CHAPTER – 4

ON B*-CONTINUOUS AND B*-CLUSTER CONTINUOUS MULTIFUNCTIONS

4.1 INTRODUCTION:

In this chapter we introduce new notions of upper and lower B*- continuous multifunctions as well as upper and lower B*- cluster continuous multifunctions defined on a topological space X, and obtain some characterizations alongwith some properties of such functions in connection with B*- closed or B*-open sets.

The multifunctions considered here are defined on X and assume their values in $P(Y) \setminus \varphi$, where P(Y) is the power set of Y. Multifunctions are denoted by capital letters F, G, H etc.

In case of multifunctions we write simply $F: X \to Y$ instead of $F: X \to P(Y) \setminus \varphi$.

If $F: X \to Y$ is a multifunction then for $A \subset Y$ we denote $F^+(A) = \{x \in X: F(x) \subset A\}$ and $F^-(A) = \{x \in X: F(x) \cap A \neq \varphi\}$.

It is clear that $F^+(Y \setminus A) = X \setminus F^-(A)$ and $F^-(Y \setminus A) = X \setminus F^+(A)$.

4.2 THE B*-CONTINUOUS MULTIFUNCTION:

Definition 4.2.1: A multifunction F: X →Y is lower (upper) B*- continuous at a point x if for every open sets U, V with $x \in U$, $F(x) \cap V \neq \varphi$ ($F(x) \subset V$), there exists a B*- set B such that $B \subset F^{-}(V) \cap U$ ($B \subset F^{+}(V) \cap U$).

F is B* Continuous at x if it is both lower and upper B* - continuous at x.

F is lower B* Continuous, upper B* Continuous and B* Continuous over X if it is respectively so at any point x.

It is clear that upper (lower) quasi-continuity implies upper (lower) B*-continuity. But the converse is not true which follows from the example below.

Example 4.2.1: Let X = Y = [0, 1] with the usual topology.

Define F: X →Y by,

$$F(x) = \begin{cases} \{0\} & \text{if x is irrational} \\ [0, 1] & \text{if x is rational} \end{cases}$$

F is not upper quasi-continuous at x if x is irrational. But F is upper B*-continuous at any irrational point x.

We now give some characterisation of upper B*-continuous multifunction:

Theorem 4.2.1: *For a multifunction* F: X→Y *the following conditions are equivalent:*

(1) F *is upper B*-continuous at a point* $x \in X$.

(2) *For each open neighbourhood* U *of* x *and any open set* V *of* Y *with* $x \in F^{+}(V)$, $F^{+}(V) \cap U$ *is not nowhere dense.*

(3) *For each open set* U *containing* x *and each open set* V *of* Y *with* $x \in F^{+}(V)$, *there exists a nonempty open set* W *of* X *with* $W \subset U$ *such that* $W \subset Cl(F^{+}(V))$.

(4) *For each open set* V *of* Y *with* $x \in F^{+}(V)$, *there exists* $U_0 \in S.O.(X, x)$ *such that* $U_0 \subset Cl(F^{+}(V))$.

(5) $F^{+}(V) \subset Cl(Int(Cl(F^{+}(V))))$, *for each open set* V *of* Y.

(6) $Int(Cl(Int(F^{-}(A)))) \subset F^{-}(A)$, *for every closed set* A *in* Y.

Proof: (1) ⇒(2) Obvious.

(2) ⇒(3): From (2) it follows that for every open neighbourhood U of x and any open set V of Y with $x \in F^{+}(V)$, $F^{+}(V) \cap U$ is not nowhere dense. Hence, there exists an open set $W \subset U$, such that $W' \cap F^{+}(V) \cap U \neq \varphi$ for every open subset W' of W. This implies that $W \subset Cl(F^{+}(V))$.

(3) ⇒(4): From (3) it follows that for every open neighbourhood U of x and any open set V of Y with $x \in F^{+}(V)$, there exists a nonempty open set G of X such that $G \subset U$ and $G \subset Cl(F^{+}(V))$. Let V be an open set of Y containing F(x). Let $\mathcal{U}(x)$ be a family of open neighbourhoods of x. For each $U \in \mathcal{U}(x)$, there exists a nonempty open set G(U) of X such that $G(U) \subset U$ and $G(U) \subset Cl(F^{+}(V))$. Set $W = \bigcup_{U} G(U)$.

Then W is an open set of X, $x \in Cl(W)$ and $W \subset Cl(F^{+}(V))$. Now take $U_0 = W \cup \{x\}$. Then $W \subset U_0 \subset Cl(W)$ and $U_0 \in S.O.(X, x)$ and also $U_0 \subset Cl(F^{+}(V))$.

(4)⇒(5): Let V be any open set in Y and $x \in F^+(V)$. Then there exists $U \in S.O.(X,x)$ such that $U \subset Cl(F^+(V))$. Again $x \in U \subset Cl(Int(U))$, as U is semi open.

Again, $Cl(Int(U)) \subset Cl(Int(Cl(F^+(V))))$.

i.e. $x \in Cl(Int(Cl(F^+(V))))$. Therefore, $F^+(V) \subset Cl(Int(Cl(F^+(V))))$.

(5) ⇒(6): Let A be any closed set in Y. Then $Y \setminus A$ is open.

Then, $F^+(Y \setminus A) \subset Cl(Int(Cl(F^+(Y \setminus A))))$.

This implies, $X \setminus F^-(A) \subset Cl(Int(Cl(X \setminus F^-(A))))$

$= Cl(Int(X \setminus Int(F^-(A))))$

$= Cl(X \setminus Cl(Int(F^-(A))))$

$= X \setminus Int(Cl(Int(F^-(A))))$.

Therefore, $Int(Cl(Int(F^-(A)))) \subset F^-(A)$.

(6) ⇒(5): Similar.

(5) ⇒(1): Let $x \in X$ and U be any open set of X containing x and V be any open set of Y such that $F(x) \subset V$. Then, $x \in F^+(V) \subset Cl(Int(Cl(F^+(V))))$.

Hence, $\varphi \neq U \cap Int(Cl(F^+(V))) = Int(U \cap Cl(F^+(V))) \subset Int(Cl(U \cap F^+(V))) \subset Cl(U \cap F^+(V))$.

This implies, $Cl(U \cap F^+(V)) \neq \varphi$ and $U \cap F^+(V)$ is not nowhere dense.

Thus, $(U \cap F^+(V)) \cap (U \cap Int(Cl(F^+(V)))) \neq \varphi$. $\Rightarrow U \cap F^+(V) \cap Int(Cl(F^+(V))) \neq \varphi$.

i.e. $U \cap H \neq \varphi$, where $H = F^+(V) \cap Int(Cl(F^+(V)))$ is a pre open set [26].

Let $B = U \cap H$. Then B is a nonempty pre open set [26] and hence is a B*-set.

Also, $B \subset (U \cap F^+(V))$. Hence, F is upper B*-continuous at x.

Theorem 4.2.2: *For a multifunction* F: X $\rightarrow$ Y *the following conditions are equivalent;*

1) F *is lower B*-continuous at a point* $x \in X$.

2) *For each open neighbourhood* U *of* x *and any open set* V *of* Y *with* $x \in F^-(V)$, $F^-(V) \cap U$ *is not nowhere dense.*

3) *For each open set* U *containing* x *and each open set* V *of* Y *with* $x \in F^-(V)$, *there exists a nonempty open set* $W \subset U$ *such that* $W \subset Cl(F^-(V))$.

4) *For each open set* V *of* Y *with* $x \in F^-(V)$, *there exists* $O \in S.O.(X, x)$ *such that* $O \subset Cl(F^-(V))$.

5) $F^-(V) \subset Cl(Int(Cl(F^-(V))))$, *for each open set* V *of* Y.

6) $Int(Cl(Int(F^+(A)))) \subset F^+(A)$, *for every closed set* A *in* Y.

The proof is similar to theorem 4.2.1.

4.3 UPPER (LOWER) B*-CLUSTER CONTINUOUS MULTIFUNCTION AND ITS CONVERGENCE

In this section we introduce the concept of B*-open and B*-closed set; and B*-cluster continuous multifunction is defined with the help of these sets. Also we introduced the notion of upper and lower semi-uniform convergence and study some related properties.

Definition 4.3.1: Let X be a topological space and P be a subset of X. $x \in X$ is said to be B*-cluster point of the set P if for every B*-set B including x, $P \cap B \neq \varphi$.

The set of all B*-cluster points of P is called the cluster derived set of P and denoted by cls-d-P.

A set P is said to be a B*-closed set, if P = cls-d-P.

The compliment of a B*-closed set is B*-open.

A multifunction F is said to be lower (upper) B*-cluster continuous if $F^-(V)$ ($F^+(V)$) is B*-closed for every closed set V in Y.

Example 4.3.1: In the set $\mathbb{R}$ of real numbers with usual topology, the set Q of rational numbers and $\mathbb{R}\backslash Q$ of irrational numbers are B*- closed as well as B*- open.

Example 4.3.2: The set $\mathbb{R} \setminus \{1, 2, \ldots, n\}$ is not B*- closed as well as not B*- open.

In what follows $(\Lambda, \geq)$ is a directed set, $\{F_a\}$ is a net of multifunctions $F_a : X \to Y$, $a \in \Lambda$ and F is a multifunction on X into Y.

Definition 4.3.2: $\{F_a : a \in \Lambda\}$ is said to be upper semi-uniformly convergent to F on X if

(i) For every open set U of Y with $F^+(U) \neq \varphi$, and for every $a \in \Lambda$ there exists $a_0 \in \Lambda$ with $a_0 \geq a$ such that $x \in F_{a_0}^+(U)$ for all $x \in F^+(U)$.

(ii) For every $x \in X$ and every open set U of Y such that $x \in F^-(U)$ there exists $\alpha_0 \in \Lambda$ such that $x \in F_\alpha^-(U)$ for all $\alpha \geq \alpha_0$.

Definition 4.3.3: $\{F_\alpha : \alpha \in \Lambda\}$ is said to be lower semi-uniformly convergent to F on X if

(i) For every open set U of Y with $F^-(U) \neq \varphi$, and for every $\alpha \in \Lambda$ there exists $\alpha_0 \in \Lambda$ with $\alpha_0 \geq \alpha$ such that $x \in F_{\alpha_0}^-(U)$ for all $x \in F^-(U)$.

(ii) For every $x \in X$ and every open set U of Y such that $x \in F^+(U)$ there exists $\alpha_0 \in \Lambda$ such that $x \in F_\alpha^+(U)$ for all $\alpha \geq \alpha_0$.

Definition 4.3.4: $\{F_\alpha : \alpha \in \Lambda\}$ is said to be semi-uniformly convergent to F on X if it is upper as well as lower semi-uniformly convergent to F on X.

Theorem 4.3.1: *The following conditions are equivalent:*

(1) F *is lower B^*- cluster continuous.*

(2) *For each open set* V *of* Y, $F^+(V)$ *is B^*- open in* X.

(3) *For each* $x \in X$ *and for every open set* V *of* Y, *such that* $x \in F^+(V)$ *there is a B^*-set* B *in* X *containing* x *such that* $B \subset F^+(V)$.

Proof: (1) ⇔ (2): F is lower B*- cluster continuous. Let V be any open set in Y. Then $(Y \setminus V)$ is closed in Y. Then $F^-(Y \setminus V)$ is B*-closed.

But $F^+(V) = X \setminus F^-(Y \setminus V)$, i.e. $F^+(V)$ is B*- open in X; and conversely.

(2) ⇒(3) : Let $x \in X$ and V be an open set in Y containing F(x). By hypothesis, $F^+(V)$ is B*- open in X. But $F^+(V) = X \setminus F^-(Y \setminus V)$. So, $F^-(Y \setminus V)$ is B*-closed in X. Obviously $x \notin F^-(Y \setminus V)$. Therefore x is not a B*- cluster point of $F^-(Y \setminus V)$.

Then, there exists a B*- set B containing x, $B \cap F^-(Y \setminus V) = \varphi$.

$\Rightarrow F(B) \cap (Y \setminus V) = \varphi$. $\Rightarrow F(B) \subset V \Rightarrow B \subset F^+(V)$.

(3) ⇒(2): Let V be an open set in Y and let $x \notin F^-(Y \setminus V)$. i.e. $F(x) \subset V$. Then, $x \in F^+(V)$. By hypothesis there exists a B*- set B containing x such that $B \subset F^+(V)$.

So, $F(B) \cap (Y \setminus V) = \varphi$. Hence, $B \cap F^-(Y \setminus V) = \varphi$.

Consequently, $F^-(Y \setminus V)$ is B*-closed.

But $F^+(V) = X \setminus F^-(Y \setminus V)$ and hence $F^+(V)$ is B*- open.

Theorem 4.3.2: *The following conditions are equivalent:*

1) F *is upper B*- cluster continuous.*

2) *For each open set* V *of* Y, $F^-(V)$ *is B*- open in* X.

3) *For each* $x \in X$ *and for every open set* V *of* Y, *such that* $x \in F^-(V)$ *there is a B*- set* B *such that* $B \subset F^-(V)$.

Proof is similar to theorem 4.3.1.

Theorem 4.3.3: *Let* $\{F_\alpha\}_{\alpha\in\Lambda}$ *be a net of lower B*-cluster continuous multifunctions from* X *to a normal space* Y. *If* $\{F_\alpha\}_{\alpha\in\Lambda}$ *is lower semi-uniformly convergent to a* *multifunction* F: X →Y *such that* F(x) *is closed for each* x ∈ X, *then* F *is lower B*-cluster continuous.*

Proof: We suppose that F be not lower B*-cluster continuous but all F_a are lower B*-cluster continuous. Then there exists a point $x_0 \in X$ and an open set U of Y containing $F(x_0)$ such that for every B*-set B containing x_0, there exists $x \in B$ so that $F(x) \not\subset U$. It is evident that $x \notin F^+(U)$.

Since F(x) is closed in Y, then by the normality of Y there exists an open set V of Y such that $F(x_0) \subset V \subset Cl(V) \subset U$. Let $V_1 = Y \setminus Cl(V)$. Then, $Y \setminus U \subset V_1$.

As $\{F_a\}_{a \in \Lambda}$ is lower semi-uniformly convergent to F, there exists $a_0 \in \Lambda$ such that $x_0 \in F_a^+(V)$ for all $a \geq a_0$. Since $F(x) \not\subset U$ then $F(x) \cap (Y \setminus U) \neq \varphi \Rightarrow F(x) \cap V_1 \neq \varphi$. Therefore, $x \in F^-(V_1)$ i.e. $F^-(V_1) \neq \varphi$. As $\{F_a\}_{a \in \Lambda}$ is lower semi-uniformly convergent to F, there exists $a_1 \in \Lambda$ with $a_1 \geq a_0$ we have $y \in F_{a_1}^-(V_1)$ for all $y \in F^-(V_1)$. Hence, $x \in F_{a_1}^-(V_1)$. Since $V \cap V_1 = \varphi$, hence $F_{a_1}(x) \not\subset V$.

According to theorem 4.3.1 it follows that F_{a_1} is not lower B*-cluster continuous, which is a contradiction. Hence, the theorem.

Theorem 4.3.4: *Let* $\{F_a\}_{a \in \Lambda}$ *be a net of upper B*-cluster continuous multifunctions from* X *to a regular space* Y. *If* $\{F_a\}$ *is upper semi-uniformly convergent to a multifunction* F: X →Y, *then* F *is upper B*- cluster continuous.*

Proof: We suppose that F be not upper B*- cluster continuous but all F_α are upper B*- cluster continuous. Then by theorem 4.3.2, there exists a point $x_0 \in X$ and an open set U of Y intersecting $F(x_0)$ such that for every B*- set B containing x_0, there exists a point $x \in B$ with $F(x) \cap U = \varphi$..

Then for each B*- set B there exists a point $x \in B$ so that $x \notin F^-(U)$.

Since $F(x_0) \cap U \neq \varphi$ let us take an arbitrary point z of $F(x_0) \cap U$. Then by the regularity of Y there exists an open set V of Y such that $z \in V \subset Cl(V) \subset U$. Let $V_1 = Y \setminus Cl(V)$. Then, evidently $x_0 \in F^-(V)$.

Since $\{F_\alpha\}_{\alpha\in\Lambda}$ is upper semi-uniformly convergent to F, there exists $\alpha_0 \in \Lambda$ such that $x_0 \in F_\alpha^-(V)$ for all $\alpha \geq \alpha_0$. Since $x \notin F^-(V)$, then $x \in F^+(Y \setminus V)$.

Therefore, $x \in F^+(V_1)$. Since $\{F_\alpha\}_{\alpha\in\Lambda}$ is upper semi-uniformly convergent to F, there exists $\alpha_1 \in \Lambda$ with $\alpha_1 \geq \alpha_0$ such that $F_{\alpha_1}(y) \subset V_1$ for each $y \in F^+(V_1)$.

Hence, $F_{\alpha_1}(x) \subset V_1$. Therefore $x \in F_{\alpha_1}^+(V_1)$. Since $V \cap V_1 = \varphi$, hence $x \notin F_{\alpha_1}^-(V)$. According to theorem 4.3.2 it follows that F_{α_1} is not upper B*- cluster continuous, which is a contradiction. Hence the theorem.

The paper (revised according to referee's suggestion) containing the contents of this chapter has been sent for publication in SOOCHOW JOURNAL OF MATHEMATICS.