6.1 Introduction

Various aspects of the problems of radiation by a charged particle moving with a uniform velocity within a plasma in presence of external magnetic field have been examined [105-112]. The radiation characteristics of a point charge moving with uniform velocity along the direction of the static magnetic field in an anisotropic plasma are studied. It is found that the modified electron-plasma mode, in addition to the usual ordinary and extra ordinary modes, is excited as Cerenkov radiation at frequencies greater than the plasma frequency. McKenzie investigated the Cerenkov radiation in a magnetoionic medium with an emphasis on the evaluation of radiation losses suffered by a charged particle moving through the plasma and on its application to the generation of low-frequency electromagnetic radiation in the upper atmosphere due to the passage of streaming charged particles [110,111]. In these works, Cerenkov excitation of the Whistler and ion-cyclotron waves, and their critical dependence on some particular characteristic wave speed relative to the particle's velocity component parallel to the magnetic field are studied. Also, the intensity of the emitted radiation has been found to decrease with the decrease of the particle speed along the magnetic field.
When a plasma wave is slowed and approaches the electron thermal velocity, its interaction with the thermal electron increases and gives rise to Landau damping attaining eventually Cerenkov resonance conditions. Cerenkov condition becomes significant when the phase velocity approaches the electron thermal velocity which is comparable for pure and modified Alfven waves in the ELF range of the ionosphere around 300 Km to 500 Km heights. The possibility of VLF Cerenkov emission in the ionosphere by electron beams has been investigated [113,114].

The resonance cone phenomena in plasmas have been widely investigated [115,116]. Balmain [117] has given an exhaustive list of references on the subject.

In this chapter, a model calculation has been made to investigate the radiation characteristics of a point charge moving with uniform velocity through the ionospheric plasma along the direction of geomagnetic field in which the influences of time-varying irregularities and motion of heavy ions have been taken into account [61]. In the mathematical analysis, coupled equations are obtained. Using suitable transformation for decoupling, the equations are solved. The emitted radiation is found to be consisted of two modes. The dispersion relation of these modes may be analysed to examine the effects of time-varying irregularities and motion of heavy ions on the low frequency spectrum of the two modes.
6.2 Mathematical Formulation

In the stated model, a line charge will be assumed to move with a uniform velocity \( \hat{\mathbf{u}} \) along the z-direction. Maxwell's equations under Fourier transform yield

\[
\nabla \times \mathbf{E}(\mathbf{r}, \omega) = j \omega \mathbf{H}(\mathbf{r}, \omega) \quad \ldots (1)
\]

\[
\nabla \times \mathbf{H}(\mathbf{r}, \omega) = -j \omega (\bar{\varepsilon}) \mathbf{E} + \hat{\mathbf{z}} \mathbf{J}(\mathbf{r}, \omega) \quad \ldots (2)
\]

(\( \bar{\varepsilon} \)) is the dielectric tensor of the medium in presence of electrons and ions. Following the method given in Chapter II, the components of \( \bar{\varepsilon} \) can be obtained as

\[
\varepsilon_{11} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} + j \eta \frac{\omega_{pe}^2 (\omega^2 + \omega_{ce}^2)}{\omega (\omega^2 - \omega_{ce}^2)^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2} + j \eta \frac{\omega_{pi}^2 (\omega^2 + \omega_{ci}^2)}{\omega (\omega^2 - \omega_{ci}^2)^2} - \frac{2 \omega_{pe} \omega_{ce} \omega_{cz}}{(\omega^2 - \omega_{ce}^2)^2} \delta (\omega - \omega_0)
\]

\[
\varepsilon_{12} = - \frac{\omega_{pe} \omega_{ce}}{\omega (\omega^2 - \omega_{ce}^2)} - \frac{2 \eta \omega_{pe} \omega_{ce}}{(\omega^2 - \omega_{ce}^2)^2} - \frac{j \omega_{pi} \omega_{ci}}{\omega (\omega^2 - \omega_{ci}^2)} - \frac{2 \eta \omega_{pi} \omega_{ci}}{(\omega^2 - \omega_{ci}^2)^2} + \frac{j \omega_{pe} \omega_{cz} (\omega^2 + \omega_{ce}^2)}{\omega (\omega^2 - \omega_{ce}^2)^2} \delta (\omega - \omega_0)
\]

\[
\varepsilon_{13} = \frac{\omega_p^2 (\omega_{ce} \omega_{cx} - j \omega \omega_{cy})}{\omega^2 (\omega^2 - \omega_{ce}^2)} \delta (\omega - \omega_0)
\]

\[
\varepsilon_{21} = - \varepsilon_{12}, \quad \varepsilon_{22} = \varepsilon_{11}
\]
\[ \varepsilon_{23} = \frac{\omega_p^2 (\omega_{ce} \omega_{cy} + j \omega \omega_{cx})}{\omega^2 (\omega^2 - \omega_{ce}^2)} \delta (\omega - \omega_o) \]

\[ \varepsilon_{31} = \frac{\omega_p^2 (\omega_{ce} \omega_{cx} + j \omega \omega_{cy})}{\omega^2 (\omega^2 - \omega_{ce}^2)} \delta (\omega - \omega_o) \]

\[ \varepsilon_{32} = \frac{\omega_p^2 (\omega_{ce} \omega_{cy} - j \omega \omega_{cx})}{\omega^2 (\omega^2 - \omega_{ce}^2)} \delta (\omega - \omega_o) \]

\[ \varepsilon_{33} = 1 - \frac{\omega_{pe}^2}{\omega^2} - j \eta \frac{\omega_{pi}^2}{\omega^3} \]

where

- \( \omega_{pe} \) = electron plasma frequency
- \( \omega_{pi} \) = ion plasma frequency
- \( \omega_{ce} \) = electron gyrofrequency
- \( \omega_{ci} \) = ion gyrofrequency

The other symbols are explained in Chapter II. In this deduction, the influence of ions on time-varying irregularities has been neglected.

The external magnetic field is considered to be traversed along the z-direction, where \( f, \phi \) and \( Z \) would be assumed to form a cylindrical co-ordinate system. The expression for the point charge moving along the direction of the external magnetic field is given by

\[ q = q_o \frac{\delta(f)}{2\pi f} \delta(z - ut) \quad \ldots (3) \]
The corresponding current density can be written as
\[ \mathbf{J}(\mathbf{r}, t) = \mathbf{z} q_0 u \frac{\delta(f)}{2\pi f} \delta(z - ut) \] \hspace{1cm} (4)

\[ \mathbf{r} \] is the position vector of the point charge in the \((\mathbf{f}, \phi, z)\) system. The Fourier transform of (2) yields
\[ \mathbf{J}_z(\mathbf{r}, \omega) = q_0 \frac{\delta(f)}{2\pi f} e^{j\omega z/u} \] \hspace{1cm} (5)

The field components are independent of \(\phi\), but dependent on \(z\) through the phase factor \(e^{j\omega z/u}\) and can be represented as
\[ \begin{align*}
\mathbf{E}(\mathbf{r}, \omega) &= \mathbf{E}(\mathbf{f}, \omega) e^{j\omega z/u} \\
\mathbf{H}(\mathbf{r}, \omega) &= \mathbf{H}(\mathbf{f}, \omega) e^{j\omega z/u}
\end{align*} \] \hspace{1cm} (6)

The current density
\[ \mathbf{J}_y(\mathbf{r}, \omega) = \mathbf{J}_y(\mathbf{f}, \omega) e^{j\omega z/u} \] \hspace{1cm} (7)

From eqs. (1), (2), (6) and (7), the quantities \(\mathbf{E}_\phi\), \(\mathbf{E}_z\), \(\mathbf{H}_\phi\), \(\mathbf{H}_z\) can be expressed in terms of \(\mathbf{H}_\phi\) and \(\mathbf{E}_\phi\). Moreover, the coupled equations for \(\mathbf{E}_\phi\) and \(\mathbf{H}_\phi\) would be deduced as
\[ \frac{\partial}{\partial \phi} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \mathbf{E}_\phi) \right] + \left[ \frac{\omega^2}{c^2} - \frac{\epsilon_{e_4} - \epsilon_{e_2} + \epsilon_{e_1} e_{42}}{\epsilon_{e_1}} - \frac{\omega^2}{u^2} \right] \mathbf{E}_\phi = - \frac{j\omega^2}{c^2} \frac{\epsilon_{e_33}}{\epsilon_{e_11}} \mathbf{H}_\phi \] \hspace{1cm} (8)

\[ \frac{\partial}{\partial \phi} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \mathbf{H}_\phi) \right] + \left[ \frac{\omega^2}{c^2} \epsilon_{e_55} - \frac{\omega^2}{u^2} \frac{\epsilon_{e_33}}{\epsilon_{e_44}} + \left(1 + \frac{\epsilon_{e_12}}{\epsilon_{e_11}} \right) \right] \mathbf{H}_\phi = - \frac{j\omega^2}{c^2} \frac{\epsilon_{e_42} e_{e_33}}{\epsilon_{e_11} u} \mathbf{E}_\phi - \frac{\partial}{\partial \rho} \mathbf{J}_z \] \hspace{1cm} (9)
These coupled wave equations can be solved by Hankel transform technique \([108,118]\). The transform of order 1 can be defined as

\[
\tilde{E}_\phi (\xi, \omega) = \int_0^\infty E_\phi (\xi, \omega) J_1 (\xi \xi') \xi' d\xi' 
\]

... (10)

\[
E_\phi (\xi, \omega) = \int_0^\infty \tilde{E}_\phi (\xi, \omega) J_1 (\xi \xi') \xi d\xi 
\]

... (11)

\[
\tilde{H}_\phi (\xi, \omega) = \int_0^\infty H_\phi (\xi, \omega) J_1 (\xi \xi') \xi' d\xi' 
\]

... (12)

\[
H_\phi (\xi, \omega) = \int_0^\infty \tilde{H}_\phi (\xi, \omega) J_1 (\xi \xi') \xi d\xi 
\]

... (13)

\(J_1\) is the first-order Bessel function.

Applying (10) and (12) to the eqs. (8) and (9) and solving for \(\tilde{E}_\phi (\xi, \omega)\) and \(\tilde{H}_\phi (\xi, \omega)\), one can get

\[
\tilde{E}_\phi (\xi, \omega) = \frac{j q_0 \omega^2 \xi \varepsilon_{zz}}{4 \pi u \varepsilon_{ii} \Delta} 
\]

... (14)

and

\[
\tilde{H}_\phi (\xi, \omega) = \frac{q_0 \xi \left[ (\kappa_2^2 - \xi^2) + \eta^2 \right]}{4 \pi \Delta} 
\]

... (15)

where

\[
\Delta = (k_1^2 - \xi^2) (k_2^2 - \xi^2) - \frac{2 \omega^4 \varepsilon_{zz}^2 \varepsilon_{i3}^2}{c^2 u^2 \varepsilon_{ii}^2} = (\xi^2 - k_0^2)(\xi^2 - k_e^2) 
\]

... (16)
\[ K_1^2 = \frac{\omega^2}{c^2} \varepsilon_{33} - \frac{\omega^2 \varepsilon_{33}}{u^2 \varepsilon_{11}} \left( 1 + \frac{\varepsilon_{12}}{\varepsilon_{11}} \right) \]
\[ K_2^2 = \frac{\omega^2}{c^2} \varepsilon_{11} - \frac{\varepsilon^2_{22} + \varepsilon_{11} \varepsilon_{12}}{\varepsilon_{11}} - \frac{\omega^2}{u^2} \]
\[ \eta^2 = \frac{\omega^2 \varepsilon_{12}}{c u \varepsilon_{11}^2} \]

Introducing the expressions (14), (15), (16) and (17) into the eqs. (11) and (13), one can get the solutions after integration as

\[ E_\phi (\varphi, \omega) = E_{\phi_0} (\varphi, \omega) + E_{\phi_e} (\varphi, \omega) \]
\[ E_\phi (\varphi, \omega) = - \frac{j \omega^2 \varepsilon_{12}}{u \varepsilon_{11} (K_0^2 - K_e^2)} H_{\phi_0} (\varphi, \omega) - \frac{j \omega^2 \varepsilon_{22}}{u \varepsilon_{11} (K_0^2 - K_e^2)} H_{\phi_e} (\varphi, \omega) \quad \ldots (18) \]

\[ H_\phi (\varphi, \omega) = H_{\phi_0} (\varphi, \omega) + H_{\phi_e} (\varphi, \omega) \]
\[ H_\phi (\varphi, \omega) = - \frac{j q_0}{8} \hat{K}_0 \frac{K_0^2 - K_e^2}{K_0^2 - K_e^2} H_1^{(1)} (\hat{K}_0 \varphi) + \frac{j q_0}{8} \hat{K}_e \frac{K_e^2 - K_e^2}{K_0^2 - K_e^2} H_1^{(1)} (\hat{K}_e \varphi) \quad \ldots (19) \]

\[ \hat{K}_0 = \pm K_0, \quad \hat{K}_e = \pm K_e \quad \ldots (20) \]

Here, \( H_1^{(1)} \) is the Hankel function of first kind. The two possible modes are denoted by the subscripts 0 and e.
6.3 Dispersion Relation

To obtain the frequency spectrum of radiated energy, the frequency ranges where \( \kappa_0 \) and \( \kappa_e \) are positive should be determined. The functional dependence of \( \kappa_0 \) and \( \kappa_e \) on \( \omega \) can be secured from (16) and (17) as

\[
\kappa_{0e}^2 = \frac{\omega^2}{2u^2} \left[ -\lambda_1 \pm \sqrt{\lambda_1^2 - 4\lambda_2} \right] \quad \ldots (21)
\]

where

\[
\lambda_1 = \left[ 1 + \frac{\varepsilon_{33}}{\varepsilon_{11}} - \frac{u^2}{c^2\varepsilon_{11}} \left( \varepsilon_{11} - \varepsilon_{42} + \varepsilon_{11}\varepsilon_{21} \right) \right] \quad \ldots (22)
\]

\[
\lambda_2 = \left( \frac{\varepsilon_{33}}{\varepsilon_{11}} - 2\varepsilon_{33} \left( \frac{u}{c} \right)^2 + \right.
+ \left. \left( \frac{u}{c} \right)^4 \varepsilon_{33} \left( \varepsilon_{11} - \varepsilon_{42} + \varepsilon_{33}^2 \right) \right] \quad \ldots (23)
\]

Using (22), (23) and the values of the elements of the dielectric tensor, one can obtain the dispersion relation from (21).

6.4 Direction of Cerenkov Radiation

The powers radiated in the ordinary and extra-ordinary modes per unit length and per unit frequency are given by

\[
\begin{align*}
I_0(\omega) & = 2\pi f \left\langle S_{p0} \right\rangle \\
I_e(\omega) & = 2\pi f \left\langle S_{pe} \right\rangle
\end{align*}
\]

\( \left\langle S_{p0} \right\rangle \) and \( \left\langle S_{pe} \right\rangle \) are the radial components of time-averaged Poynting vectors for the two modes. \( I_0(\omega) \) and \( I_e(\omega) \) have been
obtained as

\[
I_0(\omega) = \pm \frac{q^2}{32 \omega \varepsilon_{33}} \frac{K_0^2 - K_2^2 + \eta^2}{K_0^2 - K_e^2 + \eta^2} \quad \ldots \ (25)
\]

\[
I_e(\omega) = \pm \frac{q^2}{32 \omega \varepsilon_{33}} \frac{K_e^2 - K_2^2 - \eta^2}{K_e^2 - K_0^2 - \eta^2} \quad \ldots \ (26)
\]

The upper and lower signs correspond to the upper and lower signs of (20). These must be chosen correctly so that \( I_0(\omega) > 0 \) and \( I_e(\omega) > 0 \), within the frequency range where the radiation is possible.

For Cerenkov ray, the z-component of the time-averaged Poynting vector is necessary. This yields, after some algebraic simplifications, for the ordinary wave as

\[
\langle \mathbf{S}_{z0} \rangle = \frac{1}{2\pi^2} \frac{q^2}{32 \varepsilon_{33}(K_0^2 - K_e^2 + \eta^2)^2} \left[ \frac{\omega^2 \varepsilon_{11} \left( K_0^2 - K_e^2 + \eta^2 \right)}{u^2} + \frac{\omega^4 \varepsilon_{12}^2 \varepsilon_{44}}{c^2 \varepsilon_{11}} \left( \varepsilon_{11} \varepsilon_{12} + \varepsilon_{11}^2 - \varepsilon_{12}^2 \right) \frac{K_e}{\varepsilon_{33}} \right] \quad \ldots \ (27)
\]

For the extra-ordinary wave, the expression of \( \langle \mathbf{S}_{ze} \rangle \) can be obtained by interchanging \( K_0 \) and \( K_e \) in (27). The angle between the Cerenkov ray and the direction of motion of the source is given by
\[ \tan \theta_o = \frac{\langle S_{fe} \rangle}{\langle S_{ze} \rangle} \]

and

\[ \tan \theta_e = \frac{\langle S_{fe} \rangle}{\langle S_{ze} \rangle} \]

where \( \theta_o \) and \( \theta_e \) refer to ordinary and extra-ordinary waves.

6.5 Discussion

The direction of the Cerenkov radiation may be computed numerically using expressions (24), (27) and (28). In the analysis, the influences of time-varying irregularities and motion of heavy ions have been taken into account through the dielectric tensor. The effect of motion of heavy ions is important in the F-region of the ionosphere. Thus the contributions of these effects on the frequency spectrum of Cerenkov radiation from the upper atmosphere may be examined from the present results. Although an idealised plasma model has been considered, it is expected that the present study may provide some physical insight into the possible radiation from a moving charge particle in the anisotropic ionosphere traversed by the geomagnetic field and time-varying irregularities.