CHAPTER – 2

SOME REMARKS ON B*-CONTINUOUS FUNCTIONS

2.1 INTRODUCTION AND BASIC PROPERTIES:

In this chapter we study some properties of B*- continuous functions and characterize B*-continuous function by a new oscillation-type function.

Proposition 2.1.1: *If* $\{A_n\}_n$ *is a family of B*-sets, then* $\bigcup_{n=1}^{\infty} A_n$ *is also a B*-set.*

Proposition 2.1.2: *Any quasi-continuous function is B*- continuous.*

We now show that if a function is B*-continuous on a dense set of points it may not be so on the whole space.

Example 2.1.1: Let f: $\mathbb{R} \rightarrow [0, 1]$ be such that,

$$f(x) = \begin{cases} 0 & \text{if } x = 1, 2, \ldots, n \text{ (n is fixed and finite)} \\ 1 & \text{elsewhere} \end{cases}$$

The function is B*-continuous on $\mathbb{R} \setminus \{1, 2, \ldots, n\}$ which is dense in $\mathbb{R}$, but it fails to be B*-continuous at each of the points 1, 2,…, n.

But we have the following theorem:

Theorem 2.1.1: *If for each open set* U *in* X *and each open set* V *in* Y, $U \cap f^{-1}(V)$ *is dense in* U *and if* f *is* B*- *continuous on a dense set, it will be so in the whole of* X.

Proof: Let f be B*-continuous on E, where Cl(E) = X, and $y \in X \setminus E$. Let U be open set in X containing y and V be open set in Y containing f(y). Then $U \cap f^{-1}(V)$ does not contain a B*-set.

Then two cases may arise:

Case I: $f^{-1}(V)$ is nowhere dense in U. As $U \cap f^{-1}(V)$ is dense in U, it contradicts that $y \notin E$. Hence, E = X.

Case II: $U \cap f^{-1}(V)$ is not nowhere dense in U, but does not have the property of Baire. Then for $B \subset U \cap f^{-1}(V)$, $B \neq (O \setminus I) \cup J$, where O is open and I, J are sets of first category. Then B does not intersect an open set. This contradicts $U \cap f^{-1}(V)$ is not nowhere dense. Hence, E = X.

Remark 2.1.1: We may inquire whether the set B(f) of points of B*-continuity is closed or not. But the function in example 2.1.1 shows that B(f) may not be closed.

The following example shows that B*-continuous functions may be decomposed into two functions which are not B*-continuous.

Example 2.1.2: Let,

$$f(x) = \begin{cases} 0 & \text{if } x = 1, 2, \ldots, n \text{ (n is fixed and finite)} \\ 1 & \text{elsewhere} \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{if } x = 1, 2, \ldots, n \\ 0 & \text{elsewhere} \end{cases}$$

Then f and g are not B*-continuous, but f + g is a constant function and hence B*-continuous.

2.2 ON POINTWISE B*-CONTINUOUS FUNCTION:

Definition 2.2.1: A function is said to be pointwise B*-continuous in X if the set of points where the function is non-B*-continuous is everywhere dense in X but is not closed relative to X.

Definition 2.2.2: A function is said to be pointwise non-B*-continuous in X if the set of points where the function is B*-continuous is everywhere dense in X but is not closed relative to X.

Theorem 2.2.1: *The set of points at which a pointwise B*-continuous function is B*-continuous is nowhere dense.*

Proof: Suppose A, the set of points at which f is B*-continuous, were not nowhere dense in X. Then there would exist atleast one open set G such that A would be everywhere dense in G. Thus f would be B*-continuous on a dense set in G. Hence by theorem 2.1.1 f would be B*-continuous at every point of G. This contradicts the hypothesis that the set of points at which f is non-B*-continuous is everywhere dense in X. Hence the set of points at which f is B*-continuous is nowhere dense.

Theorem 2.2.2: *Let* X *be a Baire space and* Y *be a second countable topological space and* f: X $\rightarrow$ Y *be an arbitrary function. The function* f *will be B*-continuous if the set* D_f *of points at which* f *is discontinuous, is of the first category.*

Proof: Suppose that f is not B*-continuous at a point $p \in X$. Then there exists an open set V_p in the countable basis of Y such that there exists a nonempty open set U_p of X containing p such that for each nonempty B*-set $B \subset U_p$ implies $f(B) \not\subset V_p$. Hence f is not B*-continuous at each point of U_p and therefore not continuous at each point of U_p. Thus $U_p \subset D_f$ and hence U_p is of the first category as D_f is so. But U_p is a nonempty open set of X, which is a Baire space. Therefore U_p is of the second category which is a contradiction. Hence f is B*-continuous at p.

Remark 2.2.2.1: The Baire property of X can not be omitted as will be clear from the following example:

Example 2.2.2.1: Let X be the set of rational numbers of the interval [0, 1] and Y = $\mathbb{R}$ (with the Euclidean metric). Let f: X $\rightarrow$ Y, with f(x) = 1/q for x = p/q, where p, q are integers prime to each other and q >0. Then D_f is a set of the first category but f is not B*-continuous.

2.3 CHARACTERIZATION OF THE CLASS OF B*-CONTINUOUS FUNCTIONS:

Definition 2.3.1: Let f: X $\rightarrow$ (Y, d). Let us define Ω_f: X $\rightarrow$ $[0, \infty)$ as follows:

$$\Omega_f(x) = \sup_{U} \inf_{B \subset U} \sup_{z \in B} \{d(f(x), f(z))\}$$

where *U* is the family of open neighbourhoods of x and *B* runs over all nonempty B*-sets in *U*. The above function is called the oscillation-type function associated with the function f.

Theorem 2.3.1: *A function* f: X $\rightarrow$ Y *is* B*-*continuous at a point* x $\in$ X *if and only if* $\Omega_f(x) = 0$.

Proof: Let $\Omega_f(x) = 0$ and $\varepsilon > 0$. It follows from the definition of $\Omega_f(x)$ that for an arbitrary neighbourhood U(x) of the point x we have $\Omega_f(x, U(x)) < \varepsilon$. Hence, there exists a nonempty open set G $\subset$ U(x) and a set B ($\subset$ G) having the property of Baire which is dense in G, such that f(B) $\subset$ V where V is any neighbourhood of f(x). Hence, B $\subset$ U $\cap$ $f^{-1}(V)$. Therefore, f is B*-continuous.

Conversely, let f be B*-continuous at x and let $\eta > 0$.

Then, $x \in Cl(Int(Cl(f^{-1}(V)))$ [theorem 1.2.1 (iv)], for every open set V with $f(x) \in V$.

Take $V = S(f(x), \eta/2)$ is an open sphere with centre at f(x) and radius $\eta/2$. Hence, an arbitrary neighbourhood U(x) of the point x contains a point $y \in Int(Cl(f^{-1}(V))$.

This implies that there exists a neighbourhood G of y such that $G \subset Cl(f^{-1}(V))$.

We may suppose $G \subset U(x)$. It follows that the set B of all those points $z \in G$ which is a member of $f^{-1}(V)$ is dense in G and possesses the property of Baire. If $y \in B$, then $d(f(y), f(x)) \leq \eta/2$.

Therefore, $\sup_{y \in B} d(f(y), f(x)) \leq \eta/2$.

Then by above definition $\Omega_f(x, U(x)) \leq \eta/2$ for every arbitrary neighbourhood U(x) of the point x. Therefore, $\Omega_f(x) = \sup_{U(x)} \Omega_f(x, U(x)) \leq \eta/2$.

Hence $\Omega_f(x) = 0$, as η is arbitrary.

Remark 2.3.1: We know that for a continuous function f, the set C(f) of points of continuity of f is a Borel set of class 1 which is determined by the oscillation of the function and hence is Lebesgue measurable. Thus the topological structure of the set C(f) is known. Now, we construct a non-dense, non-measurable set M in $\mathbb{R}$. Let us define a function f: $\mathbb{R} \to \mathbb{R}$, by $f(x) = \chi_M$. clearly f is B*-continuous at each point of $\mathbb{R} \setminus M$. But we have $\Omega_f(x) = 1$ for $x \in M$. Hence, f is not B*-continuous at the points of M. Now, the points of B*-continuity is not a Borel set because, $\mathbb{R} \setminus M$ is not Lebesgue measurable. Thus, we cannot predict the topological structure of the set B(f) of points of B*-continuity by means of the function $\Omega_f(x)$.

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