Chapter 4

$M^{[X]}/G/1$ Queue with Bernoulli Schedule Vacations and Random Breakdowns

4.1 Introduction

Research studies on queues with server breakdowns and vacations have become as an indispensable area in queueing theory and have been studied extensively and successfully due to their various applications in production, communication systems and computer networks. Most recently the research studies on queueing systems with breakdowns and server vacations have been attracted by numerous researchers in queueing theory literature. Li et al. (1997) analyzed the reliability of M/G/1 queues with server breakdowns and vacations. Gray (2000) studied vacation queueing model with service breakdowns. Ke (2003) analyzed the optimal strategy of the controllable $M^{[X]}/G/1$ queue with breakdowns and multiple vacations. Neuts and Lucantoni (1979) studied multi server Markovian queue with breakdowns and repairs. Tang (1997) published the paper devoted to unreliable single server queue with generally distributed service times. Maraghi et
al. (2009) obtained steady state solution of batch arrival queueing system with random breakdowns and Bernoulli schedule server vacations having general vacation time. Transient analysis is useful to the behavior of the queueing system over time. Garg (2003) derived the time dependent solution of a single channel queueing system with service in batches of variable size where inter arrival times and service times are exponentially distributed.

This chapter consists of two models. In Model 1, we study $M^{[X]}/G/1$ Queue with feedback, Bernoulli schedule vacations and random breakdowns and in model 2, we study $M^{[X]}/G/1$ queue with second optional service, Bernoulli schedule vacations and random breakdowns.

**Model 1 : $M^{[X]}/G/1$ Queue with Feedback, Bernoulli Schedule Vacations and Random Breakdowns**

In this model, we investigate a batch arrival queueing system in which the server suffers random breakdowns from time to time, and the server has the option to take a vacation after the completion of any service. The durations of the server vacation are of random length. The study of queueing systems with feedback have been concentrated by many authors. Takacs (1963) has done a pioneer work in the study of single server queue with feedback. A tagged customer may get an unsuccessful service, then it retries to take the service until the service should be success. In this case, the customer feedback instantaneously to the tail of the queue with probability $p$ ($0 \leq p \leq 1$) or depart from the system with probability $q = (1 - p)$.

In most of the research study of queueing models, the server is assumed to be reliable such that the server works for ever, but this is not the case in most of the real scenarios that the servers are reliable such that the servers may meet
breakdowns. Whenever the system breaks down, it enters a repair process and the
customer whose service is interrupted goes back to the head of the queue. After a
service completion, the server may go for a vacation with probability \( \theta (0 \leq \theta \leq 1) \)
or may continue to serve the next customer, if any, with probability \( 1 - \theta \). The
service times and the vacation times are generally distributed, while the repair
times are exponentially distributed. We obtain time dependent results and steady
state results in terms of the queue size distribution at the random epoch as well
as the probabilities for various states of the system.

The rest of the model is organized as follows. Section 4.2 gives the assumptions
underlying the queueing system considered. Equations governing the system are
formulated in section 4.3 while the time dependent solutions to those equations
are given in section 4.4. In section 4.5, the steady state analysis has been discussed
and in section 4.6, some system performance indices are found. Some particular
cases of interest are discussed in 4.7. To validate the analytical results of the
model, we discuss numerical results and graphs in section 4.8.

### 4.2 Description of the Model

To describe the required queueing model, we assume the following.

a) Let \( \lambda \sum_{i=1}^{\infty} c_i \, dt; \ i = 1, 2, 3, \ldots \) be the first order probability of arrival of \( i \) customers
in batches in the system during a short period of time \( (t,t+dt) \) where
\( 0 \leq c_i \leq 1, \sum_{i=1}^{\infty} c_i = 1, \lambda > 0 \) is the mean arrival rate of batches.

b) There is a single server which provides service following a general(arbitrary)
distribution with distribution function \( B(v) \) and density function \( b(v) \). Let \( \mu(x) \, dx \)
be the conditional probability density function of service completion during the
interval \( (x, x+dx] \) given that the elapsed service time is \( x \), so that

\[
\mu(x) = \frac{b(x)}{1 - B(x)},
\]

(4.1)
and therefore
\[ b(v) = \mu(v)e^{-\int_0^v \mu(x)\,dx}. \quad (4.2) \]

c) After completion of service, if the customer is not satisfied with the service for certain reason or if customer received unsuccessful service, the customer may immediately join the tail of the original queue as a feedback customer for receiving another regular service with probability \( p (0 < p < 1) \). Otherwise the customer may depart forever from the system with probability \( q (= 1 - p) \). The service discipline for feedback and newly customers are first come, first served. Also service time for a feedback customer is independent of its previous service times.
d) As soon as a service is completed, the server may take a vacation of random length with probability \( \theta \) (or) he may stay in the system providing service with probability \( 1 - \theta \), where \( 0 \leq \theta \leq 1 \).
e) The vacation time of the server follows a general (arbitrary) with distribution function \( V(s) \) and the density function \( v(s) \). Let \( \nu(x)\,dx \) be the conditional probability of a completion of a vacation during the interval \( (x, x + dx] \) given that the elapsed vacation time is \( x \) so that,
\[ \nu(x) = \frac{v(x)}{1 - V(x)}, \quad (4.3) \]
and therefore
\[ v(s) = \nu(s)e^{-\int_0^s \nu(x)\,dx}. \quad (4.4) \]
f) The system may breakdown at random and the breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate \( \alpha > 0 \). Further we assume that once the system breakdown, the customer whose service is interrupted comes back to the head of queue.
g) Once the system breaks down it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate \( \beta > 0 \).
h) Various stochastic processes involved in the queueing system are assumed to be independent of each other.
4.3 Definitions and Equations Governing the System

We let,

\[ P_n(x, t) = \text{Probability that at time 't' the server is active providing service and there are 'n' (n ≥ 0) customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently } P_n(t) \text{ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in service irrespective of the value of x.} \]

\[ V_n(x, t) = \text{Probability that at time 't', the server is on vacation with elapsed vacation time x, and there are 'n' (n ≥ 0) customers waiting in the queue for service. Consequently } V_n(t) \text{ denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.} \]

\[ R_n(t) = \text{Probability that at time t, the server is inactive due to breakdown and the system is under repair while there are'n' (n ≥ 0) customers in the queue.} \]

\[ Q(t) = \text{Probability that at time 't' there are no customers in the system and the server is idle but available in the system.} \]

The model is then, governed by the following set of differential - difference equations:

\[ \frac{\partial}{\partial t} P_n(x, t) + \frac{\partial}{\partial x} P_n(x, t) + (\lambda + \mu(x) + \alpha) P_n(x, t) = \lambda \sum_{i=1}^{n-1} c_i P_{n-i}(x, t), \ n \geq 1 \quad (4.5) \]

\[ \frac{\partial}{\partial t} P_0(x, t) + \frac{\partial}{\partial x} P_0(x, t) + (\lambda + \mu(x) + \alpha) P_0(x, t) = 0 \quad (4.6) \]

\[ \frac{\partial}{\partial t} V_n(x, t) + \frac{\partial}{\partial x} V_n(x, t) + (\lambda + \nu(x)) V_n(x, t) = \lambda \sum_{i=1}^{n-1} c_i V_{n-i}(x, t), \ n \geq 1 \quad (4.7) \]

\[ \frac{\partial}{\partial t} V_0(x, t) + \frac{\partial}{\partial x} V_0(x, t) + (\lambda + \nu(x)) V_0(x, t) = 0 \quad (4.8) \]

\[ \frac{d}{dt} R_n(t) = - (\lambda + \beta) R_n(t) + \lambda \sum_{i=1}^{n-1} c_i R_{n-i}(x, t) + \alpha \int_{0}^{\infty} P_{n-1}(x, t) dx, \ n \geq 1 \quad (4.9) \]

\[ \frac{d}{dt} R_0(t) = - (\lambda + \beta) R_0(t) \quad (4.10) \]
\[
\frac{d}{dt} Q(t) = -\lambda Q(t) + \beta R_0(t) + \int_0^\infty V_0(x, t) \nu(x) dx + (1 - \theta) q \int_0^\infty P_0(x, t) \mu(x) dx
\] (4.11)

The above equations are to be solved subject to the following boundary conditions,

\[
P_n(0, t) = (1 - \theta) \left[ p \int_0^\infty P_n(x, t) \mu(x) dx + q \int_0^\infty P_{n+1}(x, t) \mu(x) dx \right]
+ \int_0^\infty V_{n+1}(x, t) \nu(x) dx + \beta R_{n+1}(t) + \lambda c_{n+1} Q(t), \ n \geq 0 \tag{4.12}
\]

\[
V_n(0, t) = \theta \int_0^\infty P_n(x, t) \mu(x) dx, \ n \geq 0 \tag{4.13}
\]

Assuming there are no customers in the system initially so that the server is idle.

\[
V_0(0) = 0, \ V_n(0) = 0, \ P_n(0) = 0, \ n = 0, 1, 2, ..., \ Q(0) = 1. \tag{4.14}
\]

### 4.4 Generating Functions of the Queue Length: The Time Dependent Solution

We define the probability generating functions,

\[
P_q(x, z, t) = \sum_{n=0}^\infty z^n P_n(x, t), \ P_q(z, t) = \sum_{n=0}^\infty z^n P_n(t), \tag{4.15}
\]

\[
V_q(x, z, t) = \sum_{n=0}^\infty z^n V_n(x, t), \ V_q(z, t) = \sum_{n=0}^\infty z^n V_n(t), \tag{4.16}
\]

\[
R_q(z, t) = \sum_{n=0}^\infty z^n R_n(t), \tag{4.17}
\]

\[
C(z) = \sum_{n=1}^\infty c_n z^n \tag{4.18}
\]

which are convergent inside the circle given by \(|z| \leq 1\) and define the Laplace transform of a function \(f(t)\) as

\[
\overline{f}(s) = \int_0^\infty f(t) e^{-st} dt, \ \text{Re}(s) \geq 0 \tag{4.19}
\]

Taking Laplace transforms of equations (4.5) to (4.11) and using the probability generating function defined above.

\[
\frac{\partial}{\partial x} \overline{P}_n(x, s) + (s + \lambda + \mu(x) + \alpha) \overline{P}_n(x, s) = \lambda \sum_{i=1}^{n-1} c_i \overline{P}_{n-i}(x, s), \ n \geq 1 \tag{4.20}
\]
\frac{\partial}{\partial x} \bar{P}_n(x, s) + (s + \lambda + \mu(x) + \alpha) \bar{P}_n(x, s) = 0 \tag{4.21}

\frac{\partial}{\partial x} \bar{V}_n(x, s) + (s + \lambda + \nu(x)) \bar{V}_n(x, s) = \lambda \sum_{i=1}^{n-1} c_i \bar{V}_{n-i}(x, s), \ n \geq 1 \tag{4.22}

\frac{\partial}{\partial x} \bar{V}_0(x, s) + (s + \lambda + \nu(x)) \bar{V}_0(x, s) = 0 \tag{4.23}

(s + \lambda + \beta) \bar{R}_n(s) = \lambda \sum_{i=1}^{n-1} c_i \bar{R}_{n-i}(s) + \alpha \int_{0}^{\infty} \bar{P}_{n-1}(x, s) dx, \ n \geq 1 \tag{4.24}

(s + \lambda + \beta) \bar{R}_0(s) = 0 \tag{4.25}

(s + \lambda) \bar{Q}(s) = 1 + \beta \bar{R}_0(s) + \int_{0}^{\infty} \bar{V}_0(x, s) \nu(x) dx

+ (1 - \theta) q \int_{0}^{\infty} \bar{P}_0(x, s) \mu(x) dx \tag{4.26}

for boundary conditions,

\bar{P}_n(0, s) = (1 - \theta) \left[ p \int_{0}^{\infty} \bar{P}_n(x, s) \mu(x) dx + q \int_{0}^{\infty} \bar{P}_{n+1}(x, s) \mu(x) dx \right]

+ \int_{0}^{\infty} \bar{V}_{n+1}(x, s) \nu(x) dx + \beta \bar{R}_{n+1}(s) + \lambda c_{n+1} \bar{Q}(s), \ n \geq 0 \tag{4.27}

\bar{V}_n(0, s) = \theta \int_{0}^{\infty} \bar{P}_n(x, s) \mu(x) dx, \ n \geq 0 \tag{4.28}

Multiplying equation (4.20) by \ z^n \ and adding (4.21) we get

\frac{\partial}{\partial x} \bar{P}_q(x, z, s) + (s + \lambda - \lambda C(z) + \mu(x) + \alpha) \bar{P}_q(x, z, s) = 0 \tag{4.29}

Performing similar operations to equations (4.22) to (4.25).

\frac{\partial}{\partial x} \bar{V}_q(x, z, s) + (s + \lambda - \lambda C(z) + \nu(x)) \bar{V}_q(x, z, s) = 0 \tag{4.30}

(s + \lambda - \lambda C(z) + \beta) \bar{R}_q(z, s) = \alpha z \int_{0}^{\infty} \bar{P}_q(x, z, s) dx \tag{4.31}

For the boundary conditions, we multiply equation (4.27) by \ z^{n+1}, \ sum over \ n \ from 0 to \ \infty \ and use generating function defined above, we get

z \bar{P}_q(0, z, s) = (1 - \theta)(pz + q) \int_{0}^{\infty} \bar{P}_q(x, z, s) \mu(x) dx + \int_{0}^{\infty} \bar{V}_q(x, z, s) \nu(x) dx

+ \beta \bar{R}_q(z, s) + (1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s) \tag{4.32}
Similarly multiplying equation (4.28) by \( z^n \) and sum over \( n \) from 0 to \( \infty \) and use generating function defined above.

\[
\bar{V}_q(0, z, s) = \theta \int_0^\infty \bar{P}_q(x, z, s)\mu(x)dx \tag{4.33}
\]

Integrating equation (4.29) from 0 to \( x \) yields

\[
\bar{P}_q(x, z, s) = \bar{P}_q(0, z, s)e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu(t)dt} \tag{4.34}
\]

where \( \bar{P}_q(0, z, s) \) is given by equation (4.32). Again integrating equation (4.34) by parts with respect to \( x \) yields

\[
\bar{P}_q(z, s) = \bar{P}_q(0, z, s) \left[ \frac{1 - \bar{B}(s + \lambda - \lambda C(z) + \alpha)}{(s + \lambda - \lambda C(z) + \alpha)} \right] \tag{4.35}
\]

where

\[
\bar{B}(s + \lambda - \lambda C(z) + \alpha) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}dB(x) \tag{4.36}
\]

is Laplace - Stieltjes transform of the service time \( B(x) \). Now multiplying both sides of equation (4.34) by \( \mu(x) \) and integrating over \( x \), we get

\[
\int_0^\infty \bar{P}_q(x, z, s)\mu(x)dx = \bar{P}_q(0, z, s)\bar{B}(s + \lambda - \lambda C(z) + \alpha) \tag{4.37}
\]

Using equation (4.37) in equation (4.33), we get

\[
\bar{V}_q(0, z, s) = \theta \bar{P}_q(0, z, s)\bar{B}(s + \lambda - \lambda C(z) + \alpha) \tag{4.38}
\]

Similarly integrating equation (4.30) from 0 to \( x \), we get

\[
\bar{V}_q(x, z, s) = \bar{V}_q(0, z, s)e^{-(s+\lambda-\lambda C(z))x-\int_0^x \nu(t)dt} \tag{4.39}
\]

Substituting the value of \( \bar{V}_q(0, z, s) \) from (4.38) in equation (4.39), we get

\[
\bar{V}_q(x, z, s) = \theta \bar{P}_q(0, z, s)\bar{B}(s + \lambda - \lambda C(z)) + \alpha)e^{-(s+\lambda-\lambda C(z))x-\int_0^x \nu(t)dt} \tag{4.40}
\]

Again integrating equation (4.40) by parts with respect to \( x \)

\[
\bar{V}_q(z, s) = \theta \bar{P}_q(0, z, s)\bar{B}(s + \lambda - \lambda C(z)) + \alpha) \left[ 1 - \bar{V}(s + \lambda - \lambda C(z)) \right] \tag{4.41}
\]
where

\[
\bar{V}(s + \lambda - \lambda C(z)) = \int_{0}^{\infty} e^{-(s+\lambda - \lambda C(z))x} dV(x)
\]  

(4.42)

is Laplace - Stieltjes transform of the vacation time $V(x)$. Now multiplying both sides of equation (4.39) by $\nu(x)$ and integrating over $x$, we have

\[
\int_{0}^{\infty} \bar{V}(x, z, s) \nu(x) dx = \theta \bar{P}_q(0, z, s) \bar{B}(s + \lambda - \lambda C(z) + \alpha) \bar{V}(s + \lambda - \lambda C(z))
\]  

(4.43)

Using equation (4.35), equation (4.31) becomes

\[
\bar{R}_q(z, s) = \frac{\alpha z \bar{P}_q(0, z, s)[1 - \bar{B}(s + \lambda - \lambda C(z) + \alpha)]}{[s + \lambda - \lambda C(z) + \beta][s + \lambda - \lambda C(z) + \alpha]}
\]

(4.44)

Now using equations (4.37), (4.43) and (4.44) in equation (4.32) and solving for $\bar{P}_q(0, z, s)$, we get

\[
\bar{P}_q(0, z, s) = \frac{f_1(z, s)f_2(z, s)[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{D(z, s)}
\]  

(4.45)

where

\[
D(z, s) = f_1(z, s)f_2(z, s) \left\{ z - (1 - \theta)(pz + q)\bar{B}[f_1(z, s)] ight. \\
- \theta \bar{V}[f_3(z, s)]\bar{B}[f_1(z, s)] - \alpha \beta z \left\{ 1 - \bar{B}[f_1(z, s)] \right\},
\]

\[
f_1(z, s) = s + \lambda - \lambda C(z) + \alpha,
\]

\[
f_2(z, s) = s + \lambda - \lambda C(z) + \beta,
\]

and $f_3(z, s) = s + \lambda - \lambda C(z)$.

Substituting the value of $\bar{P}_q(0, z, s)$ from equation (4.45) in to equations (4.35), (4.41) and (4.44), we have

\[
\bar{P}_q(z, s) = \frac{f_2(z, s)[1 - \bar{B}[f_1(z, s)]][1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{D(z, s)}
\]  

(4.46)

\[
\bar{V}_q(z, s) = \frac{\theta f_1(z, s)f_2(z, s)\bar{B}[f_1(z, s)][(1 - s\bar{Q}(s)) + \lambda(C(z)) - 1)\bar{Q}(s)]}{D(z, s)}
\]

\[
\times \left[ 1 - \bar{V}(s + \lambda - \lambda C(z)) \right]_{f_3(z, s)}
\]

(4.47)

\[
\bar{R}_q(z, s) = \frac{\alpha z[1 - \bar{B}[f_1(z, s)]][1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{D(z, s)}
\]

(4.48)

where $D(z, s)$ is given above. Thus, the Probability generating functions $\bar{P}_q(z, s)$, $\bar{V}_q(z, s)$ and $\bar{R}_q(z, s)$ are completely determined in closed form.
4.5 The Steady State Results

In this section we shall derive the steady state probability distribution for the model. To define the steady state probabilities, suppress the argument ‘t’ wherever it appears in the time dependent analysis and this can be obtained by using well known Tauberian property

\[ \lim_{t \to \infty} f(t) = \lim_{s \to 0} s f(s) \] (4.49)

Multiplying both sides of equation (4.46), (4.47), (4.48) and applying equation (4.49) and then simplifying, we get

\[ P_q(z) = f_2(z)(1 - \bar{B}[f_1(z)])[\lambda(C(z) - 1)]Q \] (4.50)

\[ V_q(z) = \theta f_1(z)f_2(z)\bar{B}[f_1(z)][\bar{V}[f_3(z)] - 1]Q \] (4.51)

\[ R_q(z) = \theta \alpha z[1 - \bar{B}[f_1(z)]][\lambda(C(z) - 1)]Q \] (4.52)

where

\[ D(z) = f_1(z)f_2(z) \left\{ z - (1 - \theta)(pz + q)\bar{B}[f_1(z)] - \theta \bar{V}[f_3(z)]\bar{B}[f_1(z)] \right\} \]

\[ - \alpha \beta z\{1 - \bar{B}[f_1(z)]\}, \] (4.53)

\[ f_1(z) = \lambda - \lambda C(z) + \alpha, \]

\[ f_2(z) = \lambda - \lambda C(z) + \beta, \]

and \( f_3(z) = \lambda - \lambda C(z). \)

Let \( W_q(z) \) denotes the probability generating function of queue size irrespective of the state of the system. Then adding (4.50), (4.51) and (4.52) we get

\[ W_q(z) = P_q(z) + V_q(z) + R_q(z) \] (4.54)

\[ W_q(z) = \frac{f_2(z)[1 - B[f_1(z)]]}{D(z)} \left[ \lambda(C(z) - 1)Q \right] + \frac{\theta f_1(z)f_2(z)B[f_1(z)][\bar{V}[f_3(z)] - 1]}{D(z)}Q \]

\[ + \frac{\alpha z[1 - B[f_1(z)]]\lambda(C(z) - 1)Q}{D(z)} \] (4.55)
In order to obtain $Q$, we use the normalization condition as follows.

$$W_q(1) + Q = 1$$  \hspace{1cm} (4.56)$$

We see that at $z = 1$, $W_q(z)$ is indeterminate of the form $0/0$. We apply L’Hopital rule in equation (4.55)

$$W_q(1) = \frac{\lambda Q E(I) [(\alpha + \beta) [1 - \bar{B}(\alpha)] + \theta \alpha \beta B(\alpha) E[V]]}{(q + p\theta) \alpha \beta B(\alpha) - \lambda (\alpha + \beta)(1 - B(\alpha)) E(I) - \theta \lambda \alpha \beta B(\alpha) E(I) E[V]}$$  \hspace{1cm} (4.57)$$

where $\bar{B}(0) = 1, \bar{V}(0) = 1, -V'(0) = E[V]$, the mean vacation time, $C(1)=1$, $C'(1)= E(I)$ is the mean batch size of the arriving customers. Using equation (4.57) in equation (4.56), we get

$$Q = 1 - \frac{\lambda E(I)}{q + p\theta} \left\{ \frac{1}{\beta B(\alpha)} + \frac{1}{\alpha B(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right\}$$  \hspace{1cm} (4.58)$$

and the utilization factor $\rho$ of the system is given by

$$\rho = \frac{\lambda E(I)}{q + p\theta} \left\{ \frac{1}{\beta B(\alpha)} + \frac{1}{\alpha B(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right\}$$  \hspace{1cm} (4.59)$$

where $\rho < 1$ is the stability condition under which the steady state exists, equation (4.58) gives the probability that the server is idle. Substitute $Q$ from equation (4.58) in equation (4.55), $W_q(z)$, the probability generating function of the queue size in steady state has been completely and explicitly determined.

### 4.6 The Average Queue Size and Average Waiting Time

Let $L_q$ denote the mean number of customers in the queue under the steady state, then

$$L_q = \frac{d}{dz} W_q(z) \big|_{z=1}$$

as this formula gives $0/0$ form, we write

$$W_q(z) = \frac{N(z)}{D(z)}$$
where \( N(z) \) and \( D(z) \) are the numerator and denominator of the right hand side of equation (4.54) respectively, then we use

\[
L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2[D'(1)]^2}
\]

(4.60)

where primes and double primes in equation (4.60) denote first and second derivatives at \( z=1 \) respectively. Carrying out the derivatives at \( z=1 \), we have

\[
N'(1) = \lambda E(I)Q \left\{ (\alpha + \beta) + \bar{B}(\alpha)(\theta \alpha \beta E(V) - \alpha - \beta) \right\}
\]

(4.61)

\[
N''(1) = 2Q[\lambda E(I)]^2 \left\{ \left( \frac{\alpha}{\lambda E(I)} - 1 \right) + \bar{B}(\alpha) \left[ 1 - \frac{\alpha}{\lambda E(I)} - \theta \alpha E(V) - \theta \beta E(V) + \frac{1}{2}\theta \alpha \beta E(V^2) \right] + \bar{B}'(\alpha)(\alpha + \beta - \theta \alpha \beta E(V)) \right\} + \lambda QE(I(I-1)) \left\{ (\alpha + \beta) + \bar{B}(\alpha)(\theta \alpha \beta E(V) - \alpha - \beta) \right\}
\]

(4.62)

\[
D'(1) = -\lambda E(I)(\alpha + \beta) + \bar{B}(\alpha) \left\{ \alpha \beta (q + p \theta) + \lambda E(I) [ (\alpha + \beta) - \theta \alpha \beta E(V) ] \right\}
\]

(4.63)

\[
D''(1) = 2[\lambda E(I)]^2 \left\{ \left( 1 - \frac{\alpha + \beta}{\lambda E(I)} \right) + \bar{B}(\alpha) \left[ -(q + p \theta) + \theta \alpha E(V) + \theta \beta E(V) - \frac{1}{2}\alpha \beta \theta E(V^2) \right] + \bar{B}'(\alpha) \left[ -(q + p \theta)(\alpha + \beta) - \frac{\alpha \beta}{\lambda E(I)} + \alpha \beta \theta E(V) \right] \right\} + \lambda E(I(I-1)) \left\{ -(\alpha + \beta) + \bar{B}(\alpha)(\alpha + \beta - \theta \alpha \beta E(V)) \right\}
\]

(4.64)

where \( E(V) \) is the mean vacation time, \( E(V^2) \) is the second moment of the vacation time, \( E(I) \) is the mean of the batch size of arriving customers, \( E(I(I-1)) \) is the second factorial moment of the batch size of arriving customers and \( Q \) which has been found in equation (4.58). Then if we substitute the values of \( N'(1) \), \( N''(1) \), \( D'(1) \) and \( D''(1) \) from equations (4.61), (4.62), (4.63) and (4.64) in to equation (4.60),
we obtain $L_q$ in a closed form.

The mean waiting time of a customer could be found as follows:

$$W_q = \frac{L_q}{\lambda} \quad (4.65)$$

Also other performance measures, average number of customers in the system $L_s$ and average waiting time $W_s$ are obtained by using Little’s formula as follows:

$$L_s = L_q + \rho \quad (4.66)$$

$$W_s = \frac{L}{\lambda} \quad (4.67)$$

### 4.7 Particular Cases

#### 4.7.1 No Feed Back

If all customers do not require feedback, that is $p = 0$ and $q = 1$, then from the main results of the model, we obtain

$$W_q(z) = \frac{f_2(z)[1 - B[f_1(z)][\lambda(C(z) - 1)]Q}{D(z)} + \frac{\theta f_1(z)f_2(z)B[f_1(z)][\bar{V}[f_3(z)] - 1]Q}{D(z)} + \frac{\alpha z[1 - B[f_1(z)]][\lambda[(C(z) - 1)Q]}{D(z)} \quad (4.68)$$

where

$$D(z) = f_1(z)f_2(z) \left\{ z - (1 - \theta)\bar{B}[f_1(z)] - \theta\bar{V}[f_3(z)]\bar{B}[f_1(z)] \right\} - \alpha\beta z[1 - \bar{B}[f_1(z)]],$$

$$f_1(z) = \lambda - \lambda C(z) + \alpha,$$

$$f_2(z) = \lambda - \lambda C(z) + \beta,$$

and $f_3(z) = \lambda - \lambda C(z)$.

$$Q = 1 - \lambda E(I) \left\{ \frac{1}{\beta B(\alpha)} + \frac{1}{\alpha B(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right\} \quad (4.69)$$

and the utilization factor $\rho$ of the system is given by

$$\rho = \lambda E(I) \left\{ \frac{1}{\beta B(\alpha)} + \frac{1}{\alpha B(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right\} \quad (4.70)$$
\[ N'(1) = \lambda E(I)Q \left\{ (\alpha + \beta) + \bar{B}(\alpha)(\theta \alpha \beta E(V) - \alpha - \beta) \right\} \] (4.71)

\[ N''(1) = 2Q[\lambda E(I)]^2 \left\{ \left( \frac{\alpha}{\lambda E(I)} - 1 \right) \right. \\
+ \bar{B}(\alpha) \left[ 1 - \frac{\alpha}{\lambda E(I)} - \theta \alpha E(V) - \theta \beta E(V) + \frac{1}{2}\theta \alpha \beta E(V^2) \right] \\
+ \bar{B}'(\alpha)(\alpha + \beta - \theta \alpha \beta E(V)) \left\} \right. \\
+ \lambda QE(I)(I - 1)) \left\{ (\alpha + \beta) + \bar{B}(\alpha)(\theta \alpha \beta E(V) - \alpha - \beta) \right\} \] (4.72)

\[ D'(1) = -\lambda E(I)(\alpha + \beta) + \bar{B}(\alpha) \left\{ \alpha \beta + \lambda E(I) [(\alpha + \beta) - \theta \alpha \beta E(V)] \right\} \] (4.73)

\[ D''(1) = 2[\lambda E(I)]^2 \left\{ \left( 1 - \frac{\alpha + \beta}{\lambda E(I)} \right) \right. \\
+ \bar{B}(\alpha) \left[ -1 + \theta \alpha E(V) + \theta \beta E(V) - \frac{1}{2}\alpha \beta \theta E(V^2) \right] \\
+ \bar{B}'(\alpha) \left[ -(\alpha + \beta) - \frac{\alpha \beta}{\lambda E(I)} + \alpha \beta \theta E(V) \right] \right. \\
+ \lambda E(I)(I - 1)) \left\{ -(\alpha + \beta) + \bar{B}(\alpha)(\alpha + \beta - \theta \alpha \beta E(V)) \right\} \] (4.74)

The results obtained in (4.68) to (4.74) agree with the results obtained by Maraghi et al. (2009)

### 4.7.2 Single Poisson Arrivals and No Feed Back

If customers arrive singly instead of batches, that is, \( c_1 = 1, c_i = 0 \) for \( i \neq 0 \), \( C(z) = z \), \( E(I) = 1 \), \( E(I(I - 1)) = 0 \) and there is no feed back, that is, \( p = 0 \) and \( q = 1 \), then we obtain

\[
W_q(z) = \frac{f_2(z)[1 - B[f_1(z)]][\lambda(z - 1)Q]}{D(z)} \\
+ \theta f_1(z)f_2(z)B[f_1(z)][\bar{V}[f_3(z)]Q] \\
+ \alpha z[1 - B[f_1(z)]][\lambda(z - 1)Q]
\]

where

\[
D(z) = f_1(z)f_2(z) \left\{ z - (1 - \theta)\bar{B}[f_1(z)] - \theta \bar{V}[f_3(z)]\bar{B}[f_1(z)] \right\} - \alpha \beta z \{ 1 - \bar{B}[f_1(z)] \}
\]

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\[ f_1(z) = \lambda - \lambda z + \alpha, \]
\[ f_2(z) = \lambda - \lambda z + \beta, \]
\[ f_3(z) = \lambda - \lambda z. \]

\[ Q = 1 - \lambda \left \{ \frac{1}{\beta B(\alpha)} + \frac{1}{\alpha B(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right \} \quad (4.76) \]
\[ \rho = \lambda \left \{ \frac{1}{\beta B(\alpha)} + \frac{1}{\alpha B(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right \} \quad (4.77) \]
\[ N'(1) = \lambda Q \left \{ (\alpha + \beta) + B(\alpha)(\theta \alpha \beta E(V) - \alpha - \beta) \right \} \quad (4.78) \]
\[ N''(1) = 2Q[\lambda]^2 \left \{ \left( \frac{\alpha}{\lambda} - 1 \right) \right. \]
\[ + B(\alpha) \left[ 1 - \frac{\alpha}{\lambda} - \theta \alpha E(V) - \theta \beta E(V) + \frac{1}{2} \theta \alpha \beta E(V^2) \right] \]
\[ + \bar{B}'(\alpha) [ (\alpha + \beta - \theta \alpha \beta E(V))] \} \quad (4.79) \]
\[ D'(1) = -\lambda (\alpha + \beta) + \bar{B}(\alpha) \{ \alpha \beta + \lambda [ (\alpha + \beta - \theta \alpha \beta E(V)]) \} \quad (4.80) \]
\[ D''(1) = 2[\lambda]^2 \left \{ \left( 1 - \frac{\alpha + \beta}{\lambda} \right) \right. \]
\[ + \bar{B}(\alpha) \left[ -1 + \theta \alpha E(V) + \theta \beta E(V) - \frac{1}{2} \alpha \beta \theta E(V^2) \right] \]
\[ + \bar{B}'(\alpha) \left[ -(\alpha + \beta) - \frac{\alpha \beta}{\lambda E(I)} + \alpha \beta \theta E(V) \right] \} \quad (4.81) \]

4.8 Numerical results

In order to validate the analytical results of this model, we consider the service times, vacation times and repair times to be exponentially distributed. We choose the following values: \( \lambda = 2, \mu = 5, \nu = 7, \beta = 8, p = 0.2, q = 0.8, E(I) = 1 \) and \( E(I(I - 1)) = 0 \), we choose that \( \alpha \) takes the values 1, 2, 3 and 4 while \( \theta \) takes the values 0.25, 0.5, 0.75 and 1. All values were chosen arbitrarily in order that the stability condition is satisfied. Table 4.1 give the computed values of the proportion of idle time, the utilization factor, and the performance measures. It is clear from table 4.1 that increasing the value of \( \alpha \) or \( \theta \) increases the traffic intensity, the average queue length and the average response time, while the server idle time decreases. All the trends shown by this table are as expected.
Table 4.1: Some queue performance measures values computed when $\lambda = 2$, $\mu = 5$, $\nu = 7$, $\beta = 8$, $p = 0.2$, $q = 0.8$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\theta$</th>
<th>$Q$</th>
<th>$\rho$</th>
<th>$L_q$</th>
<th>$L_s$</th>
<th>$W_q$</th>
<th>$W_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.3689</td>
<td>0.6311</td>
<td>1.3208</td>
<td>1.952</td>
<td>0.6604</td>
<td>0.976</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.3345</td>
<td>0.6655</td>
<td>1.3757</td>
<td>2.0412</td>
<td>0.6879</td>
<td>1.0206</td>
</tr>
<tr>
<td>1</td>
<td>0.75</td>
<td>0.3037</td>
<td>0.6963</td>
<td>1.4171</td>
<td>2.1134</td>
<td>0.7085</td>
<td>1.0567</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.276</td>
<td>0.724</td>
<td>1.446</td>
<td>2.1699</td>
<td>0.723</td>
<td>1.085</td>
</tr>
</tbody>
</table>

| 2        | 0.25     | 0.3061| 0.6939 | 1.778  | 2.4719| 0.889  | 1.236  |
| 2        | 0.5      | 0.2752| 0.7248 | 1.8974 | 2.6222| 0.9487 | 1.3111 |
| 2        | 0.75     | 0.2475| 0.7525 | 2.0143 | 2.7668| 1.0072 | 1.3834 |
| 2        | 1        | 0.2227| 0.7773 | 2.1308 | 2.9081| 1.0654 | 1.4541 |

| 3        | 0.25     | 0.2402| 0.7598 | 2.6099 | 3.3697| 1.305  | 1.6849 |
| 3        | 0.5      | 0.213 | 0.787  | 2.8329 | 3.6199| 1.4164 | 1.81   |
| 3        | 0.75     | 0.1886| 0.8114 | 3.072  | 3.8834| 1.536  | 1.9417 |
| 3        | 1        | 0.1667| 0.8333 | 3.3333 | 4.1667| 1.6667 | 2.0833 |

| 4        | 0.25     | 0.1713| 0.8287 | 4.2308 | 5.0595| 2.1154 | 2.5297 |
| 4        | 0.5      | 0.1479| 0.8521 | 4.7101 | 5.5622| 2.355  | 2.7811 |
| 4        | 0.75     | 0.1269| 0.8731 | 5.2715 | 6.1446| 2.6358 | 3.0723 |
| 4        | 1        | 0.1081| 0.8919 | 5.9465 | 6.8384| 2.9732 | 3.4192 |
Graph 4.1 shows the effect of $\theta$ on mean system size $L_s$ and mean waiting time of a customer in the system $W_s$ respectively where we consider $\alpha=4$. Graph 4.2 shows that the effect of $\alpha$ on mean system size $L_s$ and mean waiting time of a customer in the system $W_s$ when $\theta = 0.25$.

Fig 4.1: Effect of $\theta$ on $L_s$ and $W_s$

Fig 4.2: Effect of $\alpha$ on $L_s$ and $W_s$
Model 2 : $M^{[X]}/G/1$ Queue with Second Optional Service, Bernoulli Vacations and Random Breakdowns

In this model, we consider a queueing model with unreliable server such that server provides first essential service to all arriving customers and after completing the first essential service, second optional service will be provided to some customers those who demand such service. Madan and Baklizi (2002a) and Choudhury and Paul (2005) studied queue with second optional service by considering feed back also. Krishna kumar et al. (2002b) investigated an M/G/1 queue with second optional service and server breakdowns such that breakdowns may encounter with a fixed rate while providing service in either phase and also customer departs the system when server breakdown. Madan, Abu-Dayyeh and Saleh (2002b) studied M/G/1 queue with second a optional service and Bernoulli schedule vacations. Wang (2004) studied both transient and steady behavior of an M/G/1 queueing system with second optional service and server breakdowns based on supplementary variable technique. Thangaraj and Vanitha (2010) studied two stage heterogenous service of M/G/1 queue system with random breakdowns and compulsory vacations. Chakravarthy (2013) discussed multi server queue with second optional service and vacations.

In this model we consider a batch arrival queueing system with a single server, second optional service, Bernoulli schedule server vacations and random breakdowns. We assume general (arbitrary) distributions for the first essential service times, second optional service times, vacation times whereas the breakdown times and repair times are exponentially distributed. The customer whose service is interrupted due to the server breakdown, goes back to the head of the queue. Customers arrive in batches to the system and are served on a first come, first served basis.
The remaining part of the model is structured as follows. In section 4.9, mathematical description of the model is briefly explained and in section 4.10, governing equations of the model are formulated. Section 4.11 gives the transient solution of the system and in section 4.12, the steady state results are found. Some system performance measures are obtained in section 4.13. Particular cases of interest are discussed in 4.14 and in section 4.15, numerical illustrations and graphs are carried out to test validity of analytical results of the model.

### 4.9 Description of the Model

a) Customers arrive at the system in batches of variable size in a compound Poisson process. Let \( \lambda c_i \Delta t (i = 1, 2, 3, \ldots) \) be the first order probability that a batch of \( i \) customers arrives at the system during a short interval of time \((t, t + \Delta t)\), where \( 0 \leq c_i \leq 1 \) and \( \sum_{i=1}^{\infty} c_i = 1 \) and \( \lambda > 0 \) is the mean arrival rate of batches. The customers are served one-by-one on a “first come, first served” basis.

b) The single server provides the ‘first essential service’ to all arriving customers. Let \( B_1(v) \) and \( b_1(v) \) be the distribution function and the density function of the first service times respectively.

c) As soon as the first service of a customer is complete, the customer may opt for the ‘second optional service’ with probability \( r \), in which case his second service will immediately commence or else with probability \( 1-r \) he may opt to leave the system, in which case another customer at the head of the queue (if any) is taken up for his first essential service.

d) The second service time is assumed to be general with the distribution function \( B_2(v) \) and the density function \( b_2(v) \).

e) Let \( \mu_i(x) \) dx be the conditional probability of completion of the \( i^{th} \) service during the interval \((x, x+ dx]\) given that elapsed service time is \( x \), so that

\[
\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2, \quad (4.82)
\]
and therefore,

\[ b_i(v) = \mu_i(v) e^{-\int_0^v \mu_i(x) dx}, \quad i = 1, 2. \] (4.83)

f) As soon as the customer’s service is completed, the server may go for a vacation of random length V with probability \( \theta \) (0 ≤ \( \theta \) ≤ 1) or it may continue to serve the next customer with probability (1 − \( \theta \)).

g) The vacation time follow general (arbitrary) distribution with distribution function \( V(s) \) and the density function \( v(s) \). Let \( \gamma(x)dx \) be the conditional probability of a completion of a vacation during the interval \((x, x + dx]\) given that the elapsed vacation time is \( x \), so that

\[ \gamma(x) = \frac{v(x)}{1 - V(x)}, \] (4.84)

and therefore

\[ v(s) = \gamma(s) e^{-\int_0^s \gamma(x) dx}. \] (4.85)

h) On returning from vacation the server instantly starts serving the customer at the head of the queue, if any.

i) The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate \( \alpha > 0 \).

j) Once the system breaks down, it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate \( \beta > 0 \).

k) Various stochastic processes involved in the system are assumed to be independent of each other.

4.10 Definitions and Equations Governing the System

We let,

\[ P_{n}^{(1)}(x, t) = \text{Probability that at time } 't' \text{ the server is active providing first essential service and there are } 'n' \text{ (} n \geq 0 \text{) customers in the queue excluding the one being } \]
served and the elapsed service time for this customer is \( x \). Consequently, \( P_n^{(1)}(t) \) denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in the first stage service irrespective of the value of \( x \).

\( P_n^{(2)}(x,t) = \) Probability that at time 't' the server is active providing second optional service and there are 'n' \((n \geq 0)\) customers in the queue excluding the one being served and the elapsed service time for this customer is \( x \). Consequently \( P_n^{(2)}(t) \) denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in the second stage service irrespective of the value of \( x \).

\( V_n(x,t) = \) Probability that at time 't', the server is on vacation with elapsed vacation time \( x \), and there are 'n' \((n \geq 0)\) customers waiting in the queue for service. Consequently \( V_n(t) \) denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of \( x \).

\( R_n(t) = \) Probability that at time \( t \), the server is inactive due to breakdown and the system is under repair while there are 'n' \((n \geq 0)\) customers in the queue.

\( Q(t) = \) Probability that at time 't' there are no customers in the system and the server is idle but available in the system.

The queueing model is then, governed by the following set of differential - difference equations:

\[
\frac{\partial}{\partial t} P_n^{(1)}(x,t) + \frac{\partial}{\partial x} P_n^{(1)}(x,t) + (\lambda + \mu_1(x) + \alpha) P_n^{(1)}(x,t) = \lambda \sum_{i=1}^{n-1} c_i P_{n-i}^{(1)}(x,t), \quad n \geq 1
\]  

\[(4.86)\]

\[
\frac{\partial}{\partial t} P_0^{(1)}(x,t) + \frac{\partial}{\partial x} P_0^{(1)}(x,t) + (\lambda + \mu_1(x) + \alpha) P_0^{(1)}(x,t) = 0
\]  

\[(4.87)\]

\[
\frac{\partial}{\partial t} P_n^{(2)}(x,t) + \frac{\partial}{\partial x} P_n^{(2)}(x,t) + (\lambda + \mu_2(x) + \alpha) P_n^{(2)}(x,t) = \lambda \sum_{i=1}^{n-1} c_i P_{n-i}^{(2)}(x,t), \quad n \geq 1
\]  

\[(4.88)\]

\[
\frac{\partial}{\partial t} P_0^{(2)}(x,t) + \frac{\partial}{\partial x} P_0^{(2)}(x,t) + (\lambda + \mu_2(x) + \alpha) P_0^{(2)}(x,t) = 0
\]  

\[(4.89)\]

\[
\frac{\partial}{\partial t} V_n(x,t) + \frac{\partial}{\partial x} V_n(x,t) + (\lambda + \gamma(x)) V_n(x,t) = \lambda \sum_{i=1}^{n-1} c_i V_{n-i}(x,t), \quad n \geq 1
\]  

\[(4.90)\]
\[
\frac{\partial}{\partial t} V_0(x,t) + \frac{\partial}{\partial x} V_0(x,t) + (\lambda + \gamma(x)) V_0(x,t) = 0 \tag{4.91}
\]

\[
\frac{d}{dt} R_n(t) = - (\lambda + \beta) R_n(t) + \lambda \sum_{i=1}^{n-1} c_i R_{n-i}(t) + \alpha \int_0^\infty P_{n-1}^{(1)}(x,t) dx
\]

\[
+ \alpha \int_0^\infty P_{n-1}^{(2)}(x,t) dx, \quad n \geq 1, \tag{4.92}
\]

\[
\frac{d}{dt} R_0(t) = - (\lambda + \beta) R_0(t) \tag{4.93}
\]

\[
\frac{d}{dt} Q(t) = - \lambda Q(t) + \beta R_0(t) + (1 - \theta)(1 - r) \int_0^\infty P_0^{(1)}(x,t) \mu_1(x) dx
\]

\[
+ (1 - \theta) \int_0^\infty P_0^{(2)}(x,t) \mu_2(x) dx + \int_0^\infty V_0(x,t) \gamma(x) dx \tag{4.94}
\]

Equations (4.86) to (4.94) are to be solved subject to the following boundary conditions.

\[
P_0^{(1)}(0, t) = \lambda c_1 Q(t) + \beta R_1(t) + \int_0^\infty V_1(x,t) \gamma(x) dx
\]

\[
+ (1 - \theta)(1 - r) \int_0^\infty P_1^{(1)}(x,t) \mu_1(x) dx + (1 - \theta) \int_0^\infty P_1^{(2)}(x,t) \mu_2(x) dx \tag{4.95}
\]

\[
P_n^{(1)}(0, t) = \int_0^\infty V_{n+1}(x,t) \gamma(x) dx
\]

\[
+ (1 - \theta)(1 - r) \int_0^\infty P_{n+1}^{(1)}(x,t) \mu_1(x) dx + (1 - \theta) \int_0^\infty P_{n+1}^{(2)}(x,t) \mu_2(x) dx
\]

\[
+ \lambda c_{n+1} Q(t) + \beta R_{n+1}(t), \quad n \geq 1 \tag{4.96}
\]

\[
P_n^{(2)}(0, t) = r \int_0^\infty P_n^{(1)}(x,t) \mu_1(x) dx, \quad n \geq 0 \tag{4.97}
\]

\[
V_n(0, t) = (1 - r) \theta \int_0^\infty P_n^{(1)}(x,t) \mu_1(x) dx + \theta \int_0^\infty P_n^{(2)}(x,t) \mu_2(x) dx, \quad n \geq 0 \tag{4.98}
\]

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

\[
P_n^{(j)}(0) = 0, \quad j = 1, 2, \quad n = 0, 1, 2, \ldots, \quad V_0(0) = V_n(0) = 0, \quad Q(0) = 1. \tag{4.99}
\]
4.11 Generating Functions of the Queue Length: The Time Dependent Solution

We define the probability generating function

\[ P_q^{(j)}(x, z, t) = \sum_{n=0}^{\infty} z^n P_n^{(j)}(x, t), \quad j = 1, 2, \]

\[ P_q^{(j)}(x, z, t) = \sum_{n=0}^{\infty} z^n P_n^{(j)}(t), \quad j = 1, 2, \]

\[ P_q^{(j)}(x, z, t) = \sum_{n=0}^{\infty} z^n P_n^{(j)}(t), \quad j = 1, 2, \]

\[ V_q(x, z, t) = \sum_{n=0}^{\infty} z^n V_n(x, t), \quad V_q(z, t) = \sum_{n=0}^{\infty} z^n V_n(t), \]

\[ R_q(z, t) = \sum_{n=0}^{\infty} z^n R_n(t), \]

\[ C(z) = \sum_{n=1}^{\infty} c_n z^n \]

which are convergent inside the circle given by \(|z| \leq 1\) and define the Laplace transform of a function \(f(t)\) as

\[ \tilde{f}(s) = \int_{0}^{\infty} f(t) e^{-st} dt, \, R(s) \geq 0 \]

Taking Laplace transforms of equations (4.86) to (4.98)

\[ \frac{\partial}{\partial x} \tilde{P}_n^{(1)}(x, s) + (s + \lambda + \mu_1(x) + \alpha) \tilde{P}_n^{(1)}(x, s) = \lambda \sum_{i=1}^{n-1} c_i \tilde{P}_{n-i}^{(1)}(x, s), \quad n \geq 1 \]

\[ \frac{\partial}{\partial x} \tilde{P}_0^{(1)}(x, s) + (s + \lambda + \mu_1(x) + \alpha) \tilde{P}_0^{(1)}(x, s) = 0 \]

\[ \frac{\partial}{\partial x} \tilde{P}_n^{(2)}(x, s) + (s + \lambda + \mu_2(x) + \alpha) \tilde{P}_n^{(2)}(x, s) = \lambda \sum_{i=1}^{n-1} c_i \tilde{P}_{n-i}^{(2)}(x, s), \quad n \geq 1 \]

\[ \frac{\partial}{\partial x} \tilde{P}_0^{(2)}(x, s) + (s + \lambda + \mu_2(x) + \alpha) \tilde{P}_0^{(2)}(x, s) = 0 \]

\[ \frac{\partial}{\partial x} \tilde{V}_n(x, s) + (s + \lambda + \gamma(x)) \tilde{V}_n(x, s) = \lambda \sum_{i=1}^{n-1} c_i \tilde{V}_{n-i}(x, s), \quad n \geq 1 \]

\[ \frac{\partial}{\partial x} \tilde{V}_0(x, s) + (s + \lambda + \gamma(x)) \tilde{V}_0(x, s) = 0 \]
(s + \lambda + \beta) \bar{R}_n(s) = \lambda \sum_{i=1}^{n-1} c_i \bar{R}_{n-i}(s) + \alpha \int_{0}^{\infty} \bar{F}_{n-1}^{(1)}(x,s)dx \\
+ \alpha \int_{0}^{\infty} \bar{F}_{n-1}^{(2)}(x,s)dx, \ n \geq 1 \tag{4.111}

(s + \lambda + \beta) \bar{R}_0(s) = 0 \tag{4.112}

(s + \lambda) \bar{Q}(s) = 1 + \beta \bar{R}_0(s) + (1 - r)(1 - \theta) \int_{0}^{\infty} \bar{F}_0^{(1)}(x,s)\mu_1(x)dx \\
+ (1 - \theta) \int_{0}^{\infty} \bar{P}_0^{(2)}(x,s)\mu_2(x)dx + \int_{0}^{\infty} \bar{V}_0(x,s)\gamma(x)dx \tag{4.113}

\bar{P}_0^{(1)}(0, s) = (1 - \theta) \left\{ (1 - r) \int_{0}^{\infty} \bar{F}_1^{(1)}(x,s)\mu_1(x)dx + \int_{0}^{\infty} \bar{F}_1^{(2)}(x,s)\mu_2(x)dx \right\} \\
+ \int_{0}^{\infty} \tilde{V}_1(x,s)\gamma(x)dx + \beta \bar{R}_1(s) + \lambda c_1 \bar{Q}(s) \tag{4.114}

\bar{P}_n^{(1)}(0, s) = (1 - \theta) \left\{ (1 - r) \int_{0}^{\infty} \bar{F}_{n+1}^{(1)}(x,s)\mu_1(x)dx + \int_{0}^{\infty} \bar{F}_{n+1}^{(2)}(x,s)\mu_2(x)dx \right\} \\
+ \int_{0}^{\infty} \tilde{V}_{n+1}(x,s)\gamma(x)dx + \beta \bar{R}_{n+1}(s) + \lambda c_{n+1} \bar{Q}(s) \tag{4.115}

\bar{P}_n^{(2)}(0, s) = r \int_{0}^{\infty} \bar{P}_n^{(1)}(x,s)\mu_1(x)dx, \ n = 0, 1, 2, ... \tag{4.116}

\bar{V}_n(0, s) = (1 - r)\theta \int_{0}^{\infty} \bar{P}_n^{(1)}(x,s)\mu_1(x)dx + \theta \int_{0}^{\infty} \bar{P}_n^{(2)}(x,s)\mu_2(x)dx, \ n = 0, 1, 2, ... \tag{4.117}

Now multiplying equation (4.105) by \( z^n \) and summing over \( n \) from 1 to \( \infty \), adding to equation (4.106) and using the definition of probability generating function, we obtain

\[
\frac{\partial}{\partial x} \bar{P}_q^{(1)}(x, z, s) + (s + \lambda - \lambda C(z) + \mu_1(x) + \alpha)\bar{P}_q^{(1)}(x, z, s) = 0 \tag{4.118}
\]

Performing similar operations on equations (4.107) to (4.112)

\[
\frac{\partial}{\partial x} \bar{P}_q^{(2)}(x, z, s) + (s + \lambda - \lambda C(z) + \mu_2(x) + \alpha)\bar{P}_q^{(2)}(x, z, s) = 0 \tag{4.119}
\]

\[
\frac{\partial}{\partial x} \bar{V}_q(x, z, s) + (s + \lambda - \lambda C(z) + \gamma(x))\bar{V}_q(x, z, s) = 0 \tag{4.120}
\]
\[(s + \lambda - \lambda C(z) + \beta)\bar{R}_q(z, s) = \alpha z \left[ \int_0^\infty \bar{P}_q^{(1)}(x, z, s)dx + \int_0^\infty \bar{P}_q^{(2)}(x, z, s)dx \right] \quad (4.121)\]

For the boundary conditions, multiply both sides of equation (4.114) by \(z\), multiply both sides of equation (4.115) by \(z^{n+1}\), summing over 1 to \(\infty\), adding the two results and using the definition of probability generating function equation, we get

\[
z\bar{P}_q^{(1)}(0, z, s) = (1 - \theta) \left\{ (1 - r) \int_0^\infty \bar{P}_q^{(1)}(x, z, s)\mu_1(x)dx \right. \\
+ \int_0^\infty \bar{P}_q^{(2)}(x, z, s)\mu_2(x)dx \right\} + \beta\bar{R}_q(z, s) \\
+ \int_0^\infty \bar{V}_q(x, z, s)\gamma(x)dx + [1 - s\bar{Q}(s)] + \lambda[C(z) - 1]\bar{Q}(s) \quad (4.122)\]

Performing similar operation on equations (4.116) and (4.117) we obtain

\[
\bar{P}_q^{(2)}(0, z, s) = r \int_0^\infty \bar{P}_q^{(1)}(x, z, s)\mu_1(x)dx \quad (4.123)\]

\[
\bar{V}_q(0, z, s) = (1 - r)\theta \int_0^\infty \bar{P}_q^{(1)}(x, z, s)\mu_1(x)dx + \theta \int_0^\infty \bar{P}_q^{(2)}(x, z, s)\mu_2(x)dx \quad (4.124)\]

Integrating the equation (4.118) from 0 to \(x\) yields

\[
\bar{P}_q^{(1)}(0, z, s) = \bar{P}_q^{(1)}(0, z, s)e^{-(s + \lambda - \lambda C(z) + \alpha)x-f_0^{\infty} \mu_1(t)dt} \quad (4.125)\]

where \(\bar{P}_q^{(1)}(0, z, s)\) is given by equation (4.122). Again integrating equation (4.125) by parts with respect to \(x\) yields

\[
\bar{P}_q^{(1)}(z, s) = \bar{P}_q^{(1)}(0, z, s) \left[ \frac{1 - \bar{B}_1(s + \lambda - \lambda C(z) + \alpha)}{(s + \lambda - \lambda C(z) + \alpha)} \right] \quad (4.126)\]

where

\[
\bar{B}_1(s + \lambda - \lambda C(z) + \alpha) = \int_0^\infty e^{-(s + \lambda - \lambda C(z) + \alpha)x}dB_1(x) \quad (4.127)\]

is Laplace - Stieltjes transform of the first essential service time \(B_1(x)\). Now multiplying both sides of equation (4.125) by \(\mu_1(x)\) and integrating over \(x\), we get

\[
\int_0^\infty \bar{P}_q^{(1)}(x, z, s)\mu_1(x)dx = \bar{P}_q^{(1)}(0, z, s)\bar{B}_1(s + \lambda - \lambda C(z) + \alpha) \quad (4.128)\]
Similarly, on integrating equation (4.119) and (4.120) from 0 to \(x\), we get

\[
P_q^{(2)}(x, z, s) = \bar{P}_q^{(2)}(0, z, s)e^{-(s+\lambda-\lambda C(z)+\alpha)x - \int_0^x \mu_2(t)dt}
\]  

(4.129)

\[
\bar{V}_q(x, z, s) = \bar{V}_q(0, z, s)e^{-(s+\lambda-\lambda C(z))x - \int_0^x \gamma(t)dt}
\]  

(4.130)

where \(\bar{P}_q^{(2)}(0, z, s)\) and \(\bar{V}_q(0, z, s)\) are given by equations (4.123) and (4.124). Again integrating equations (4.129) and (4.130) by parts with respect to \(x\) yields

\[
\bar{P}_q^{(2)}(z, s) = \bar{P}_q^{(2)}(0, z, s) \left[1 - \bar{B}_2(s + \lambda - \lambda C(z) + \alpha)\right]
\]  

(4.131)

where

\[
\bar{B}_2(s + \lambda - \lambda C(z) + \alpha) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}dB_2(x)
\]  

(4.132)

is Laplace - Stieltjes transform of the second optional service time \(B_2(x)\).

\[
\bar{V}_q(z, s) = \bar{V}_q(0, z, s) \left[1 - \bar{V}(s + \lambda - \lambda C(z))\right]
\]  

(4.133)

where

\[
\bar{V}(s + \lambda - \lambda C(z)) = \int_0^\infty e^{-(s+\lambda-\lambda C(z))x}dV(x)
\]  

(4.134)

is Laplace - Stieltjes transform of the vacation time \(V(x)\). Now multiplying both sides of equation (4.129) by \(\mu_2(x)\) and integrating over \(x\), we get

\[
\int_0^\infty \bar{P}_q^{(2)}(x, z, s)\mu_2(x)dx = \bar{P}_q^{(2)}(0, z, s)\bar{B}_2(s + \lambda - \lambda C(z) + \alpha)
\]  

(4.135)

Now using equation (4.128), equation (4.123) reduces to

\[
\bar{P}_q^{(2)}(0, z, s) = r\bar{P}_q^{(1)}(0, z, s)\bar{B}_1(s + \lambda - \lambda C(z) + \alpha)
\]  

(4.136)

Now multiplying both sides of equation (4.130) by \(\gamma(x)\) and integrating over \(x\), we get

\[
\int_0^\infty \bar{V}_q(x, z, s)\gamma(x)dx = \bar{V}_q(0, z, s)\bar{V}(s + \lambda - \lambda C(z))
\]  

(4.137)

Now equation (4.131) becomes

\[
\bar{P}_q^{(2)}(z, s) = r\bar{P}_q^{(1)}(0, z, s) \left[\frac{\bar{B}_1(s + \lambda - \lambda C(z) + \alpha)(1 - \bar{B}_2(s + \lambda - \lambda C(z) + \alpha))}{(s + \lambda - \lambda C(z) + \alpha)}\right]
\]
Now using equations (4.128) and (4.135), equation (4.124) can be written as

\[
\bar{V}_q(0, z, s) = \theta(1 - r) \bar{P}_q^{(1)}(0, z, s) B_1(s + \lambda - \lambda C(z) + \alpha)
+ \theta r \bar{P}_q^{(1)}(0, z, s) B_1(s + \lambda - \lambda C(z) + \alpha) B_2(s + \lambda - \lambda C(z) + \alpha)
\]

(4.139)

Using above equation (4.139), equation (4.133) becomes

\[
\bar{V}_q(z, s) = \bar{P}_q^{(1)}(0, z, s) \theta H(z, s) \left[ 1 - \bar{V}[f_3(z, s)] \right] \]

(4.140)

where

\[
\begin{align*}
    f_1(z, s) &= s + \lambda - \lambda C(z) + \alpha, \\
    f_2(z, s) &= s + \lambda - \lambda C(z) + \beta, \\
    f_3(z, s) &= s + \lambda - \lambda C(z),
\end{align*}
\]

and \( H(z, s) = (1 - r) B_1[f_1(z, s)] + r B_1[f_1(z, s)] B_2[f_1(z, s)] \).

Therefore (4.121) becomes

\[
\bar{R}_q(z, s) = \alpha z \bar{P}_q^{(1)}(0, z, s) \left[ 1 - H(z, s) \right] \left[ \frac{1 - \bar{V}[f_3(z, s)]}{[f_2(z, s)]} \right] \]

(4.141)

Now equation (4.122) becomes

\[
\bar{P}_q^{(1)}(0, z, s) = \frac{f_1(z, s)f_2(z, s)[(1 - s \bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{D(z, s)}
\]

(4.142)

where

\[
D(z, s) = f_1(z, s)f_2(z, s) \{ z - [(1 - \theta) + \theta \bar{V}[f_3(z, s)])H(z, s) \} - \alpha \beta z[1 - H(z, s)]
\]

(4.143)

Substituting the value for \( \bar{P}_q^{(1)}(0, z, s) \) in equations (4.126), (4.138), (4.140) and (4.141), we get

\[
\bar{P}_q^{(1)}(z, s) = \frac{f_2(z, s)[(1 - s \bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{D(z, s)} \left[ 1 - \bar{B}_1[f_1(z, s)] \right]
\]

(4.144)
\[ \bar{P}^{(2)}_q(z, s) = \frac{rf_2(z, s)[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]B_1[f_1(z, s)][1 - \bar{B}_2[f_1(z, s)]]}{D(z, s)} \]  

(4.145)

\[ \bar{V}_q(z, s) = \frac{\theta f_1(z, s)f_2(z, s)[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]H(z, s)}{D(z, s)} \times \left[ 1 - \bar{V}[f_3(z, s)] \right] \]  

(4.146)

\[ \bar{R}_q(z, s) = \frac{\alpha z[1 - H(z, s)][(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{D(z, s)} \]  

(4.147)

where \(D(z,s)\) is given by the equation (4.143). Thus, the Probability generating functions \(\bar{P}^{(1)}_q(z, s), \bar{P}^{(2)}_q(z, s), \bar{V}_q(z, s)\) and \(\bar{R}_q(z, s)\) are completely determined in closed form.

### 4.12 The Steady State Results

In this section we shall derive the steady state probability distribution for the model. To define the steady state probabilities, suppress the argument 't' wherever it appears in the time dependent analysis and this can be obtained by using well known Tauberian property

\[ \lim_{t \to \infty} f(t) = \lim_{s \to 0} \frac{s f(s)}{s} \]  

(4.148)

Multiplying both sides of equation (4.144), (4.145), (4.146) and (4.147) by \(s\) and applying property (4.148) and simplifying, we get

\[ P^{(1)}_q(z) = \frac{[f_2(z)][\lambda(C(z) - 1)Q][1 - \bar{B}_1[f_1(z)]]}{D(z)} \]  

(4.149)

\[ P^{(2)}_q(z) = \frac{rf_2(z)[\lambda(C(z) - 1)Q][\bar{B}_1[f_1(z)]][1 - \bar{B}_2[f_1(z)]]}{D(z)} \]  

(4.150)

\[ V_q(z) = \frac{\theta[f_1(z)][f_2(z)]H(z)Q[\bar{V}[f_3(z)] - 1]}{D(z)} \]  

(4.151)

\[ R_q(z) = \frac{\alpha z[\lambda(C(z) - 1)]Q[1 - H(z)]}{D(z)} \]  

(4.152)

where

\[ D(z) = f_1(z)f_2(z) \left\{ z - [(1 - \theta) + \theta\bar{V}[f_3(z)]H(z)] \right\} - \alpha \beta z[1 - H(z)], \quad (4.153) \]
\[ f_1(z) = \lambda - \lambda C(z) + \alpha, \]
\[ f_2(z) = \lambda - \lambda C(z) + \beta, \]
\[ f_3(z) = \lambda - \lambda C(z), \]

and \( H(z) = (1 - r)\bar{B}_1[f_1(z)] + r\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)] \)

Let \( W_q(z) \) denotes the probability generating function of queue size irrespective of the state of the system. Then adding (4.149), (4.150), (4.151) and (4.152) we get

\[ W_q(z) = P_q^{(1)}(z) + P_q^{(2)}(z) + V_q(z) + R_q(z) \quad (4.154) \]

\[
W_q(z) = f_2(z)[\lambda(C(z) - 1)Q][1 - \bar{B}_1[f_1(z)]] \\
+ r f_2(z)[\lambda(C(z) - 1)Q][\bar{B}_1[f_1(z)][1 - \bar{B}_2[f_1(z)]] \\
+ \theta [f_1(z)][f_2(z)]H(z)Q[\bar{V}^2(\lambda - \lambda C(z)) - 1] \\
+ \alpha z[\lambda(C(z) - 1)Q[1 - H(z)] \\
D(z) \quad (4.155) \]

where \( D(z) \) is given above. In order to obtain \( Q \), using the normalization condition,

\[ W_q(1) + Q = 1 \quad (4.156) \]

We see that for \( z = 1 \), \( W_q(z) \) is indeterminate of the form 0/0. We apply L’Hopital’s rule in equation (4.155), where \( \bar{B}_i(0) = 1, i = 1, 2; \bar{V}(0) = 1, -V'(0) = E[V], \) the mean vacation time. Now

\[ P_q^{(1)}(1) = \frac{\lambda \beta Q[1 - \bar{B}_1(\alpha)]E(I)}{Dr} \quad (4.157) \]
\[ P_q^{(2)}(1) = \frac{r \lambda \beta Q\bar{B}_1(\alpha)[1 - \bar{B}_2(\alpha)]E(I)}{Dr} \quad (4.158) \]
\[ V_q(1) = \theta \frac{\lambda \alpha \beta Q[(1 - r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]E(I)E(V)}{Dr} \quad (4.159) \]
\[ R_q(1) = \frac{\lambda \alpha Q[1 - (1 - r)\bar{B}_1(\alpha) - r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]E(I)}{Dr} \quad (4.160) \]
\[ W_q(1) = \frac{\lambda QE(I) \{(\alpha + \beta) + [\bar{B}(\alpha)(-(\alpha + \beta) + \alpha \beta \theta E(V))]\}}{Dr} \quad (4.161) \]
\[ Dr = \alpha \beta \bar{B} - \lambda E(I)[(\alpha + \beta)] - \lambda \alpha \beta E(I)E(V)\bar{B}(\alpha) \]

where \( \bar{B}(\alpha) = (1 - r)\bar{B_1}(\alpha) + r \bar{B_1}(\alpha)\bar{B_2}(\alpha) \)

\[ Q = 1 - \frac{\lambda}{E(I)} \left[ \frac{1}{\beta B(\alpha)} + \frac{1}{\alpha B(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right] \]  

(4.162)

and the utilization factor \( \rho \) of the system is given by

\[ \rho = \frac{\lambda}{E(I)} \left[ \frac{1}{\beta B(\alpha)} + \frac{1}{\alpha B(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right] \]  

(4.163)

where \( \rho < 1 \) is the stability condition under which the steady state exists, equation (4.162) gives the probability that the server is idle. Substitute \( Q \) from equation (4.162) in equation (4.155), probability generating function of the queue size \( W_q(z) \) has been completely and explicitly determined.

### 4.13 The Average Queue Size and Average Waiting Time

Let \( L_q \) denote the mean number of customers in the queue under the steady state, then

\[ L_q = \frac{d}{dz} W_q(z) \bigg|_{z=1} \]  

(4.164)

and this formula gives 0/0 form, we write

\[ W_q(z) = \frac{N(z)}{D(z)} \]  

(4.165)

where \( N(z) \) and \( D(z) \) are the numerator and denominator of the right hand side of equation (4.165) respectively, then we use

\[ L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2[D'(1)]^2} \]  

(4.166)

where primes and double primes in equation (4.166) denote first and second derivatives at \( z = 1 \) respectively. Carrying out the derivatives at \( z = 1 \), we have

\[ N'(1) = \lambda E(I)Q \left\{ (\alpha + \beta) + \bar{B}(\alpha)[(\theta \alpha \beta E(V) - \alpha - \beta)] \right\} \]  

(4.167)
\[ N''(1) = 2Q[\lambda E(I)]^2 \left\{ \left( \frac{\alpha}{\lambda E(I)} - 1 \right) + \bar{B}(\alpha) \left[ 1 - \frac{\alpha}{\lambda E(I)} - \theta E(V)(\alpha + \beta) + \frac{1}{2} \theta \alpha \beta E(V^2) \right] + \left[ (1-r)\bar{B}'_1(\alpha) + r(\bar{B}'_1(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}'_2(\alpha)) \right] \left[ \alpha + \beta - \theta \alpha \beta E(V) \right] \right\} + \lambda Q E(I(I-1)) \left\{ (\alpha + \beta) + \bar{B}_1(\alpha)(\theta \alpha \beta E(V) - \alpha - \beta) \right\} \] (4.168)

\[ D'(1) = -\lambda E(I)(\alpha + \beta) + \bar{B}(\alpha) \{ \alpha \beta + \lambda E(I)(\alpha + \beta - \theta \alpha \beta E(V)) \} \] (4.169)

\[ D''(1) = 2[\lambda E(I)]^2 \left\{ \left( 1 - \frac{\alpha + \beta}{\lambda E(I)} \right) + \bar{B}(\alpha)[-1 + \theta E(V)(\alpha + \beta) - \frac{1}{2} \alpha \beta E(V^2)] + \left[ (1-r)\bar{B}'_1(\alpha) + r(\bar{B}'_1(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}'_2(\alpha)) \right] \left[ -(\alpha + \beta) - \frac{\alpha \beta}{\lambda E(I)} + \alpha \beta \theta E(V) \right] \right\} + \lambda E(I(I-1)) \left\{ -(\alpha + \beta) + \bar{B}(\alpha)[(\alpha + \beta - \theta \alpha \beta E(V))] \right\} \] (4.170)

where \( E(V^2) \) is the second moment of the vacation time and \( Q \) has been found in equation (4.162). Then if we substitute the values of \( N'(1), N''(1), D'(1) \) and \( D''(1) \) from equations (4.167), (4.168), (4.169) and (4.170) in to equation (4.166) we obtain \( L_q \) in a closed form.

Mean waiting time of a customer could be found as

\[ W_q = \frac{L_q}{\lambda} \] (4.171)

Also other performance measures such as average number of customer in the system

\[ L_s = L_q + \rho \]

and average waiting time of a customer in the system

\[ W_s = \frac{L_s}{\lambda} \]

can be obtained by using Little’s formula.
4.14 Particular Cases

4.14.1 No Second Optional Service

In this case, we assume that all customers want only the first essential service and they do not demand the second optional service. Accordingly, we set \( r = 0 \) and \( P_q^{(2)}(z) = 0 \) in the above main results of this model and we obtain

\[
W_q(z) = \frac{f_2(z)[\lambda(C(z) - 1)Q][1 - \bar{B}_1[f_1(z)]]}{D(z)}
+ \frac{\theta[f_1(z)][f_2(z)]\bar{B}_1[f_1(z)]Q[\bar{V}f_3(z)] - 1}{D(z)}
+ \frac{\alpha z[\lambda(C(z) - 1)]Q[1 - \bar{B}_1[f_1(z)]]}{D(z)}
\]

(4.172)

where

\[
D(z) = f_1(z)f_2(z) \{ z - [(1 - \theta) + \theta \bar{V}[f_3(z)]\bar{B}_1[f_1(z)]] \\
- \alpha \beta z \{ 1 - \bar{B}_1[f_1(z)] \} \}.
\]

(4.173)

\[
Q = 1 - \frac{\lambda}{E(I)} \left[ \frac{1}{\beta[B_1(\alpha)]} + \frac{1}{\alpha[B_1(\alpha)]} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right]
\]

(4.174)

\[
\rho = \frac{\lambda}{E(I)} \left[ \frac{1}{\beta[B_1(\alpha)]} + \frac{1}{\alpha[B_1(\alpha)]} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right]
\]

(4.175)

\[
N'(1) = \lambda E(I)Q \{ (\alpha + \beta) + (\bar{B}_1(\alpha)) \} \{ (\theta \alpha \beta E(V) - \alpha - \beta) \}
\]

(4.176)

\[
N''(1) = 2Q[\lambda E(I)]^2 \left\{ \left( \frac{\alpha}{\lambda E(I)} - 1 \right) \\
+ (\bar{B}_1(\alpha)) \left[ 1 - \frac{\alpha}{\lambda E(I)} - \theta \alpha E(V) - \theta \beta E(V) + \frac{1}{2} \theta \alpha \beta E(V^2) \right] \\
+ \bar{B}_1(\alpha)[\alpha + \beta - \theta \alpha \beta E(V)] \} \\
+ \lambda Q E(I)(I - 1) \{ (\alpha + \beta) + (\bar{B}_1(\alpha)) \} \{ (\theta \alpha \beta E(V) - \alpha - \beta) \}
\]

(4.177)

\[
D'(1) = -\lambda E(I)(\alpha + \beta) + (\bar{B}_1(\alpha)) \{ \alpha \beta + \lambda E(I)(\alpha + \beta - \theta \alpha \beta E(V)) \}
\]

(4.178)
4.14.2 Both services are essential

When all customers demand both services that is $r = 1$, then from the main results of this model, we get

\[
D''(1) = 2[\lambda E(I)]^2 \left\{ \left(1 - \frac{\alpha + \beta}{\lambda E(I)}\right) \right.
\]
\[
+ \left(\bar{B}_1(\alpha)\right) \left[-1 + \theta \alpha E(V) + \theta \beta E(V) - \frac{1}{2} \alpha \beta \theta E(V^2)\right]
\]
\[
+ \left(\bar{B}_1'(\alpha)\right) \left[-(\alpha + \beta) - \frac{\alpha \beta}{\lambda E(I)} + \alpha \beta \theta E(V)\right]
\]
\[
+ \lambda E(I(I - 1)) \left\{[-(\alpha + \beta) + (\bar{B}_1(\alpha))(\alpha + \beta - \theta \alpha \beta E(V))\right]\} \tag{4.179}
\]

4.14.2 Both services are essential

When all customers demand both services that is $r = 1$, then from the main results of this model, we get

\[
W_d(z) = \frac{f_2(z)[\lambda(C(z) - 1)Q][1 - \bar{B}_1[f_1(z)]]}{D(z)}
\]
\[
+ \frac{f_2(z)[\lambda(C(z) - 1)Q][\bar{B}_1[f_1(z)][1 - \bar{B}_2[f_1(z)]]]}{D(z)}
\]
\[
+ \frac{\theta[f_1(z)][f_2(z)][\bar{B}_1[f_1(z)][\bar{B}_2[f_1(z)]]Q[\bar{V}[f_3(z)] - 1]}{D(z)}
\]
\[
+ \frac{\alpha \bar{z}[\lambda(C(z) - 1)]Q[1 - \bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]]}{D(z)} \tag{4.180}
\]

\[
D(z) = f_1(z)f_2(z) \left\{z - [(1 - \theta) + \theta \bar{V}[f_3(z)]\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]]\right\}
\]
\[
- \alpha \beta z \left\{1 - \bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]\right\} . \tag{4.181}
\]

\[
Q = 1 - \frac{\lambda}{E(I)} \left[\frac{1}{\beta[B_1(\alpha)B_2(\alpha)]} + \frac{1}{\alpha[B_1(\alpha)B_2(\alpha)]} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V)\right] \tag{4.182}
\]
and the utilization factor $\rho$ of the system is given by

\[
\rho = \frac{\lambda}{E(I)} \left[\frac{1}{\beta[B_1(\alpha)B_2(\alpha)]} + \frac{1}{\alpha[B_1(\alpha)B_2(\alpha)]} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V)\right] \tag{4.183}
\]

\[
N'(1) = \lambda E(I)Q \left\{(\alpha + \beta) + (\bar{B}_1(\alpha)\bar{B}_2(\alpha))[(\theta \alpha \beta E(V) - \alpha - \beta)]\right\} \tag{4.184}
\]

\[
N''(1) = 2Q[\lambda E(I)]^2 \left\{\left(\frac{\alpha}{\lambda E(I)} - 1\right)\right.
\]
\[
+ \left(\bar{B}_1(\alpha)\bar{B}_2(\alpha)\right) \left[1 - \frac{\alpha}{\lambda E(I)} - \theta E(V)(\alpha + \beta) + \frac{1}{2} \alpha \beta \theta E(V^2)\right]
\]
\[
+ \left((\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha))\right) \left[\alpha + \beta - \theta \alpha \beta E(V)\right]
\]
\[
+ \lambda Q E(I(I - 1)) \left\{\left(\alpha + \beta + (\bar{B}_1(\alpha)\bar{B}_2(\alpha)) \theta \alpha \beta E(V) - \alpha - \beta\right)\right\} \tag{4.185}
\]
\[ D'(1) = -\lambda E(I)(\alpha + \beta) + \left( \bar{B}_1(\alpha) \bar{B}_2(\alpha) \right) \{ \alpha \beta + \lambda E(I)(\alpha + \beta - \theta \alpha \beta E(V)) \} \]

\[ D''(1) = 2[\lambda E(I)]^2 \left\{ \left( 1 - \frac{\alpha + \beta}{\lambda E(I)} \right) \right. \]
\[ \left. + \left( \bar{B}_1(\alpha) \bar{B}_2(\alpha) \right) \left[ -1 + \theta \alpha E(V) + \theta \beta E(V) - \frac{1}{2} \alpha \beta E(V^2) \right] \right. \]
\[ \left. + \left( \bar{B}_1(\alpha) \bar{B}_2(\alpha) \right) \left[ -(\alpha + \beta) - \frac{\alpha \beta}{\lambda E(I)} + \alpha \beta E(V) \right] \right\} \]
\[ + \lambda E(I)(I - 1) \left\{ -(\alpha + \beta) + \left( \bar{B}_1(\alpha) \bar{B}_2(\alpha) \right) (\alpha + \beta - \theta \alpha \beta E(V)) \right\} \]

\[ (4.186) \]

4.14.3 Single Poisson Arrivals, Both Stage Services are Essential, Compulsory vacation

If we assume that customers arrive singly, both stage services are provided to all customers and after the completion of each second stage service, the server will take compulsory vacation, then we have \( c_1 = 1, c_i = 0, \) if \( i \neq 0, C(z) = z, E(I) = 1, E(I(I - 1)) = 0, r = 1 \) and \( \theta = 1. \)

\[ W_q(z) = \frac{f_2(z)[\lambda(z - 1)Q][1 - \bar{B}_1[f_1(z)]]}{D(z)} \]
\[ + \frac{f_2(z)[\lambda(z - 1)Q][\bar{B}_1[f_1(z)]][1 - \bar{B}_2[f_1(z)]]}{D(z)} \]
\[ + \frac{f_1(z)f_2(z)[\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]][\bar{V}[f_3(z)] - 1]}{D(z)} \]
\[ + \frac{\alpha z[\lambda(z - 1)Q][1 - \bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]]}{D(z)} \]

\[ (4.188) \]

\[ D(z) = f_1(z)f_2(z) \left\{ z - [\bar{V}[f_3(z)]\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]] - \alpha \beta z \{ 1 - \bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)] \} \right\}. \]

\[ (4.189) \]

\[ Q = 1 - \lambda \left[ \frac{1}{\beta[B_1(\alpha)B_2(\alpha)]} + \frac{1}{\alpha[B_1(\alpha)B_2(\alpha)]} - \frac{1}{\beta} - \frac{1}{\alpha} + E(V) \right] \]

\[ (4.190) \]

\[ \rho = \lambda \left[ \frac{1}{\beta[B_1(\alpha)B_2(\alpha)]} + \frac{1}{\alpha[B_1(\alpha)B_2(\alpha)]} - \frac{1}{\beta} - \frac{1}{\alpha} + E(V) \right] \]

\[ (4.191) \]

\[ N'(1) = \lambda Q \left\{ (\alpha + \beta) + \left( \bar{B}_1(\alpha) \bar{B}_2(\alpha) \right) [(\alpha \beta E(V) - \alpha - \beta)] \right\} \]

\[ (4.192) \]
\[ N''(1) = 2Q(\lambda)^2 \left\{ \left( \frac{\alpha}{\lambda} - 1 \right) \right. \]
\[ + \left( \bar{B}_1(\alpha)\bar{B}_2(\alpha) \right) \left[ 1 - \frac{\alpha}{\lambda} - \alpha E(V) - \beta E(V) + \frac{1}{2} \alpha\beta E(V^2) \right] \]
\[ + \left( \left( \bar{B}_1(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2(\alpha) \right) \left\{ \lambda + \alpha - \alpha\beta E(V) \right\} \right) \right\} \] (4.193)

\[ D'(1) = -\lambda(\alpha + \beta) + \left( \bar{B}_1(\alpha)\bar{B}_2(\alpha) \right) \left\{ \alpha\beta + \lambda(\alpha + \beta - \alpha\beta E(V)) \right\} \] (4.194)

\[ D''(1) = 2\lambda^2 \left\{ \left( 1 - \frac{\alpha + \beta}{\lambda} \right) \right. \]
\[ + \left( \bar{B}_1(\alpha)\bar{B}_2(\alpha) \right) \left[ -1 + \alpha E(V) + \beta E(V) - \frac{1}{2} \alpha\beta E(V^2) \right] \]
\[ + \left[ \bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha) \right] \left[ -\alpha + \frac{\alpha\beta}{\lambda} + \alpha\beta E(V) \right] \right\} \] (4.195)

The results obtained in this case agree with the results discussed by Thangaraj and Vanitha (2010).

### 4.14.4 No Server Breakdowns

In this case, we assume that server does not meet any breakdown, hence \( \alpha = 0 \) and \( R_q(z) = 0 \). By using these assumptions in the main results of this model, we have

\[ W_q(z) = \frac{Q[1 - \bar{B}_1[f(z)]]}{\{ z - [(1 - \theta) + \theta V[f(z)]][1 - r]\bar{B}_1[f(z)] + r\bar{B}_1[f(z)]\bar{B}_2[f(z)] \}} \]
\[ + r \frac{Q[\bar{B}_1[f(z)][1 - \bar{B}_2[f(z)]]}{\{ z - [(1 - \theta) + \theta V[f(z)][1 - r]\bar{B}_1[f(z)] + r\bar{B}_1[f(z)]\bar{B}_2[f(z)] \}} \]
\[ + \frac{\theta[(1 - r)\bar{B}_1[f(z)] + r\bar{B}_1[f(z)]\bar{B}_2[f(z)]][1 - \lambda C(z) - 1]}{\{ z - [(1 - \theta) + \theta V[f(z)][1 - r]\bar{B}_1[f(z)] + r\bar{B}_1[f(z)]\bar{B}_2[f(z)] \}} \] (4.196)

\[ Q = 1 - \{ \lambda E(I)Q[E(S_1) + rE(S_2) + \theta E(V)] \} \] (4.197)

\[ N'(1) = \lambda E(I)Q[E(S_1) + rE(S_2) + \theta E(V)] \] (4.198)
\[ N''(1) = Q \left\{ [\lambda E(I)]^2 [E(S_1^2) + 2rE(S_1)E(S_2) + rE(S_2^2) + \theta E(V^2)] \\
+ 2[\lambda E(I)]^2 \theta E(V)[E(S_1) + rE(S_2)] \\
+ E(I(I - 1))[E(S_1) + rE(S_2) + \theta E(V)] \right\} \]
\[ D'(1) = 1 - \lambda E(I)[E(S_1) + rE(S_2) + \theta E(V)] \]
\[ D''(1) = [\lambda E(I)]^2 \left\{ E(S_1^2) + 2rE(S_1)E(S_2) + rE(S_2^2) + \theta E(V^2) \\
+ 2\theta E(V)[E(S_1) + rE(S_2)] + E(I(I - 1))[E(S_1) + rE(S_2) + \theta E(V)] \right\} \]

where \(-B_1'(0) = E(S_1)\) is the mean of the first essential service time, \(B_2'(0) = E(S_2)\) is the mean of the second optional service time, \(-V'(0) = E(V)\) is the mean vacation time, \(B_1''(0) = E(S_1^2)\) is the second moment of the first essential service time, \(B_2''(0) = E(S_2^2)\) is the second moment of the second optional service time and \(V''(0) = E(V^2)\) is the second moment of the vacation time. Using above equations we can easily find \(L_q\) and \(W_q\).

If the server has no option of taking vacations, that is \(\theta = 0\), \(V_q(z) = 0\) and arrivals are single, the model will be reduced to the one considered by Al-Jaraha and Madan (2003) and we get similar results.

### 4.15 Numerical Illustration

For the numerical illustration purpose, we use the results which have been obtained in this model. We assume that the essential service times, optional service times, vacation times and repair times are all exponential such that \(\alpha = 2\), \(\mu_1 = 6\), \(\mu_2 = 8\), \(\nu = 7\), \(\beta = 8 r = 0.5\), while both \(\lambda\) takes the values 0.2, 0.4, 0.6 and 0.8 while \(\theta\) takes 0.25, 0.5, 0.75 and 1. Also we assume single arrivals by taking \(C(z) = z\), \(E(I) = 1\), \(E(I(I - 1)) = 0\). All the values of parameters of the system were chosen so that the steady state condition is satisfied. The table 4.2 gives the computed values for the proportion of idle time, the utilization factor, the mean queue size, mean number of customers in the system, the mean waiting time in
the queue and the mean waiting time in the system. Table 4.2 shows that as long
as increasing the $\lambda$ and $\theta$, the proportion of server's idle time decreases while the
utilization factor, the average system size and average waiting time of queue are
all increase.

Table 4.2: Some queue performance measures values computed
when $\alpha = 2$, $\mu_1 = 6$, $\mu_2 = 8$, $\nu = 7$, $\beta = 8$ $r = 0.5$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\theta$</th>
<th>$Q$</th>
<th>$\rho$</th>
<th>$L_q$</th>
<th>$L_s$</th>
<th>$W_q$</th>
<th>$W_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.25</td>
<td>0.8237</td>
<td>0.1763</td>
<td>0.0596</td>
<td>0.2359</td>
<td>0.2982</td>
<td>1.1795</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>0.8166</td>
<td>0.1834</td>
<td>0.0626</td>
<td>0.246</td>
<td>0.3131</td>
<td>1.2301</td>
</tr>
<tr>
<td>0.2</td>
<td>0.75</td>
<td>0.8095</td>
<td>0.1905</td>
<td>0.0656</td>
<td>0.2562</td>
<td>0.3282</td>
<td>1.2809</td>
</tr>
<tr>
<td>0.2</td>
<td>1</td>
<td>0.8023</td>
<td>0.1977</td>
<td>0.0687</td>
<td>0.2664</td>
<td>0.3436</td>
<td>1.332</td>
</tr>
<tr>
<td>0.4</td>
<td>0.25</td>
<td>0.6475</td>
<td>0.3525</td>
<td>0.202</td>
<td>0.5545</td>
<td>0.505</td>
<td>1.3863</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5</td>
<td>0.6332</td>
<td>0.3668</td>
<td>0.2191</td>
<td>0.5859</td>
<td>0.5477</td>
<td>1.4647</td>
</tr>
<tr>
<td>0.4</td>
<td>0.75</td>
<td>0.6189</td>
<td>0.3811</td>
<td>0.2369</td>
<td>0.618</td>
<td>0.5922</td>
<td>1.5449</td>
</tr>
<tr>
<td>0.4</td>
<td>1</td>
<td>0.6046</td>
<td>0.3954</td>
<td>0.2555</td>
<td>0.6509</td>
<td>0.6388</td>
<td>1.6273</td>
</tr>
<tr>
<td>0.6</td>
<td>0.25</td>
<td>0.4712</td>
<td>0.5288</td>
<td>0.5261</td>
<td>1.0549</td>
<td>0.8768</td>
<td>1.7581</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5</td>
<td>0.4498</td>
<td>0.5502</td>
<td>0.5903</td>
<td>1.1405</td>
<td>0.9838</td>
<td>1.9008</td>
</tr>
<tr>
<td>0.6</td>
<td>0.75</td>
<td>0.4284</td>
<td>0.5716</td>
<td>0.6608</td>
<td>1.2324</td>
<td>1.1013</td>
<td>2.054</td>
</tr>
<tr>
<td>0.6</td>
<td>1</td>
<td>0.4069</td>
<td>0.5931</td>
<td>0.7386</td>
<td>1.3316</td>
<td>1.2309</td>
<td>2.2194</td>
</tr>
<tr>
<td>0.8</td>
<td>0.25</td>
<td>0.295</td>
<td>0.705</td>
<td>1.3675</td>
<td>2.0726</td>
<td>1.7094</td>
<td>2.5907</td>
</tr>
<tr>
<td>0.8</td>
<td>0.5</td>
<td>0.2664</td>
<td>0.7336</td>
<td>1.6303</td>
<td>2.3639</td>
<td>2.0379</td>
<td>2.9549</td>
</tr>
<tr>
<td>0.8</td>
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<td>0.2378</td>
<td>0.7622</td>
<td>1.9556</td>
<td>2.7177</td>
<td>2.4445</td>
<td>3.3972</td>
</tr>
<tr>
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<td>1</td>
<td>0.2093</td>
<td>0.7907</td>
<td>2.369</td>
<td>3.1598</td>
<td>2.9613</td>
<td>3.9497</td>
</tr>
</tbody>
</table>
Graph 4.3 shows the effect of vacation rate $\theta$ on the proportion of idle time of the server and on the mean number of customers in the queue respectively, where we consider that $\lambda=0.6$. Graph 4.4 shows the effect of breakdown rate $\lambda$ on mean queue size and mean waiting time in the queue respectively, where we consider that $\theta=0.75$.

Fig 4.3: Proportion of idle time of server and $L_q$ versus $\theta$

Fig 4.4: $L_q$ and $W_q$ versus $\lambda$