CHAPTER – 5

ON SOME WEAKER FORMS OF B*-CONTINUITY FOR MULTIFUNCTIONS

5.1 INTRODUCTION:

In this chapter we introduce the concepts of upper and lower weakly B*-continuous multifunctions, and obtain some characterizations and several properties concerning upper and lower B*-continuous as well as upper and lower weakly B*-continuous multifunctions. The relationship between these multi functions and their graphs are investigated.

5.2 UPPER AND LOWER B*- CONTINUOUS MULTIFUNCTIONS AND THEIR GRAPHS

Definition 5.2.1: Let F: $X \to Y$ be a multifunction, the graph of F, $G_F : X \to X\times Y$ is defined as $G_F(x) = \{x\}\times F(x)$ for all $x \in X$.

Remark 5.2.1: We shall use the following result due to Popa and Noiri [68].

For multifunction F: $X \to Y$,

(a) $G_F^+(A\times B) = A \cap F^+(B)$ and

(b) $G_F^-(A\times B) = A \cap F^-(B)$ for any $A \subset X$ and $B \subset Y$.

Theorem 5.2.1: *Let* F: X $\rightarrow$ Y *be a multifunction such that* F(x) *is compact for each* x $\in$ X. *Then* F *is upper B*-continuous if and only if* G_F : X $\rightarrow$ X$\times$Y *is upper B*-continuous.*

Proof: Suppose F: X $\rightarrow$ Y be upper B*-continuous. Let x $\in$ X and W $\subset$ X$\times$Y is an open set containing $G_F(x)$. For any y $\in$ F(x) there exist open sets U(y) $\subset$ X and V(y) $\subset$ Y such that (x, y) $\in$ U(y) $\times$V(y) $\subset$ W. Then the family {V(y) : y $\in$ F(x)} is an open cover of F(x) and F(x) is compact. Then there exist a finite number of points $y_1, y_2, \ldots, y_n \in F(x)$ such that $F(x) \subset \cup\{V(y_i) : 1 \leq i \leq n\}$.

Set $U = \cap\{U(y_i) : 1 \leq i \leq n\}$, $V = \cup\{V(y_i) : 1 \leq i \leq n\}$. Then U and V are open in X and Y respectively. Now $\{x\}\times F(x) \subset U\times V \subset W$. Since F is upper B*- continuous there exists a B*-set B containing x such that B $\subset$ U and F(B) $\subset$ V. Then $B \subset U \cap F^+(V) = G_F^+(U\times V) \subset G_F^+(W)$. This shows that G_F is upper B*- continuous.

Conversely, suppose that G_F is upper B*- continuous. Let x $\in$ X and U be any open set containing x and V be any open set of Y such that F(x) $\subset$ V. Then we have $G_F(x) = \{x\}\times F(x) \subset U\times V$. Then there exists a B*-set B containing x such that B $\subset$ U and $G_F(B) \subset U\times V$. Hence by remark 5.2.1(a), $B \subset G_F^+(U\times V) = U \cap F^+(V)$. Hence, F is upper B*- continuous.

Theorem 5.2.2: *Let* F: X $\rightarrow$ Y *be a multifunction such that* F(x) *is compact for each* x $\in$ X. *Then* F *is lower B*- continuous if and only if* G_F : X $\rightarrow$ X$\times$ Y *is lower B*- continuous.*

Proof: Suppose F: X $\rightarrow$Y be lower B*- continuous. Let $x \in X$ and $W \subset X\times Y$ is an open set such that $W \cap G_F(x) \neq \varphi$. Let $(x, y) \in W \cap G_F(x)$. Then there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y)\times V(y)$ and $U(y)\times V(y) \cap W \neq \varphi$. Then the family $\{V(y) : y \in F(x)\}$ is an open cover of F(x) and F(x) is compact. Then there exists a finite number of points $y_1, y_2, \ldots, y_n \in F(x)$ such that $F(x) \subset \cup\{V(y_i) : 1 \leq i \leq n\}$.

Set $U = \cap\{U(y_i) : 1 \leq i \leq n\}$, $V = \cup\{V(y_i) : 1 \leq i \leq n\}$. Then U and V are open in X and Y. Now $U\times V \cap W \neq \varphi$ and let $W = W_1\times W_2$, where W_1, W_2 are open sets in X and Y respectively. As F is lower B*- continuous so there exists a B*-set B containing x such that $B \subset U$ such that $B \subset F^-(V \cap W_2) \cap (U \cap W_1) = G_F^-(U\times V \cap W)$. i.e. $G_F(B) \cap W \neq \varphi$. This shows that G_F is lower B*- continuous.

Conversely, suppose that G_F is lower B*- continuous. Let $x \in X$ and U be any open set containing x and V be any open set of Y such that $F(x) \cap V \neq \varphi$. Then we have $G_F(x) \cap U\times V = (\{x\}\times F(x)) \cap U\times V = \{x\}\times(F(x) \cap V) \neq \varphi$. Then there exists a B*-set B containing x such that $B \subset U$ and $B \subset G_F^-(U\times V) = U \cap F^-(V)$. Hence, F is lower B*- continuous.

Theorem 5.2.3: *Let* X *be a Baire space and* F *be a multifunction from* X *into* Y. *Then* F *is lower B*-continuous If for every open set* $V\subset Y$, *there exist open set* $G\subset X$ *and two sets of first category* I, J *in* X *with* $J \subset Cl(G)$ *such that* $F^-(V) = (G \setminus I) \cup J$.

Proof: Let $F^-(V) = (G \setminus I) \cup J$ whenever V is open in Y, G is open and I, J are sets of first category with $J \subset Cl(G)$.We will show that F is lower B*- continuous. Let p $\in$ X. Then for any open sets U, V in X and Y respectively with $p \in U \cap F^-(V)$, we have $p \in U\cap \{(G \setminus I) \cup J\} = \{U\cap (G \setminus I)\} \cup (U \cap J) \subset \{U\cap (G \setminus I)\} \cup (U \cap Cl(G))$. $(U \cap G) \setminus I$ is a B*-set because it is a not nowhere dense set having the property of Baire. Again, $(U\cap G) \setminus I \subset U\cap F^-(V)$. Hence F is lower B*- continuous at the point p.

Theorem 5.2.4: *Let* X *be a Baire space and* F *be a multifunction from* X *into* Y. *Then* F *is upper B*-continuous if for every open set* $V \subset Y$, *there exist open set* $G \subset X$ *and two sets of first category* I, J *in* X *with* $J \subset Cl(G)$ *such that* $F^+(V) = (G \setminus I) \cup J$.

The proof is similar to Theorem 5.2.3.

5.3 UPPER (LOWER) WEAKLY B*-CONTINUOUS MULTIFUNCTIONS:

Definition 5.3.1: A multifunction $F: X \to Y$ is called upper weakly B*-continuous if for each point $x \in X$ and for each open set V of Y with $F(x) \subset V$, there exists a B*-set B containing x such that $F(B) \subset Cl(V)$.

Definition 5.3.2: A multifunction $F: X \to Y$ is called lower weakly B*-continuous if for each point $x \in X$ and for each open set V of Y with $F(x) \cap V \neq \varphi$, there exists a B*-set B containing x such that $F(y) \cap Cl(V) \neq \varphi$ for each $y \in B$.

Definition 5.3.3: [19] A set $A \subset X$ is said to be β-open if $A \subset Cl(Int(Cl(A)))$.

The complement of a β-open set is said to be β-closed.

Definition 5.3.4: The intersection of all β-closed subsets of X containing A is called the β-closure of A and is denoted by $\beta cl(A)$.

Definition 5.3.5: The union of all β-open subsets of X contained in A is called the β-interior of A and is denoted by $\beta int(A)$.

In [1] the following properties were established:

Let A be a subset of X. Then

(i) $\beta cl(A) = A \cup Int(Cl(Int(A)))$. (ii) $\beta int(A) = A \cap Cl(Int(Cl(A)$.

Theorem 5.3.1: *For a multifunction* $F: X \rightarrow Y$ *the following are equivalent:*

(1) F *is upper weakly B*-continuous.*

(2) *For each* $x \in X$ *and every open set* V *of* Y *containing* F(x) *there exists* $H \in$ S.O.(X, x) *such that* $H \subset Cl(F^{+}(Cl(V)))$.

(3) $F^{+}(V) \subset Cl(Int(Cl(F^{+}(Cl(V)))))$ *for each open set* V *of* Y.

(4) $Int(Cl(Int(F^{-}(Int (C))))) \subset F^{-}(C)$, *for every closed set* C *in* Y.

(5) $\beta cl(F^{-}(Int(C))) \subset F^{-}(C)$, *for every closed set* C *in* Y.

(6) β cl(F^-(Int(Cl(B)))) $\subset$ F^-(Cl(B)), *for any arbitrary set* B *in* Y,.

(7) F^+(Int B) $\subset$ β int(F^+(Cl(Int(B)))) *for every subset* B *of* Y.

(8) F^+(V) $\subset$ β int(F^+(Cl(V))) *for every open set* V *of* Y.

Proof: (1) $\Rightarrow$(2): Let U be any open set containing x and V be any open set of Y such that x $\in$ F^+(V). Then there exists a B*- set B containing x such that B $\subset$ U $\cap$ F^+(Cl(V)). Then U $\cap$ F^+(Cl(V)) is not nowhere dense and hence we have $\varphi \neq$ Int(Cl(U $\cap$ F^+(Cl(V)))) $\subset$ Cl(U). Put W = Int(Cl(U $\cap$ F^+(Cl(V)))) $\cap$ U. Then W is a nonempty open set such that W $\subset$ U and W $\subset$ Cl(F^+(Cl(V).))

Let $\mathcal{U}$(x) be a family of open neighbourhoods of x and G = $\cup$\{W_U : U $\in$ $\mathcal{U}$(x)\} such that $W_U \subset$ Cl(F^+(Cl(V))). Then G is open in X and x $\in$ Cl(G).

Let H = G $\cup$ \{x\}. Then G $\subset$ H $\subset$ Cl(G). Therefore H $\in$ S.O.(X, x). Clearly it follows that H $\subset$ Cl(F^+(Cl(V))).

(2) $\Rightarrow$(3): Let V be any open set in Y and x $\in$ F^+(V). As F is upper weakly B*-continuous, by (2) there exists $U_0 \in$ S.O.(X, x) such that $U_0 \subset$ Cl(F^+(Cl(V))). Since $U_0 \in$ S.O.(X, x), $U_0 \subset$ Cl(Int(U_0)). Hence x $\in$ Cl(Int(U_0)) $\subset$ Cl(Int(Cl(F^+(Cl(V))))).

Hence F^+(V) $\subset$ Cl(Int(Cl(F^+(Cl(V))))).

(3) $\Rightarrow$(4): Let C be any closed set in Y. This implies Y \ C is open in Y. Then from (3) we have F^+(Y \ C) $\subset$ Cl(Int(Cl(F^+(Cl(Y \ C))))).

Hence, X \ F^-(C) $\subset$ Cl(Int(Cl(F^+(Y \ Int(C))))) = Cl(Int(Cl(X \ F^-(Int(C)))))

= Cl(Int(X \ Int(F^-(Int(C))))) = Cl(X \ Cl(Int(F^-(Int(C)))))

$= X \setminus Int(Cl(Int(F^-(Int(C)))))$.

Hence, $Int(Cl(Int(F^-(Int(C))))) \subset F^-(C)$.

(4) ⇒(5): Let C be any closed set in Y.

Then we have $Int(Cl(Int(F^-(Int\ (C))))) \subset F^-(C)$.

We know $\beta cl(C) = C \cup Int(Cl(Int(C)))$ for any subset C of X.

Also $F^-(Int(C)) \subset F^-(C)$. Thus $\beta cl\ (F^-(Int(C)) \subset F^-(C)$.

(5) ⇒(6): Obvious.

(6) ⇒(7): Let B be any arbitrary subset of Y.

Then $X \setminus F^+(Int(B)) = F^-(Cl(Y \setminus B)) \supset \beta cl(F^-(Int(Cl(Y \setminus B))))$

$= \beta cl(X \setminus (F^+(Cl(Int(B))))) = X \setminus \beta int(F^+(Cl(Int(B))))$.

(7) ⇒(8): Obvious.

(8) ⇒(1): Let $x \in X$ and U be any open set containing x. Let V be any open set of Y such that $F(x) \subset V$. Then by (8) we have, $x \in F^+(V) \subset \beta int(F^+(Cl(V)))$.

Then there exists a β-open set H containing x such that $H \subset F^+(Cl(V))$.

Let G = Int(H) and put $O = G \cap U$. Then O is a nonempty open subset of U and hence is a B*-set. Also $F(O) \subset Cl(V)$, i.e. $O \subset F^+(Cl(V))$.

Hence F is upper weakly B*-continuous.

Theorem 5.3.2: *For a multifunction* F: X →Y *the following are equivalent:*

(1) F *is lower weakly B*-continuous.*

(2) *For each* $x \in X$ *and every open set* V *of* Y *such that* $F(x) \cap V \neq \varphi$ *there exists* H $\in$ S.O.(X, x) *such that* $F(u) \cap Cl(V) \neq \varphi$ *for every* $u \in H$.

(3) $F^-(V) \subset Cl(Int(Cl(F^-(Cl(V)))))$ *for each open set* V *of* Y.

(4) $Int(Cl(Int(F^+(Int(C))))) \subset F^+(C)$, *for every closed set* C *in* Y.

(5) $\beta\, cl(F^+(Int(C))) \subset F^+(C)$, *for every closed set* C *in* Y.

(6) $\beta\, cl(F^+(Int(Cl(B)))) \subset F^+(Cl(B))$, *for any arbitrary set* B *in* Y.

(7) $F^-(Int\ B) \subset \beta\, int(F^-(Cl(Int(B))))$ *for every subset* B *of* Y.

(8) $F^-(V) \subset \beta\, int(F^-(Cl(V)))$ *for every open set* V *of* Y.

The proof is similar to that of the theorem 5.3.1.

Theorem 5.3.3: *Let a multifunction* F: X → Y *be upper weakly B*-continuous and* Y *is Hausdorff, and* F(x) *is paracompact at each point* $x \in X$. *Then the graph* $G_F(x)$ *is B*-closed in* X× Y.

Proof: Let x_0 be any arbitrary point of X and $(x_0, y_0) \notin G_F(x_0)$. Then $y_0 \notin F(x_0)$. Since Y is Hausdorff for each $y \in F(x_0)$ there exist open sets V(y) and $W(y_0)$ containing y and y_0 respectively such that $V(y) \cap W(y_0) = \varphi$. Then the family $\{V(y) : y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is paracompact. Therefore it has a locally finite open refinement $\mathcal{U} = \{U_a : a \in \Lambda\}$ which covers $F(x_0)$.

Let W_0 be an open neighbourhood of y_0 such that W_0 intersects only a finitely many members $U_{a_1}, U_{a_2}, \ldots, U_{a_n}$ of $\mathcal{U}$. Choose $y_1, y_2, \ldots, y_n$ in $F(x_0)$ such that $U_{a_i} \subset$

$V(y_i)$, $1 \le i \le n$. Set $W_1 = \bigcap_{1}^{n} W(y_i)$, and $W = W_1 \cap W_0$. Then W is a neighbourhood of y_0 with $W \cap (\bigcup_{\alpha\in\Lambda} U_\alpha) = \varphi$. This implies $W \cap Cl(\bigcup_{\alpha\in\Lambda} U_\alpha) = \varphi$.

Since F is upper weakly B*-continuous, there exists a B*-set B containing x_0 such that $F(B) \subset Cl(\bigcup_{\alpha\in\Lambda} U_\alpha)$. Therefore $(B\times W) \cap G_F(x_0) = \varphi$.

Hence, $G_F(x_0)$ is B*-closed in $X\times Y$.

Corollary 5.3.3.1: *If a compact valued multifunction* F: X →Y *is upper weakly B*-continuous and* Y *is Hausdorff then* $G_F(x)$ *is B*-closed.*

Proof follows easily from theorem 5.3.3.

5.4 NETS FOR MULTIFUNCTIONS:

Let $(\Lambda, >)$ is a directed set, $\{F_\alpha\}$ is a net of multifunctions F_α: X → Y, $\alpha\in\Lambda$ and F is a multifunction on X into Y.

Definition 5.4.1: [18] $\{F_\alpha : \alpha \in \Lambda\}$ is said to be,

i) Upper pointwise convergent to F if for each $x \in X$ and each open set $U \subset Y$ containing F(x), there exists $\beta \in \Lambda$ such that $x \in F_\alpha^+(U)$ for all $\alpha > \beta$.

ii) Lower pointwise convergent to F if for each $x \in X$ and each open set $U \subset Y$ such that $x \in F^-(U)$ there exists $\beta \in \Lambda$ such that $x \in F_\alpha^-(U)$ for all $\alpha > \beta$.

iii) Pointwise convergent if it is both upper pointwise convergent and lower pointwise convergent.

Theorem 5.4.1: *Let* $\{F_\alpha\}_{\alpha\in\Lambda}$ be *a net of upper weakly B*-continuous multifunctions* $F_\alpha : X \rightarrow Y$, *converging upper pointwise to* $F: X \rightarrow Y$ *such that the point images under* F *are closed sets. If* Y *is normal and for each open set* G *of* Y *with* $F^-(G) \neq \varphi$ *and* $\beta \in \Lambda$ *there exists* $\alpha > \beta$ *such that* $F_\alpha(x) \cap G \neq \varphi$ *for all* $x \in F^-(Cl(G))$, *then* F *is also upper weakly B*- continuous.*

Proof: We suppose that F is not upper weakly B*-continuous at $x_0 \in X$. Then there exists an open set G of Y containing $F(x_0)$ such that for every B*-set B of X containing x_0 there exists a point $x_B \in B$ such that $F(x_B) \not\subset Cl(G)$ and $F(x_0) \cap (Y \setminus G) = \varphi$.

Since $F(x_0)$ is closed in Y and $F(x_0) \subset G$, then by normality of Y there exists an open set G_1 such that $F(x_0) \subset G_1 \subset Cl(G_1) \subset G \subset Cl(G)$.

Let $G_2 = Y \setminus Cl(G_1)$. Then $Y \setminus Cl(G) \subset Y \setminus Cl(G_1) = G_2$.

From the upper pointwise convergence of $\{F_\alpha\}$ to F at x_0 it follows that there exists $\alpha_0 \in \Lambda$ such that $F_\alpha(x_0) \subset G_1 \subset Cl(G_1)$ for all $\alpha > \alpha_0$.

This implies $x_0 \in F_\alpha^{+}(Cl(G_1))$ for all $\alpha > \alpha_0$.

But, $F(x_B) \cap (Y \setminus Cl(G)) \neq \varphi$ i.e., $F(x_B) \cap G_2 \neq \varphi$. This implies $F(x_B) \cap Cl(G_2) \neq \varphi$.

Hence there exists $\gamma > \alpha_0$ such that $F_\gamma(x) \cap G_2 \neq \varphi$ for all $x \in F^-(Cl(G_2))$.

This implies $F_\gamma(x_B) \cap G_2 \neq \varphi$ as $x_B \in F^-(Cl(G_2))$. Then $F_\gamma(x_B) \not\subset Cl(G_1)$.

Hence F_γ is not upper weakly B*- continuous at x_0. This contradicts the hypothesis. Hence, F is upper weakly B*- continuous.

Theorem 5.4.2: *Let* $\{F_a\}_{a\in\Lambda}$ be *a net of lower weakly B*-continuous multifunctions* $F_a: X \rightarrow Y$ *converging lower pointwise to* $F: X \rightarrow Y$. *If* Y *is regular and for each open set* G *of* Y *with* $F^+(G) \neq \varphi$ *and* $\beta \in \Lambda$ *there exists* $a > \beta$ *such that* $F_a(x) \subset G$ *for all* $x \in F^+(Cl(G))$, *then* F *is also lower weakly B*- continuous.*

Proof: We suppose that F is not lower weakly B*- continuous at $x_0 \in X$. Then there exists an open set G of Y with $G \cap F(x_0) \neq \varphi$ such that for every B*-set B of X containing x_0 there exists a point $x_B \in B$ such that $F(x_B) \cap Cl(G) = \varphi$, i.e. $F(x_B) \cap G = \varphi$. Let $y_0 \in G \cap F(x_0)$. Then by regularity of Y there exists an open set G_1 such that $y_0 \in G_1 \subset Cl(G_1) \subset G$. Let $G_2 = Y \setminus Cl(G_1)$. As $y_0 \in G_1 \cap F(x_0)$, $x_0 \in F^-(G_1)$.

From lower pointwise convergence of $\{F_a\}$ to F it follows that there exists $a_0 \in \Lambda$ such that $F_a(x_0) \cap G_1 \neq \varphi$ for all $a > a_0$. This implies $F_a(x_0) \cap Cl(G_1) \neq \varphi$ for all $a > a_0$. Since $F(x_B) \cap G = \varphi$, then $x_B \in F^+(Y \setminus G)$. i.e. $F(x_B) \subset Y \setminus G \subset G_2 \subset Cl(G_2)$. This implies $F^+(G_2) \neq \varphi$, Hence there exists $\gamma > a_0$ such that $F_\gamma(x) \subset G_2$ for all $x \in F^+(Cl(G_2))$. This implies $F_\gamma(x_B) \subset G_2$. Hence $F_\gamma(x_B) \cap Cl(G_1) = \varphi$.

Then F_γ is not lower weakly B*-continuous at x_0. This contradicts the hypothesis. Hence, F is lower weakly B*-continuous.

The paper (revised according to referee's suggestion) containing the contents of this chapter has been sent for publication in SOOCHOW JOURNAL OF MATHEMATICS.