CHAPTER – 6

ON CONTINUITY OF A REAL FUNCTION ASSOCIATED WITH A REGULAR TYPE OF INFINITE MATRIX

6.1 INTRODUCTION AND PRELIMINARIES:

In the present chapter we establish some results relating to a real-valued A-continuous function satisfying some conditions of regular infinite matrix A and the nature of the set of points of A-discontinuity of the function is investigated.

In 1948, R. C. Buck [14] gave the idea of C-continuity of a real-valued function, where C is the Cesaro matrix of order 1 (Defn. 0.33).

Definition 6.1.1: Let $C = (c_{mn})$ be a Cesaro matrix. Then a real-valued function $f:\mathbb{R} \to \mathbb{R}$ is said to be C-continuous at the point $x_0 \in \mathbb{R}$ if $f(x_n) \xrightarrow{C} f(x_0)$ when $x_n \xrightarrow{C} x_0$. Otherwise f is said to be C-discontinuous at the point x_0.

If f is C-continuous at every point, then the function is called C-continuous function.

Buck [14] established the following result.

Result 6.1.1: If f: $\mathbb{R} \to \mathbb{R}$ is C-continuous atleast at one point of $\mathbb{R}$ then f is linear.

More generalised version of this type of continuity is A-continuuity where, A is an infinite regular matrix (Defn. 0.31).

Definition 6.1.2:[2] Let A be a regular matrix and f: $\mathbb{R} \to \mathbb{R}$ be a real-valued function. Then f is A-continuous at the point $x_0 \in \mathbb{R}$ if $f(x_n) \xrightarrow{A} f(x_0)$ whenever $x_n \xrightarrow{A} x_0$. Otherwise f is said to be A-discontinuous at the point x_0.

Josef Antoni and Tibor Salat found the existence of a regülar matrix A for which a non-linear function is A-continuous.

Result 6.1.2: [2] There exists a regular matrix A for which a non-linear function f is A-continuous.

E. C. Posner [76] generalised the notion of continuity of functions in the following result:

Result 6.1.3: Let A be a regular matrix and f: $\mathbb{R} \to \mathbb{R}$ be a real-valued function such that $\{f(x_n)\}_n$ is A-summable whenever $\{x_n\}_n$ converges. Then f is continuous.

Definition 6.1.3: A regular infinite matrix A = (a_{mn}) is said to satisfy the property (α) if there exists a (C, 1) summable sequence $\{s_n\}_1^\infty$ of 0's and 1's such that A-lim s_n is 1/2.

6.2 ON A-CONTINUOUS FUNCTIONS:

Theorem 6.2.1: *Let* A = (a_{mn}) *be a regular infinite matrix with the property* (α). *If* f *is A-continuous atleast at one point* x_0 *in* $\mathbb{R}$ *then* f *is continuous on* $\mathbb{R}$.

Proof: Let f be A-continuous at a point x_0. We first show that f is continuous at any $x \in \mathbb{R}$. Since A is a regular matrix having the property (α) so there exists a sequence $\{\alpha_n\}$ of 0's and 1's which is A-summable to 1/2. We consider the sequence $\{x_n\}$ in $\mathbb{R}$ such that $x_n = \alpha_n x + (1-\alpha_n)[2x_0 - x]$, where $\alpha_n = 0$ or 1 for all n. Therefore, A-lim $x_n = x_0$.

If possible, let f be not continuous at x. Then there exists a sequence $\{h_n\}$, $h_n \to 0$ such that $f(x+h_n) \to y \neq f(x)$ as $h_n \to 0$. (y may be $+\infty$ or $-\infty$.)

Now we take another sequence $\{y_n\}$ given by $y_n = \alpha_n (x+s_n) + (1-\alpha_n) [2x_0-x]$. Then for every sequence $\{s_n\}$, $s_n \to 0$, A-lim $y_n = x_0$.

In particular, for $z_n = \alpha_n(x+h_n) + (1 - \alpha_n)[2x_0 - x]$,

$f(z_n) = \alpha_n f(x+h_n) + (1-\alpha_n) f(2x_0 - x)$,

Hence, A-lim $f(z_n) = 1/2\, y + 1/2\, f(2x_0 - x)$.

Since f is A-continuous at x_0, A-lim $f(x_n) = f(x_0) =$ A-lim $f(z_n)$.

Also A-lim $f(x_n) = 1/2\, f(x) + 1/2\, f(2x_0 - x)$.

Therefore y = f(x) which is a contradiction. Hence f is continuous at the point x. But x is arbitrary. Therefore f is continuous on ℝ.

Note 6.2.1.1: The property (α) of the regular matrix A is essential for holding theorem 6.2.1. We give the following example in the support of this remark.

Example 6.2.1.1: Let A be the infinite identity matrix. Then clearly A is regular. But A does not have the property (α).

Now we consider a real-valued function f defined by

$$f(x) = \begin{cases} 0 \text{ for } x < 1/2 \\ 1 \text{ for } x \geq 1/2 \end{cases}$$

Then f is discontinuous at x = 1/2.

But f is A-continuous at 0, i.e. A-lim $f(x_n) = f(0) = 0$ whenever A-lim $x_n = 0$, because A-lim $x_n = 0$ implies $\{x_n\} \to 0$, and hence lim $f(x_n) = 0$.

Therefore A-lim $f(x_n) = 0$.

Theorem 6.2.2: *Let* A = (a_{mn}) *be a regular matrix with the property* (α). *If* f *is A-continuous at every point of* ℝ, *then* f *is linear.*

Proof: Let x, y $\in$ ℝ. Since A is regular having property (α), there exists a sequence $\{v_n\}_1^\infty$ of 0's and 1's such that A-lim $v_n = 1/2$.

We now consider a sequence $\{t_n\}_1^\infty$ defined by $t_n = v_n x + (1 - v_n)y$.

Then $\text{A-lim } t_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \; t_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \; [v_k x + (1 - v_k)y]$

$= x. \text{ A-lim } v_n + y. \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \; (1 - v_k)$

$= x. \text{ A-lim } v_n + y.(1 - \text{A-lim } v_n)$, (as A is regular, $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.)

$= x. 1/2 + y(1 - 1/2) = (x+y)/2.$

Again $f(t_n) = v_n f(x) + (1 - v_n)f(y)$. As above, $\text{A-lim } f(t_n) = [f(x) + f(y)]/2$.

Since f is A-continuous at (x+y)/2, so $f([x+y]/2) = [f(x) + f(y)]/2$ i.e. f is half-point linear. By theorem 6.2.1, f is also continuous. Hence by a well-known result of functional equations, we conclude that f is linear.

Note 6.2.2.1: In theorem 6.2.2, A-continuity of f at every point is essential for the linearity of f. In this regard we mention the example given by Antoni and Salat [2], of the function which is everywhere continuous but non-linear and which fails to be A-continuous at more than one point.

Example 6.2.2.1: Let us consider the regular matrix,

$$A = \begin{pmatrix} ½, ½, 0, 0, 0, \ldots\ldots\ldots \\ 0, ½, ½, 0, 0, \ldots\ldots\ldots\ldots \\ 0, 0, ½, ½, 0, \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{pmatrix}$$

Let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by,

$$f(x) = \begin{cases} -1 \text{ for } x \leq -1 \\ x \text{ for } -1<x<1 \\ 1 \text{ for } x \geq 1 \end{cases}$$

The matrix A is regular with property(α). The function is continuous but nonlinear. Also note that the function is A-continuous only at one point 0 but is not A-continuous at any point $x \in \mathbb{R}$, $x \neq 0$.

6.3 ON POINTS OF A-DISCONTINUITY:

Theorem 6.3.1: *Let* $A = (a_{mn})$ *be a regular matrix. If the graph of a real-valued function* $f : \mathbb{R} \rightarrow \mathbb{R}$ *is a* G_δ *set then the set of points of A-discontinuity of* f *contains a closed set.*

Proof: Let G denotes the graph of f. Since G is a G_δ set therefore $G = \bigcap_1^\infty G_n$ where each G_n is open. Now Cl(G) is a linear subspace of $\mathbb{R}^2$, considered as a linear subspace over the field of rationals [89, section 10.2].

Then $Cl(G) \setminus G = \bigcup_{n=1}^{\infty} (Cl(G) \setminus Gn)$. Each G_n is a dense open subset of Cl(G) and so each $Cl(G) \setminus G_n$ is nowhere dense in Cl(G).

Therefore $Cl(G) \setminus G$ is a set of first category in Cl(G).

For any $x \in Cl(G) \setminus G$, we have $x + G \subset Cl(G) \setminus G$.

Hence x + G is of the first category and so also is G.

Thus Cl(G) = (Cl(G) \ G) ∪ G is a set of first category in Cl(G).

But $\mathbb{R}^2$ is complete, and so Cl(G) cannot be of first category. Thus no such x can exist. Hence Cl(G) = G i.e. G is closed.

Therefore D_f, the set of points of discontinuity of f is a closed set [37].

Let D_{fA} denote the set of points of A-discontinuity of f. We will show that D_{fA} contains the set D_f. Let $x_0 \in D_f$, then there exists a sequence $\{x_n\}$ of real numbers such that $x_n \to x_0$ but $f(x_n) \nrightarrow f(x_0)$. If possible, let $x_0 \notin D_{fA}$. If $t_n \to x_0$ for any arbitrary sequence $\{t_n\}$ then $t_n \xrightarrow{A} x_0$ and since $x_0 \notin D_{fA}$ we get $f(t_n) \xrightarrow{A} f(x_0)$. Hence by Posner's result [76], f is continuous. Thus x_0 is a point of continuity, which contradicts the choice of x_0. Hence $x_0 \in D_{fA}$ i.e. D_f is a subset of D_{fA}. Thus the set of points of A-discontinuity of f contains a closed set.

Note 6.3.1.1: D_f may not contain the set D_{fA} which is clear from the example 6.2.2.1

Theorem 6.3.2: *If* $S \subset \mathbb{R}$ *be an* F_σ *set then there exists a regular infinite matrix* A *and a real function* f: $\mathbb{R} \to \mathbb{R}$ *such that* $S = \{x \in \mathbb{R} : f$ *is A-discontinuous at* $x\}$.

Proof: Since S is an F_σ set it is possible to find a real function f such that $S = D_f$, the set of points of discontinuity of f. Let us consider a subset B of $\mathbb{N}$, the set of natural numbers such that $\mathbb{N} \setminus B$ is an infinite set. We now order the set B as $n_1<n_2<n_3< \ldots\ldots$. Now we define a regular matrix $A = (a_{mn})$ as follows

$$a_{mn} = 0 \text{ if } n \neq n_m \text{ (m=1, 2, ...)}$$
$$= 1 \text{ if } n = n_m.$$

Let $x \notin S$. Then x is a point of continuity of f. Let $x_n \xrightarrow{A} x$.

Then $t_m = \sum_{n=1}^{\infty} a_{mn} x_n = x_{n_m} \to x$. As f is continuous at x, hence $f(x_{n_m}) \to f(x)$ and this means that $f(x_n) \xrightarrow{A} f(x)$. This shows that x is a point of A-continuity of f.

Now $x \in S$. Then x is a point of discontinuity of f. Then there exists a sequence $\{x_n\}$ such that $x_n \to x$ but $f(x_n) \not\to f(x)$.However this means that there exists a sequence $\{x_n\}$ such that $x_n \xrightarrow{A} x$ but $f(x_n) \overset{A}{\not\to} f(x)$. Hence f is A-discontinuous at x and the theorem follows.

Note 6.3.1: We have seen that if f is A-continuous atleast at one point then f is continuous. Now a question arises: Does any A-continuous function possess derivative? We give an affirmative answer in the following theorem.

Theorem 6.3.3: *If* f *is A-continuous at a point* $x_0 \in \mathbb{R}$ *with the regular matrix* A *having property* (α) *then* f *is differentiable almost everywhere on a dense open subset of an interval.*

Proof: We have shown on the proof of theorem 6.2.1 that f is continuous on $\mathbb{R}$. Therefore f is continuous on an interval $I \subset \mathbb{R}$. We know that every horizontal line meets the graph of a continuous function on some interval J of I in finite number of points. Cech [16] proved that such function must be monotone on some interval. Therefore we can decompose the interval J in countably many closed intervals in each of which f is monotone. Thus by Baire category theorem, there exists a sequence of intervals $\{I_n\}$ whose union is dense in J, so that f is monotone in each I_n. Hence f is differentiable almost everywhere on dense open subset of I.

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