CHAPTER – 7

ON POINTS OF CONVERGENCE AND OSCILLATION OF A-CONTINUOUS FUNCTIONS

7.1 INTRODUCTION AND PRELIMINARIES:

In this chapter we study properties of some local types of uniform convergence, which are sufficient for proving the A-continuity at one point of the limit of a sequence of A-continuous functions and also introduced the concept of the A-oscillation of a function to give characterization of the A-continuity of a function.

The definition of A-continuity of a function at a point is given in definition 6.1.2.. The next definition introduces the local form of A-continuity of a function.

Definition 7.1.1: Let A be a regular matrix and f: $\mathbb{R} \to \mathbb{R}$ be a real-valued function. Let $x_0 \in \mathbb{R}$. Let $K = \{ \{x_n\}_n : x_n \xrightarrow{A} x_0\}$ .Then f is said to be locally A-continuous at the point $x_0 \in \mathbb{R}$ if there exists a neighbourhood $V(x_0)$ of x_0 such that for any sequence $\{x_n\}_n$ in $V(x_0) \cap K$, $f(x_n) \xrightarrow{A} f(x_0)$.

The following example shows that a function which is locally A-continuous at a point may not be A-continuous at that point.

Example 7.1.1: Let A be the Cesaro matrix of order 1, and $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/2 & \text{if } x \geqslant 0 \\ -1/2 & \text{if } x < 0 \end{cases}$$

Let us consider the point 0, then f(0) = 1/2.

Now consider the sequence $\{x_n\}_n$ where $x_n=(-1)^n$, obviously $x_n \xrightarrow{A} 0$.

But $f(x_n) \xrightarrow{A} 0 \neq f(0) = 1/2$. Thus f is not A-continuous at the point 0.

Now consider the neighbourhood $[0, \infty)$ of 0, and $\{x_n\}_n$ any sequence in $[0, \infty) \cap K$, where $K = \{ \{x_n\}_n : x_n \xrightarrow{A} 0\}$. Then $f(x_n) \xrightarrow{A} f(0) = 1/2$.

Thus f is locally A-continuous at 0.

7.2 ON LOCAL CONVERGENCE:

Definition 7.2.1: A sequence of functions $\{f_n\}_n$ is said to be uniformly convergent to a function f at a point a, if for any $\varepsilon > 0$, there exists $m \in N$ such that for all x in some neighbourhood N(a) of a and $n \geq m$, $| f_n(x) - f(x) | < \varepsilon$.

We know that if a sequence of continuous functions converges uniformly then the limit function is also continuous. The following theorem has been established for same type of result for locally A-continuous functions.

Theorem 7.2.1: *Let* $A = (a_{nk})$ *be a regular infinite matrix and* $\{f_n\}_n$ *is uniformly convergent to* f *at a point* a. *If infinitely many* f_n *are locally A-continuous at* a, *then* f *is also locally A-continuous at* a.

Proof: Let $\{f_n\}_n$ be uniformly convergent to f at a. So given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ and some neighbourhood N(a) of a, such that for all $x \in N(a)$, $n \geqslant m$,

$$|f_n(x) - f(x)| < \varepsilon/3 \quad \ldots\ldots(1).$$

As infinitely many f_n's are locally A-continuous at a, we can choose $s > m$ such that f_s is locally A-continuous at a.

Therefore there exists a neighbourhood V(a) of a such that, for any sequence $\{x_n\}$ in $V(a) \cap K$, where $K = \{ \{x_n\}_n : x_n \xrightarrow{A} a\}$, we have $f_s(x_n) \xrightarrow{A} f_s(a)$, i.e. there exists $m_0 \in \mathbb{N}$ such that $\left| \sum_{k=1}^{\infty} a_{nk} f_s(x_k) - f_s(a) \right| < \varepsilon/3$, for all $n > m_0$ ....(2).

Again $A = (a_{nk})$ is a regular, so there exists $m_1 \in \mathbb{N}$ and a constant G such that

$$\sum_{k=1}^{\infty} |a_{nk}| < G \text{ for all } n > m_1 \quad \ldots\ldots(3).$$

Let $M = \max. \{m_0, m_1\}$. Now, for each $\{x_k\}_k \subset V(a) \cap N(a) \cap K$ and $n > M$,

We have, $\left| \sum_{k=1}^{\infty} a_{nk} f(x_k) - f(a) \right|$

$$\leqslant \left| \sum_{k=1}^{\infty} a_{nk} f_s(x_k) - \sum_{k=1}^{\infty} a_{nk} f(x_k) \right| + \left| \sum_{k=1}^{\infty} a_{nk} f_s(x_k) - f_s(a) \right| + |f_s(a) - f(a)|$$

$$< \left| \sum_{k=1}^{\infty} a_{nk} f_s(x_k) - \sum_{k=1}^{\infty} a_{nk} f(x_k) \right| + \varepsilon/3 + \varepsilon/3$$

$$= \left| \sum_{k=1}^{\infty} a_{nk}[f_s(x_k) - f(x_k)] \right| + 2\varepsilon/3$$

$$\leqslant \sum_{k=1}^{\infty} |a_{nk}| \, | f_s(x_k) - f(x_k) | + 2\varepsilon/3$$

$$\leqslant G.\ \varepsilon/3 + 2\varepsilon/3 = \varepsilon\,(G + 2) / 3.$$

Thus, f is locally A-continuous at a.

The concept of quasi-uniform convergence of a sequence of functions is well known. This concept plays an important part in the formulation of conditions for continuity of limit functions of sequence of continuous functions. By analogy with definition of uniform convergence at a point a, we give the following local version of quasi-uniform convergence.

Definition 7.2.2: A sequence $\{f_n\}_n$ is said to be quasi-uniformly convergent to f at a point a, if there exists a neighbourhood N(a) of a such that $f_n(a)$ converges pointwise to f(a) in N(a), and for every $\varepsilon > 0$, for each $n \in N$, there exists $r(n) \in N$ such that $\min_{0 \leq i \leq r(n)} | f_{n+i}(x) - f(x) | < \varepsilon$ for all $x \in N(a)$.

We show that the concept of quasi-uniform convergence at a point enables us to give a very weak sufficient condition for A-continuity of limit function.

Theorem 7.2.2: *Let* A = (a_{nk}) *be a regular infinite matrix. If* $\{f_n\}_n$ *is a sequence of functions locally A-continuous at a point* a *and converges quasi-uniformly to* f *at the point* a, *then* f *is also locally A-continuous at* a.

Proof: Since A=(a_{nk}) is a regular, so there exists $m_0 \in \mathbb{N}$ and a constant G such that

$$\sum_{k=1}^{\infty} |a_{nk}| < G \text{ for all } n > m_0 \quad \ldots\ldots(1).$$

By the quasi-uniform convergence of $\{f_n\}$ to f at the point a, there exists a neighbourhood N(a) of a such that $f_n(a)$ converges pointwise to f(a) in N(a),

i.e. given $\varepsilon > 0$ there exists $m_1 \in \mathbb{N}$ such that $|f_n(a) - f(a)| < \varepsilon/3$ for all $n \geqslant m_1$...(2),

and for the given $\varepsilon > 0$, for each $n \in \mathbb{N}$ there exists $r(n) \in \mathbb{N}$ such that

$$\min_{0 \le i \le r(n)} |f_{n+i}(x) - f(x)| < \varepsilon/3 \text{ for all } x \in N(a) \quad \ldots\ldots (3).$$

Now let m = max.$\{m_0, m_1\}$. So from (3), for $m \in \mathbb{N}$ there exists some $p \in \mathbb{N}$ such that

$$|f_{m+t}(x) - f(x)| < \varepsilon/3 \text{ for all } x \in N(a) \text{ and } 0 \leqslant t \leqslant p.$$

Again, by locally A-continuity of functions f_{m+t} at the point a, there exists a neighbourhood V(a) of a such that for all sequence $\{x_k\}_k$ in $V(a) \cap K$, and for all integers $t \leqslant p$.

$$\left| \sum_{k=1}^{\infty} a_{nk} f_{m+t}(x_k) - f_{m+t}(a) \right| < \varepsilon/3 \quad \ldots\ldots(4).$$

Let $\{x_k\}_k$ be an arbitrary sequence in $N(a) \cap V(a) \cap K$. Owing to (4) we can choose $t \leqslant p$ such that $\left| \sum_{k=1}^{\infty} a_{nk} f_{m+t}(x_k) - f_{m+t}(a) \right| < \varepsilon/3$

Then we have, $\left| \sum_{k=1}^{\infty} a_{nk} f(x_k) - f(a) \right|$

$$\leqslant \left| \sum_{k=1}^{\infty} a_{nk} f_{m+t}(x_k) - \sum_{k=1}^{\infty} a_{nk} f(x_k) \right| + \left| \sum_{k=1}^{\infty} a_{nk} f_{m+t}(x_k) - f_{m+t}(a) \right| + \left| f_{m+t}(a) - f(a) \right|$$

$$< \left| \sum_{k=1}^{\infty} a_{nk} f_{m+t}(x_k) - \sum_{k=1}^{\infty} a_{nk} f(x_k) \right| + \varepsilon/3 + \varepsilon/3$$

$$= \left| \sum_{k=1}^{\infty} a_{nk} [f_{m+t}(x_k) - f(x_k)] \right| + 2\varepsilon/3$$

$$\leqslant \sum_{k=1}^{\infty} |a_{nk}| \, | f_{m+t}(x_k) - f(x_k) | + 2\varepsilon/3$$

$\leqslant G. \varepsilon/3 + 2\varepsilon/3 = \varepsilon(G+2)/3.$

Thus A-continuity of f at a follows.

7.3 A-OSCILLATION OF A FUNCTION:

We now introduce the concept of A-oscillation of a function and show that this A-oscillation characterizes A-continuity of a function.

Let f be a real-valued function defined on $\mathbb{R}$ and $A = (a_{mn})$ be a regular infinite matrix. For each point $a \in \mathbb{R}$ and an open neighbourhood V(a) of a and every natural number N, we define the number,

$w_A(a, V(a), N) = \sup \left| \sum_{k=1}^{\infty} a_{nk} f(x_k) - f(a) \right|$, where, the supremum is taken over all $n \geqslant N$ and all sequence $\{x_k\}_k$ in V(a) for which $x_k \xrightarrow{A} a$.

If $N < N'$ then obviously $w_A(a, V(a), N') \leqslant w_A(a, V(a), N)$.

Let, $w_A(a, V(a)) = \inf w_A(a, V(a), N)$, where infimum is taken over $N \in \mathbb{N}$.

Definition 7.3.1: Let $\mathcal{U}(a)$ be a system of all open neighbourhoods of a. Then $w_A(a) = \inf w_A(a, V(a))$, where the infimum is taken over $\mathcal{U}(a)$ is called the A-oscillation of the function f at the point a.

Theorem 7.3.1: *Let* $r > 0$ *and* f *be a real-valued function defined on* $\mathbb{R}$. *Then, the set* $H_r = \{x \in \mathbb{R}: w_A(x) < r\}$ *is an open set.*

Proof: Let x_0 be any point of H_r. We shall show that x_0 is an interior point of H_r. Since, $w_A(x_0) < r$, there is a neighbourhood $N(x_0)$ of x_0 such that $w_A(x_0, N(x_0)) < r$. By definition of $w_A(x_0, N(x_0))$ there exists an integer $m \geqslant 1$ such that

$$w_A(x_0, N(x_0), m) = \sup \left| \sum_{k=1}^{\infty} a_{nk} f(x_k) - f(x_0) \right| < r,$$

where the supremum is taken over all $n \geqslant m$ and all sequence $\{x_k\}$ in $N(x_0) \cap K$.

Since, for each $x \in N(x_0)$ the open set $N(x_0)$ is a neighbourhood of x, so, we have, $w_A(x, N(x_0), m) < r$, and this implies that $w_A(x, N(x_0)) < r$ and hence $w_A(x) < r$, for each $x \in N(x_0)$. Hence $N(x_0) \subset H_r$, which shows that x_0 is an interior point of H_r and therefore H_r is an open set.

Corollary 7.3.1.1: *The set of points of A-continuity of a function is a* G_δ *set in* $\mathbb{R}$.

Proof: Let C_f represents the set of points of A-continuity of f in $\mathbb{R}$.

Then $C_f = \{x \in \mathbb{R}: w_A(x) = 0\} = \bigcap_{k=1}^{\infty} U_k$, where $U_k = \{x \in \mathbb{R}: w_A(x) < 1/k\}$ is an open set by theorem 7.3.1. Hence, C_f is a G_δ set in $\mathbb{R}$.

We now establish a theorem, which characterises A-continuity with the help of A-oscillation of a function.

Theorem 7.3.2: *A real-valued function* f *defined on* $\mathbb{R}$ *is A-continuous at the point* a *if and only if* $w_A(a) = 0$.

Proof: Let f be A-continuous at a. Then $f(x_n) \xrightarrow{A} f(a)$ whenever $x_n \xrightarrow{A} a$.

So, given $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ and a neighbourhood V(a) taken from $\mathcal{U}(a)$ such that for $n \geq n_0$, $|\sum_{k=1}^{\infty} a_{nk}f(x_k) - f(a)| < \varepsilon$ whenever $\{x_n\}_n \in V(a) \cap K$.

From this we get $w_A(a, V(a), n_0) \leq \varepsilon$. Then, $w_A(a, V(a)) \leq \varepsilon$. So $w_A(a) \leq \varepsilon$.

Now ε being arbitrary small, $w_A(a) = 0$.

Conversely, let $w_A(a) = 0$. Then for each $\varepsilon > 0$ there exists $N(a) \in \mathcal{U}(a)$ such that $w_A(a, N(a)) < \varepsilon$. Then there exists a natural number n_0 such that $w_A(a, N(a), n_0) < \varepsilon$.

i.e. $\sup |\sum_{k=1}^{\infty} a_{nk}f(x_k) - f(a)| < \varepsilon$ for all $n \geq n_0$ and all $\{x_k\}$in $N(a) \cap K$.

Hence f is A – Continuous at the point a.

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