and \( B \) are disjoint nonempty \( S^*_g \)-open sets in \( (X, \tau) \). Since every \( S^*_g \)-open set is semi-open, \( (X, \tau) \) is not a semi-connected space which is a contradiction and hence \( (X, \tau) \) is \( S^*_g \)-connected.

**Remark 6.4.22:** The converse of the theorem 6.4.21 is not true as can be seen from the following example.

**Example 6.4.23:** Let \( X = \{a, b, c, d\} \) and \( \tau = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\} \). Then \( SO(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\} \) and \( S^*_g O(X, \tau) = \tau \) Here \( (X, \tau) \) is \( S^*_g \)-connected but not semi-connected because \( X = \{a\} \cup \{b, c, d\} \) where \( \{a\} \) and \( \{b, c, d\} \) are non-empty disjoint semi-open sets.
Theorem 6.4.24: If $f: (X, \tau) \to (Y, \sigma)$ is almost $S^*_\theta$-continuous surjection and $(X, \tau)$ is $S^*_\theta$-connected, then $(Y, \sigma)$ is almost connected.

Proof: Suppose $(Y, \sigma)$ is not almost connected. Then $Y = A \cup B$ where $A$ and $B$ are disjoint nonempty regular open sets of $(Y, \sigma)$. Since $f$ is almost $S^*_\theta$-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty $S^*_\theta$-open sets in $(X, \tau)$. This contradicts the fact that $(X, \tau)$ is $S^*_\theta$-connected and hence $(Y, \sigma)$ is almost connected.

Theorem 6.4.25: If $f: (X, \tau) \to (Y, \sigma)$ is completely $S^*_\theta$-irresolute surjective function and $(X, \tau)$ is almost connected then $(Y, \sigma)$ is $S^*_\theta$-connected.

Proof: Suppose $(Y, \sigma)$ is not $S^*_\theta$-connected. Then there exists non-empty disjoint $S^*_\theta$-open sets $A$ and $B$ such that $Y = A \cup B$. Since $f$ is completely $S^*_\theta$-irresolute and surjective, $f^{-1}(A)$ and $f^{-1}(B)$ are regular open sets in $X$. Moreover $f^{-1}(A) \cup f^{-1}(B) = X$, $f^{-1}(A) \neq \emptyset$ and $f^{-1}(B) \neq \emptyset$. This shows that $X$ is not almost connected which is a contradiction to the assumption. Hence $(Y, \sigma)$ is $S^*_\theta$-connected.