CHAPTER - 8

ON FUNCTIONS HAVING INTERMEDIATE VALUE-TYPE PROPERTIES

8.1 INTRODUCTION:

F. Roush and R. Gibson [36] introduced the notions WCIVP (Weak Cantor Intermediate Value Property) and CIVP (Cantor Intermediate Value Property) of function in their investigation of the characterization of extendable functions. These properties are intermediate value-type properties in some restricted form. Hence, functions having these properties are called Darboux-like functions. In this chapter we shall study the behaviour of such type of functions.

8.2 BASIC DEFINITIONS AND EXAMPLES:

Definition 8.2.1:[13] A function f: $\mathbb{R} \to \mathbb{R}$ is said to have the intermediate value property if for any two real numbers p, q ($p \neq q$) and $f(p) < f(q)$, and every $y \in (f(p), f(q))$ there exists a number x between p and q with $f(x) = y$. The functions having the intermediate value property are called Darboux functions.

Definition 8.2.2: [36] A function f: I → I is said to possess the Weak Cantor Intermediate Value Property (WCIVP) if for each subinterval (x, y) ⊂ I, with f(x) ≠ f(y), there exists a Cantor set C in (x, y) such that f(C) lies between f(x) and f(y).

Definition 8.2.3: [36] A function f: I → I is said to possess the Cantor Intermediate Value Property (CIVP) if for all p, q ∈ I with p ≠ q and f(p) ≠ f(q) and for every Cantor set K between f(p) and f(q) there exists a Cantor set C between p and q such that f(C) ⊂ K.

Definition 8.2.4: [12] A function f: I → I is a connectivity function if the graph of f restricted to C, denoted by $f|_C$, is connected in I × I whenever C is connected in I.

It is clear that every continuous function is a connectivity function. Also each connectivity function is a Darboux function. Again in [36], it is shown that every continuous function has the CIVP. But the converse is not always true, which follows from the following example.

Example 8.2.1: Let f(x) = sin 1/x, x ≠ 0; f(0) = 0. Clearly f is not continuous at 0. We choose x, y ∈ I such that x < y, f(x) ≠ f(y). Let K be an arbitrary Cantor set which is between f(x) and f(y).

Let y > 0 (the proof for other case is similar).

We can always find $x_1 \in (0, y)$ such that $f(x_1) = f(x)$. Since, $f|_{[x_1, y]}$ is continuous and K is between $f(x_1)$ and f(y), $C = f^{-1}(K) \cap [x_1, y]$ is closed. Again by the nature of f, C is uncountable. Hence, C contains a Cantor set P for which $f(P) \subset K$.

Therefore, f has the CIVP.

It is shown in [36] that under certain condition a function having the CIVP is continuous.

Theorem: If $f: I \rightarrow I$ is a closed function which has the CIVP then f is a continuous function.

8.3 SOME RESULTS ON DARBOUX-LIKE FUNCTIONS:

Lemma 8.3.1: *If* $f: \mathbb{R} \rightarrow \mathbb{R}$ *has the CIVP, then for each* $x \in \mathbb{R}$, *there exists a sequence* $\{y_n\}_n$ *such that* $y_n \nearrow x$ ($y_n \searrow x$) *and* $f(y_n) \rightarrow f(x)$.

Proof: Let $x \in \mathbb{R}$. Assume that there exists $\varepsilon > 0$ such that f is constant on no subinterval of $(x - \varepsilon, x)$, having x as a right-hand point. Let $\{x_n\}_n$ be an increasing sequence in $(x - \varepsilon, x)$ such that $f(x_n) \neq f(x)$ for all $n \in \mathbb{N}$. Let K_n be a sequence of Cantor sets between f(x) and $f(x_n)$ such that

$K_n \subset (f(x), \min (f(x_n), f(x) + 1/n))$ if $f(x_n) > f(x)$; and

$K_n \subset (\max (f(x_n), f(x) - 1/n), f(x))$ otherwise.

Since, f has the CIVP there exists a Cantor set C_n between x_n and x such that $f(C_n) \subset K_n$. For each n, choose $y_n \in C_n$. Then, $y_n \nearrow x$ and $f(y_n) \rightarrow f(x)$.

Theorem 8.3.1: *If* f: $\mathbb{R} \rightarrow \mathbb{R}$ *is a function of Baire class* 1 *and possesses the CIVP then f is a connectivity function.*

Proof: Let x be any point of the real line $\mathbb{R}$. First we suppose there exists no subinterval of $[x - \varepsilon, x]$ with x a right hand end point for which f is constant. By lemma 8.3.1, there exists a sequence $\{x_n\}_n$ of real numbers such that $x_n \nearrow x$ and $f(x_n)$ is convergent to f(x). Suppose $f(x_n) \neq f(x_{n+1})$ for all $n \in \mathbb{N}$, and P_n be a Cantor set between $f(x_n)$ and $f(x_{n+1})$. By the CIVP of f, there exists a Cantor set G_n between x_n and x_{n+1} such that $f(G_n) \subset P_n$ for all $n \in \mathbb{N}$.

Let $P = \cup G_n \cup \{x\}$. Then P is a perfect set and $f|_P$ is continuous at x from left.

In similar manner we can construct Q such that $f|_Q$ is continuous at x from right. If f is constant either on $[x - \varepsilon, x]$ or $[x, x + \varepsilon]$ for some $\varepsilon>0$, then $P=[x - \varepsilon, x]\cup[x, x+\varepsilon]$ is a perfect set with x as a bilateral limit point and $f|_P$ is continuous at x. Hence, f has a perfect road at every point of $\mathbb{R}$. Since, f belongs to Baire class 1 hence the graph of f is connected [13]. Therefore f is a connectivity function.

Corollary 8.3.1.1: *Let* f: $\mathbb{R} \rightarrow \mathbb{R}$ *be a function of Bounded variation and possesses the CIVP, then* f *is a connectivity function.*

Theorem 8.3.2: *Composite of two functions having the CIVP possesses the CIVP.*

Proof: Choose x, y $\in \mathbb{R}$ such that x < y and $g(f(x)) \neq g(f(y))$. Let K be any Cantor set between g(f(x)) and g(f(y)). Then, $f(x) \neq f(y)$ and there exists a Cantor set C between f(x) and f(y) with $g(C) \subset K$. Then, f having the CIVP, there exists a Cantor set $P \subset (x, y)$ with $f(P) \subset C$. So, $g(f(P)) \subset g(C) \subset K$.

Hence, $(g \circ f)$ possesses the CIVP.

It is well known that there are Darboux functions, which are not connectivity functions [12]. Gibson and Roush [36] constructed a connectivity function, which does not have the CIVP.

We are now able to construct a connectivity function, which does not possess the WCIVP.

Theorem 8.3.3: *There exists a connectivity function* f: $\mathbb{R} \rightarrow \mathbb{R}$ *such that* $f(x + y) = f(x) + f(y)$ *for each* x, y $\in \mathbb{R}$, *which does not have the WCIVP.*

Proof: Jones constructed [45] a function f: $\mathbb{R} \rightarrow \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ for each x, y $\in \mathbb{R}$, whose graph intersects every closed subset P of $\mathbb{R}^2$ with uncountable x-projection, dom.(P). Jones showed that f is a connectivity function. Let x < y, $f(x) \neq f(y)$ and C is any Cantor set between x and y. Since, f meets the closed subset $C \times \{f(x)\}$ of $\mathbb{R}^2$, so $f(C) \not\subset (f(x), f(y))$. Therefore f does not have the WCIVP.

If we consider the class of functions having the WCIVP, then this class is closed under uniform limit as is clear from the next theorem.

Theorem 8.3.4: *The uniform limit f of a sequence* $\{f_n\}_n$ *of functions having the WCIVP also possesses the WCIVP.*

Proof: Let $x < y$ and $f(x) \neq f(y)$. Choose x_1 and y_1 such that $x < x_1 < y_1 < y$; $f(x_1) \neq f(y_1)$. We assume $f(x_1)$ and $f(y_1)$ lie between $f(x)$ and $f(y)$ we may assume that $f(x) < f(x_1) < f(y_1) < f(y)$. Let $\varepsilon = 1/2 \min\{(f(x_1) - f(x)), (f(y) - f(y_1)), (f(y_1) - f(x_1))\}$. Then there exists $N \in \mathbb{N}$ such that for every $n \geqslant N$ and all $t \in \mathbb{R}$, $|f(t) - f_n(t)| < \varepsilon$. Since, each f_N has the WCIVP, there exists $C \subset (x_1, y_1)$ such that $f_N(C)$ lies between $f_N(x_1)$ and $f_N(y_1)$ (as $f_N(x_1) \neq f_N(y_1)$). Then, $C \subset (x, y)$ and $f(C) \subset (f(x), f(y))$. Therefore, f has the WCIVP.

Remark 8.3.1: Since the function f in theorem 8.3.3 does not have WCIVP, hence according to the theorem 8.3.4 f cannot be the uniform limit of a sequence of connectivity functions. Thus we can have the following theorem.

Theorem 8.3.5: *There exists a connectivity function* $f: \mathbb{R} \to \mathbb{R}$ *that is not the uniform limit of a sequence of functions from* $\mathbb{R}$ *into* $\mathbb{R}$ *having WCIVP.*

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