CHAPTER – 3

ON CONVERGENCE IN B*-CONTINUOUS FUNCTIONS

3.1 INTRODUCTION:

In this chapter it is shown that pointwise limit of transfinite sequence of B*-continuous functions and uniform limit of a sequence of B*-continuous functions is again B*-continuous; but this property is not shared by the quasi-uniform convergence. Again it is established that B*-continuity of uniform limit of a sequence of B*-continuous functions is transmitted to the oscillation-type function associated with the B*-continuous function. Also quasi-uniform and quasi-normal convergence of sequence of real-valued B*-continuous functions are compared.

3.2 ON POINTWISE AND QUASI-UNIFORM CONVERGENCE:

W. Sierpinski [83] proved that pointwise limit of transfinite sequence of continuous function is continuous and that of Baire-1 function is Baire-1 again. In 1974, A. Neubrunnova [63] showed that quasi-continuity and cliquishness of functions from separable metric space to metric spaces are preserved under the pointwise limit of transfinite sequence.

We can show by example that pointwise limit of a sequence of B*-continuous function may not be B*-continuous.

Example 3.2.1: Let $f_n(x) = x^n$, $x \in [0, 1]$.

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Then $f_n \to f$ pointwise. Now each f_n being continuous it is B*-continuous. But f is not B*-continuous at x = 1.

But, pointwise limit of transfinite sequences of B*-continuous functions remains B*-continuous.

Theorem 3.2.1: *Let* f_α: X→ Y *where* X *be a separable metric space and* Y *be any metric space, with metric* d, ($\alpha < \Omega$). *If the transfinite sequence* $\{f_\alpha\}$ *of* B*-*continuous functions pointwise converge to* f, *then* f *is also* B*-*continuous.*

Proof: Let f be not B*-continuous at the point $x_0 \in X$. Let $K(x_0, \delta)$ be a neighbourhood of x_0, where $K(x_0, \delta)$ denotes the sphere with centre x_0 and radius δ. Then $T=\{t\in X: d(f(t), f(x_0)) \geqslant \varepsilon\}$ is dense in $K(x_0, \delta)$. Let S be a countable dense subset of T. There is a $\beta < \Omega$ such that $\alpha > \beta$, $f_\alpha(x) = f(x)$ for all $x \in S \cup \{x_0\}$ [63] . Let $\gamma > \beta$ is any fixed ordinal number. Then, f_γ is B*-continuous at the point x_0.

Then, $P = \{x: d(f_y(x), f_y(x_0)) < \varepsilon\}$ is not nowhere dense in $K(x_0, \delta)$ and has Baire property. Therefore, there exists open set $U \subset K(x_0, \delta)$, such that $U \cap P \neq \varphi$, where $P = (O \setminus I) \cup J$, O is open and I, J are sets of first category (as P has Baire property). Let $z \in U \cap P \cap S$; then, $d(f_y(z), f_y(x_0)) < \varepsilon$. But, $f_y(z) = f(z)$, $f_y(x_0) = f(x_0)$. Therefore, $d(f(z), f(x_0)) < \varepsilon$, i.e. $z \notin S$, which is a contradiction. Hence, f is B*-continuous.

Like pointwise limit, the quasi-uniform limit of a sequence of B*-continuous functions is also not B*-continuous, which is clear from the following example.

Example 3.2.2: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions, where each $f_n: \mathbb{R} \to \mathbb{R}$ is defined by $f_n = \chi_{\langle 0,1/n \rangle}((-1)^n.x)$, for all n. Each f_n being quasi continuous [21], it is B*-continuous. Also $f_n \to f = \chi_{\{0\}}$. Let $\varepsilon > 0$, $m \in \mathbb{N}$. Denote $p = m + 2$.

Then for $x \geqslant 0$, we have $|f_{m+p-1}(x) - f(x)| = 0$ and for $x<0$ we have $|f_{m+p}(x) - f(x)| = 0$.

Therefore, $\min \{ |f_{m+1}(x) - f(x)|, \ldots |f_{m+p}(x) - f(x)| \} < \varepsilon$, for all $x \in \mathbb{R}$.

So, $\{f_n\}$ quasi-uniformly converges to f. But f is not B*-continuous.

3.3 UNIFORM CONVERGENCE:

It is well known that the sets of quasi continuity and cliquish functions are closed with respect to the uniform convergence [27]. The following theorem indicates that the class of B*-continuous functions is closed under uniform limit.

Theorem 3.3.1: *Let* f_n: X → Y, *where* X *is a topological space and* Y *is a metric space with metric* d, *be* B*-*continuous functions* (n ∈ ℕ), *and* $f_n \to f$ *uniformly on* X. *Then* f *is* B*-*continuous.*

Proof: Let $f_n \to f$ uniformly on X. Then given ε > 0 there exists N ∈ ℕ such that for n ⩾ N $d(f_n(x), f(x)) < \varepsilon$ for all x ∈ X. Let p ∈ X be an arbitrary point and m > N. Then, f_m being B*-continuous, for any neighbourhood U of the point p, there exists a nonempty open set G ⊂ U such that the set $H = \{ x \in X : d(f_m(x), f_m(p)) < \varepsilon/4\}$ having the property of Baire, is dense in G. Let x ∈ H.

Then, $d(f(x), f(p)) \leqslant d(f(x), f_m(x)) + d(f_m(x), f_m(p)) + d(f_m(p), f(p)) < 3\varepsilon/4 < \varepsilon$.

Hence, $f(H) \subset \{y \in Y : d(f(p), y) < \varepsilon\}$, i.e., $H \subset U \cap f^{-1}(K(f(p), \varepsilon))$.

Hence, f is B*-continuous at p.

The next theorem shows that B*-continuity of uniform limit of a sequence of B*-continuous functions $\{f_n\}_n$ is translated to that of the sequence $\{\Omega_{f_n}\}_n$ of oscillation-type functions (defn.2.3.1) associated with the sequence of functions.

Theorem 3.3.2: *Let* f: X → (Y, d), f_n: X → Y, (n = 1, 2, ...) *where* f_n *are* B*-*continuous. If* $f_n \to f$ *uniformly on* X, *then* $\Omega_{f_n} \to \Omega_f$ *uniformly on* X.

Proof: Let ε > 0. Then there exists N ∈ ℕ such that for n ⩾ N $d(f_n(x), f(x)) < \varepsilon$ for all x ∈ X. Let x be any point of X and let U(x) be a neighbourhood of x. As x is the point of B*-continuity of f, there exist a nonempty open set G and a B*-set B having the property of Baire such that $B \subset G$ is dense in G. Then for each $y \in B$, we have by triangle inequality,

$d(f(y), f(x)) \leqslant d(f(y), f_n(y)) + d(f_n(y), f_n(x)) + d(f_n(x), f(x)) < 2\varepsilon + d(f_n(y), f_n(x))$.

In the same way we can establish that $d(f_n(y), f_n(x)) < 2\varepsilon + d(f(y), f(x))$.

Therefore the above two inequalities imply that $|\Omega_{f_n}(x, U(x)) - \Omega_f(x, U(x))| \leqslant 2\varepsilon$.

Hence, $|\Omega_{f_n}(x) - \Omega_f(x)| \leqslant 2\varepsilon$ for $n \geqslant N$ and for every $x \in X$.

Note 3.3.1: It is shown in a paper [15] that the quasi-uniform convergence as well as quasi-normal convergence imply point-wise convergence, and they are both implied by the uniform convergence.

It is clear that the sequence $\{f_n\}_{n \in N}$ of functions in example 3.2.1, is quasi-normally convergent to f, but they are not quasi-uniformly convergent to f.

The sequence of functions in example 3.2.2, is quasi-uniformly convergent to f, but they are not quasi-normally convergent.

The contents of this chapter have been published in *ANALELE STIINTIFICE ALE UNIVERSITATII 'Al I Cuza', IASI, Tomul XLIX, s.I a, Matematica, 2003.*