Chapter 8

Wavelet Based Lifting Scheme for the Numerical Solution of EHL Line and Point Contact Problems

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8.1 Introduction

In chapter 7, we introduced a biorthogonal wavelet based numerical method for the solution of EHL line as well as point contact problems. In this chapter 8 developed an efficient technique i.e. wavelet based lifting scheme for the numerical solution of EHL line as well as point contact problems.

Recent years, the studies of nonlinear problems have attracted the attention of many mathematicians, engineers and physicist, since most of the physical models are inherently nonlinear in nature. The EHL model is the most important topic in fluid dynamics. The model is a form of fluid film lubrication where the elastic deformation of the contacting surface, under heavy load, plays the overriding role. Most of the common types of such bearings are contacting surfaces with low geometric conformity where load is concentrated in a small region. The deformation of the bearing surface results in changing the geometry of the lubricating film which is coupled with the variations in pressure developed. Noticeable features of EHL are presented by Dowson and Higginson (1996). The EHL model consists of Reynolds equation, film thickness equation, force balance equation with applied load, pressure dependent density and viscosity of the lubricant. These equations are considered together with boundary conditions. Usually, for the solution of EHL problems, many researchers had used finite difference, finite element and other methods. The pioneering work of Dowson, Higginson and their associates, Lubrecht, Venner and their associates and also many researchers on EHL problem has exciting history. The appearance of sharp pressure peak, the maximum pressure (the Petruservish spike (1951)), near the outlet and corresponding dip in film thickness is of special interest in EHL due to its intense impact on lubrication of bearings. The review by Lught and Moraes-Espejel (2011) presents current activities, listing various methods used in the analysis, and predicts the useful future developments in EHL. The governing equations of EHL line and point contact problem are analyzed by discretizing the equations using finite difference scheme. The resulting nonlinear system of equations is solved using the iterative schemes. As the Jacobian matrix is full in EHL problems, the convergence of the iteration is not guaranteed and takes large number of iterations to converge.

The ill-conditioned systems are arising in the solution of nonlinear system of equations. The suitable remedy is FAS for such systems. The full-approximation
scheme (FAS) is largely applicable in increasing the efficiency of the iterative methods used to solve nonlinear system of equations. In the history of numerical analysis, the development of effective error minimization techniques for nonlinear system of equations has been a significant research topic in the field of science and engineering. Recently, it is renowned that FAS iterative solvers are extremely efficient for nonlinear equations. For a detailed implementation of FAS is given in Briggs et al. (2000). An introduction of FAS is available in Trottenberg et al. (2001). Many authors used FAS to some of the differential equations. For example, Venner and Lubrecht (2000), Zargari et al. (2007) and others have significant contributions in elasto-hydrodynamic lubrication (EHL) problems using FAS. In EHL model, matrices arising from system are dense with non-smooth diagonal and smooth away from the diagonal. This smoothness of the matrix transforms into smallness using wavelet transform and it leads to design the effective wavelet based lifting techniques.

Wavelet theory is recently new and an emerging filed in the mathematics and engineering science. It has been applied in a wide range of engineering disciplines; particularly, wavelets are successfully used in signal analysis, time-frequency analysis and fast algorithms for easy implementation. Wavelet based numerical methods are used for solving the system of equations with better convergence in less computational cost. A collection of the discrete wavelet transforms (DWT) and the FAS were introduced recently in (Avudainayagam and Vani (2004), Bujurke et al. (2007) and (2006), Shiralashetti and Kantli (2016) and Shiralashetti et al. (2016)). The wavelet based full approximation scheme (WFAS) has exposed to be a very efficient and favorable method for numerous problems related to computational science and engineering fields Pereira et al. (2006). Similarly, the biorthogonal wavelet based full approximation scheme (BWFAS) is applied for the solution of EHL problems Shiralashetti et al. (2016). The method can be either used as an iterative solver or as a preconditioning technique, offering in many cases a better performance than some of the most innovative and existing FAS algorithms. Due to the efficiency and potentiality of WFAS, researches further have been carried out for its enrichment. In order to realize this task, work build that is orthogonal/biorthogonal discrete wavelet transform using lifting scheme Pereira and Nabeta (2010). Wavelet based lifting technique permits some improvements on the properties of existing wavelet transforms. The technique has some numerical benefits as a reduced number of
operations which are fundamental in the context of the iterative solvers. More details of the advantages as well as other important structural advantages of the lifting technique can be available in (Pereira and Nабета (2010) and Jensen and Cour-Harro (2001)). In addition to this, the present paper illustrates that the application of the lifting technique to the real world problems.

8.2 Wavelet filter coefficients

Orthogonal and biorthogonal wavelets system coefficients based on orthogonality and smoothness conditions that must be satisfied by scaling and wavelet functions. These conditions, in turn, impose restrictions on the value of filter coefficients through dilatation equations. Fortunately we have two distinct functions called scaling functions and wavelet functions with coefficients \( \{h_k\} \) and \( \{g_k\} \) that define the refinement relation. These coefficients decide shape of the scaling and wavelet functions and act as signal filters, in which the application where we can use the particular wavelet. The enormous amount of literature is available on filter design for specific application but it is isolated to a regular reader since practically all methods use frequency domain as well as complex analysis concepts to arrive at the filter.
The most important classes of filters are those of Finite Impulse Response (FIR). The main characteristic of these filters is the convenient time-localization properties. These filters are initiated from wavelets with compact support and are such that,

\[
h_n = 0 \quad \text{for} \quad n < 0 \quad \text{and} \quad n > L
\]

in which \( L \) is the length of the filter.

Minimal requirements for these compact FIR filters are:

1. The length of the scaling filter \( h_n \) must be even.

2. \( \sum_n h_n = \sqrt{2} \) (satisfy the condition sum of \( h_n \) is equal to \( \sqrt{2} \))

3. \( \sum_n (h_n \cdot h_{n-2k}) = \delta(k) \) (satisfy the condition sum of \( h_n \cdot h_{n-2k} \) is equal to \( \delta(k) \))

in which \( \delta(k) \) is the Kronecker delta, such that is equal to 1 when \( k = 0 \) or 0 when \( k = 1 \).
8.2.1 Haar wavelet filter coefficients

We know that low pass filter coefficients \( h = [h_0, h_1]^T = \left[ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right]^T \) and high pass filter coefficients \( g = [g_0, g_1]^T = \left[ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right]^T \) play an important role in decomposition. Thus it is natural to wonder that it possible to model the decomposition in terms of linear transformations (matrices). Moreover, since digital signals and images are composed of discrete data, we need a discrete analog of the decomposition algorithm so that we can process signal and image data.

8.2.2 Daubechies wavelet filter coefficients

Daubechies introduced scaling functions satisfying the above requirements and distinguished by having the shortest possible support. The scaling function \( \phi_L \) has support \([0, L-1]\), while the corresponding wavelet \( \psi_L \) has support in the interval \([1-L/2, L/2]\). We have filter coefficients \( h = [h_0, h_1, h_2, h_3]^T = \left[ \frac{1+\sqrt{3}}{4\sqrt{2}}, \frac{3+\sqrt{3}}{4\sqrt{2}}, \frac{3-\sqrt{3}}{4\sqrt{2}}, \frac{1-\sqrt{3}}{4\sqrt{2}} \right]^T \) are low pass filter coefficients and \( g = [g_0, g_1, g_2, g_3]^T = \left[ \frac{1-\sqrt{3}}{4\sqrt{2}}, \frac{3-\sqrt{3}}{4\sqrt{2}}, \frac{3+\sqrt{3}}{4\sqrt{2}}, \frac{1+\sqrt{3}}{4\sqrt{2}} \right]^T \) are the high pass filter coefficients.

8.2.3 Biorthogonal (CDF (2, 2)) wavelets

In many filtering applications, we need filter coefficients having symmetry to get a better accuracy. None of the orthogonal wavelet systems except Haar are having symmetrical coefficients. But Haar is too insufficient for many applications in science and engineering. Biorthogonal wavelet system can be constructed to have this feature. This is the motivation for designing such wavelet systems. The following are the biorthogonal (CDF(2, 2)) wavelet filter coefficients, Soman and Ramachandran (2005) and Ruch and Fleet (2009) are,

low pass filters: \( h = [h_0, h_1, h_2] = \left[ \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \right] \) and
\[ \tilde{h} = \left[ \tilde{h}_0, \tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_4 \right] = \left[ \frac{-\sqrt{2}}{8}, \frac{\sqrt{2}}{4}, \frac{3\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{-\sqrt{2}}{8} \right]. \]

Similarly, high pass filters: \( g_k = (-1)^k \tilde{h}_{2^{-k}} \) and \( \tilde{g}_k = (-1)^{k+1} h_{2^{-k}} \).

### 8.3 Preliminaries of lifting scheme

The wavelet transform customs averages and differences, brings us to the definition of the lifting procedure. The operations, average and difference, can be observed as distinct cases of more general operations. If two data are almost equal the difference is, of course small and it is therefore obvious to think of the first data as a prediction of the second one. It is a good prediction, if the difference is small. We also calculated the average of the two data. This can be viewed in two cases. Either as an operation, this preserves some properties of the original data, or as an extraction of an essential property of the data. The concluding viewpoint is based on the fact that the pair-wise average values containing the overall structure of the data, but with only half the number of data. The lifting procedure has three steps i.e. split, prediction and update.

**Split:** The give data are split into the even and odd entries. It is important to observe that we do this only to explain the functionality in the algorithm.

**Prediction:** The given value at the data \(2n\), we predict that the value at data \(2n+1\) is the same. Then we replace the value at \(2n+1\) with the correction to the prediction, which is the difference. Generally, the idea is to have a prediction procedure \(P\) then compute

\[ d = \text{odd} - P(\text{even}). \]

In the data \(d\), each entry is one odd data minus some prediction on an even data.

**Update:** Given an even entry we have predicted that the next odd entry has same value, and stored the difference. Then we update even entry. In general we decide an updating procedure, and then compute

\[ s = \text{even} + U(d). \]

The algorithm described here is called one step lifting. It requires the choice of a prediction \(P\) and an update \(U\). The discrete wavelet transformed is obtained by combining a number of lifting steps.
Now we look at lifting technique in general. First let us look at how we can invert the lifting procedure. It is observed by just reverse in the arrows and changing the sings. Thus the direct transform

\[ d_j = \text{odd}_j - P(\text{even}_j) \]
\[ s_j = \text{even}_j + U(d_j) \]

is inverted by the steps

\[ \text{even}_j = s_j - U(d_j) \]
\[ \text{odd}_j = d_j + P(\text{even}_j) \]

The last step, where the sequences \( \text{even}_j \) and \( \text{odd}_j \) are merged to the form of sequence \( s \) and is given to explain in the algorithm. It is observed that the generalization is crucial in applications. There are many important transforms, where the above steps do not occur in pairs. Furthermore, in last we add a new type of operation which is called normalization or sometimes rescaling, Jensen and Cour-Harbo (2001). The detailed algorithm using different wavelets is give in the next section.

### 8.4 Wavelet based lifting scheme

Consider the Reynolds equation of the form,

\[
\frac{\partial^2 P}{\partial x^2} = F(x, P, \frac{\partial P}{\partial x}, H, \eta, \rho, x_c)
\]

(8.1)

where \( P \) -pressure, \( H \) -film thickness, \( \eta \) - viscosity, \( \rho \) -density and \( x_c \) -cavitation

with

\[ P(a) = P(b) = \frac{\partial P(x_c)}{\partial x} = 0 \]

(8.2)

Taking the initial values of Hertzian pressure \( P(x) = \begin{cases} \sqrt{1-x^2} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases} \), \( x_c = 0 \) and

\[ H_0 = -\frac{1}{4} - \frac{1}{2} \log 2 \], then compute \( H \) from the film thickness equation.

By applying the finite difference scheme to Eqn. (8.1), which gives the system of nonlinear algebraic equations,

\[
A(P_j) = f_j, \quad 1 \leq j \leq N
\]

(8.3)

where \( N = 2^J \), \( N \) is the number of grid points and \( J \) is the level of resolution.
Solve Eqn. (8.3) using Gauss Seidel (GS) method with satisfying force balance equation, we get approximate solution $\tilde{P}_j$. Approximate solution containing some error, therefore required solution equals to sum of approximate solution and error. There are many methods to minimize such error to get the accurate solution. Some of them are FAS, WFAS, BWFAS etc. Now we are using the advanced technique based on different wavelets called as lifting scheme. Recently, lifting schemes are useful in the signal analysis and image processing in the area of science and engineering. But nowadays it extends to approximations in the numerical analysis Pereira et al. (2006). Here, we are discussing the algorithm, Jensen and Cour-Harbo (2001) of the lifting schemes as follows.

### 8.4.1 Lifting scheme via Haar wavelet (HWLS)

The representation of Haar wavelet via lifting form presented as;

**Decomposition:**

Consider approximate solution $S = \tilde{P}_j$ like as signal, then apply the HWLS decomposition (finer to coarser) procedure as,

\[
d_i = S(2j) - S(2j-1),
\]

\[
s_i = S(2j-1) + \frac{1}{2}d_i,
\]

\[
S_1 = \sqrt{2} s_i \quad \text{and} \quad D = \frac{1}{\sqrt{2}} d_i.
\]

In this stage finally, we get new approximation as,

\[
S = [S_1, \quad D]
\]

**Reconstruction:**

Consider Eqn. (8.5), then apply the HWLS reconstruction (coarser to finer) procedure as,

\[
d_i = \sqrt{2} D_i,
\]

\[
s_i = \frac{1}{\sqrt{2}} S_i,
\]

\[
S(2j-1) = s_i - \frac{1}{2}d_i \quad \text{and} \quad S(2j) = d_i + S(2j-1)
\]

which is the required solution of the given equation.
8.4.2 Lifting scheme via Daubechies wavelet (DWLS)

As discussed in the previous section 8.4.1, we follow the same procedure but we used different wavelet i.e., Daubechies 4th order wavelet coefficient. The DWLS procedure is as follows:

**Decomposition:**

\[ s_1 = S(2j-1) + \sqrt{3} S(2j), \]
\[ d_1 = S(2j) - \frac{\sqrt{3}}{4} s_1(j) - \frac{\sqrt{3} - 2}{4} s_1(j-1), \]
\[ s_2 = s_1 - d_1(j+1), \]
\[ S_1 = \frac{\sqrt{3} - 1}{\sqrt{2}} s_2 \quad \text{and} \]
\[ D = \frac{\sqrt{3} + 1}{\sqrt{2}} d_1. \]

Here, we get new approximation as,

\[ S = [S_1 \quad D] \quad (8.8) \]

**Reconstruction:**

Consider Eqn. (8.8), then apply the DWLS reconstruction (coarser to finer) procedure as,

\[ d_1 = \frac{\sqrt{2}}{\sqrt{3} + 1} D, \]
\[ s_2 = \frac{\sqrt{2}}{\sqrt{3} - 1} S_1, \]
\[ s_1 = s_2 + d_1(j+1), \]
\[ S(2j) = d_1 + \frac{\sqrt{3}}{4} s_1 + \frac{\sqrt{3} - 2}{4} s_1(j-1) \quad \text{and} \]
\[ S(2j-1) = s_1 - \sqrt{3} S(2j) \]

which is the required solution of the given equation.

8.4.3 Lifting scheme via biorthogonal wavelet (BWLS)

As discussed in the previous sections 8.4.1 and 8.4.2, we follow the same procedure but here also, we used another wavelet i.e., biorthogonal wavelet (CDF(2, 2)). The BWLS procedure is as follows
Decomposition:

\[ d_i = S(2j) - \frac{1}{2}[S(2j-1) + S(2j+2)], \]
\[ s_i = S(2j-1) + \frac{1}{4}[d_i(j-1) + d_j], \]
\[ D = \frac{1}{\sqrt{2}} d_i, \]
\[ S_i = \sqrt{2} s_i \]

In this stage finally, we get new signal as,
\[ S = [S_i \ D] \]  (8.10)

Reconstruction:
Consider Eqn. (8.11), then apply the BWLS reconstruction (coarser to finer) procedure as,

\[ s_i = \frac{1}{\sqrt{2}} S_i, \]
\[ d_i = \sqrt{2} D, \]
\[ S(2j-1) = s_i - \frac{1}{4}[d_i(j-1) + d_j], \]
\[ S(2j) = d_i - \frac{1}{2}[S(2j-1) + S(2j+2)], \]

which is the required solution of the given equation.

The coefficients \( s_i(j) \) and \( d_i(j) \) are the average and detailed coefficients respectively of the approximate solution \( P_j \). The new approaches are tested through some of the numerical problems and the results are shown in next section.

8.5 Numerical experiments

Here, we present some of the test problems to demonstrate the efficiency and applicability of HWLS, DWLS and BWLS.

**Test Problem 8.5.1** First, we consider the ordinary differential equation to show the efficiency of the schemes,
\[ \frac{d^2 u}{dx^2} = \frac{8}{1+2x} u^2, \quad 0 < x < 1 \]  (8.13)

subject to boundary conditions \( u(0) = 1, u(1) = \frac{1}{3} \) and the exact solution is \( u(x) = \frac{1}{1+2x} \). The wavelet based numerical solutions of Eqn. (8.13) are as follows,
Finite difference approximation of (8.13) is
\[
\frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x} = \frac{8}{1+2x_i} u_i^2, \text{ for } i = 1, 2, \ldots, 16
\] (8.14)

Solve (8.14) by iteratively, we get
\[
u = \begin{bmatrix} 0.894, & 0.808, & 0.737, & 0.677, & 0.626, & 0.582, & 0.544, & 0.510 \end{bmatrix}_T .
\]

We know that \( S = u \), then apply the HWLS as explained in section 8.4.1 as,
\[
d_i = S(1:2:15) - S(2:2:16),
\]
\[
d_i = [0.085 \ 0.059 \ 0.043 \ 0.033 \ 0.026 \ 0.020 \ 0.016 \ 0.013],
\]
\[
s_i = S(2:2:16) + \frac{1}{2} d_i,
\]
\[
s_i = [0.851 \ 0.707 \ 0.604 \ 0.527 \ 0.468 \ 0.421 \ 0.383 \ 0.353],
\]
\[
S_1 = \sqrt{2} s_i,
\]
\[
S_1 = [1.203 \ 1.00 \ 0.854 \ 0.745 \ 0.661 \ 0.595 \ 0.542 \ 0.499],
\]
\[
D = \frac{1}{\sqrt{2}} d_i,
\]
\[
D = [0.060 \ 0.042 \ 0.031 \ 0.023 \ 0.020 \ 0.019 \ 0.011 \ 0.009],
\]

We get new approximation as,
\[
S = \begin{bmatrix} 1.203, & 1.00, & 0.854, & 0.745, & 0.661, & 0.595, & 0.542, & 0.499, & 0.060, & 0.042, & 0.031, & 0.023, & 0.020, & 0.019, & 0.011, & 0.009 \end{bmatrix}_T .
\]

**Reconstruction:**

Then apply the HWLS reconstruction procedure as,
\[
d_i = \sqrt{2} D,
\]
\[
d_i = [0.085 \ 0.059 \ 0.043 \ 0.033 \ 0.026 \ 0.020 \ 0.016 \ 0.013],
\]
\[
s_i = \frac{1}{\sqrt{2}} s_i,
\]
\[
s_i = [0.851 \ 0.707 \ 0.604 \ 0.527 \ 0.468 \ 0.421 \ 0.383 \ 0.353],
\]
\[
S(1:2:15) = s_i - \frac{1}{2} d_i,
\]
\[
S(2:2:16) = d_i + S(1:2:15),
\]
\[ S = \begin{bmatrix} 0.808, & 0.894, & 0.677, & 0.737, & 0.582, & 0.626, & 0.510, & 0.544, \\ 0.455, & 0.481, & 0.410, & 0.431, & 0.375, & 0.392, & 0.346, & 0.360 \end{bmatrix}^T_{16 \times 1} \]

Which is the required HWLS solution of the given equation.

Similarly, DWLS as discussed in the section 8.4.2, we follow the same procedure as,

**Decomposition:**

\[ s_1 = S(1 : 2 : 16) + \sqrt{3} S(2 : 2 : 16), \]

\[ s_1 = \begin{bmatrix} 2.294, & 1.910, & 1.634, & 1.428, & 1.269, & 1.143, & 1.042, & 0.959 \end{bmatrix}, \]

\[ d_1 = S(2 : 2 : 16) - \frac{\sqrt{3}}{4}s_1 - \frac{\sqrt{3} - 2}{4}[s_1(8), s_1(1 : 7)], \]

\[ d_1 = \begin{bmatrix} -0.120, & 0.003, & 0.002, & 0.001, & 0.0001, & 0.0005, & 0.0004 \end{bmatrix}, \]

\[ s_2 = s_1 - [d_1(2 : 8), d_1(1)], \]

\[ s_2 = \begin{bmatrix} 2.290, & 1.907, & 1.633, & 1.427, & 1.268, & 1.143, & 1.042, & 1.080 \end{bmatrix}, \]

\[ S_1 = \frac{\sqrt{3} - 1}{\sqrt{2}} s_2, \]

\[ S_1 = \begin{bmatrix} 1.185, & 0.987, & 0.845, & 0.738, & 0.656, & 0.591, & 0.539, & 0.559 \end{bmatrix}, \]

\[ D = \frac{\sqrt{3} + 1}{\sqrt{2}} d_1, \]

\[ D = \begin{bmatrix} -0.233, & 0.007, & 0.004, & 0.003, & 0.0021, & 0.0015, & 0.0011, & 0.0008 \end{bmatrix}, \]

\[ S = \begin{bmatrix} 1.185, & 0.987, & 0.845, & 0.738, & 0.656, & 0.591, & 0.539, & 0.559, \\ -0.233, & 0.007, & 0.004, & 0.003, & 0.002, & 0.001, & 0.001, & 0.0008 \end{bmatrix}^T_{16 \times 1} \]

**Reconstruction:**

Then apply the DWLS reconstruction procedure as,

\[ d_1 = \frac{\sqrt{2}}{\sqrt{3} + 1} D, \]

\[ d_1 = \begin{bmatrix} -0.120, & 0.003, & 0.002, & 0.001, & 0.0001, & 0.0008, & 0.0005, & 0.0004 \end{bmatrix}, \]

\[ s_2 = \frac{\sqrt{2}}{\sqrt{3} - 1} S_1, \]

\[ s_2 = \begin{bmatrix} 2.290, & 1.907, & 1.633, & 1.427, & 1.268, & 1.143, & 1.042, & 1.080 \end{bmatrix}, \]

\[ s_1 = s_2 + [d_1(8), d_1(2 : 7), d_1(1)], \]
\[ s_i = [2.294 \ 1.910 \ 1.634 \ 1.428 \ 1.269 \ 1.143 \ 1.042 \ 0.959], \]
\[ S(2:2:16) = d_i + \frac{\sqrt{3}}{4}s_i + \frac{\sqrt{3}-2}{4}\left[s_i(2:8)\ s_i(1)\right], \]
\[ S(1:2:15) = s_i - \sqrt{5}S(2:2:16), \]
\[ S = \begin{bmatrix} 0.894, & 0.808, & 0.737, & 0.677, & 0.626, & 0.582, & 0.544, & 0.510, \\
                          & 0.481, & 0.455, & 0.431, & 0.410, & 0.392, & 0.375, & 0.360, & 0.346 \end{bmatrix}_{16 	imes 1}. \]

Which is the required DWLS solution of the given equation.

And also, BWLS is explained in the sections 8.4.3, we follow the similar procedure as follows

**Decomposition:**
\[ d_i = S(2:2:16) - \frac{1}{2}\left(S(1:2:15) + S(3:8)\ S(1:2)\right), \]
\[ d_i = [0.070, \ 0.053, \ 0.041, \ 0.033, \ 0.026, \ 0.022, \ -0.224, \ -0.172], \]
\[ s_i = S(1:2:15) + \frac{1}{4}\left[d_i(8)\ d_i(2:7)\ d_i(1)\right] + d_i, \]
\[ s_i = [0.868, \ 0.767, \ 0.649, \ 0.562, \ 0.496, \ 0.443, \ 0.341, \ 0.260], \]
\[ D = \frac{1}{\sqrt{2}}d_i, \]
\[ D = [0.049, \ 0.037, \ 0.029, \ 0.023, \ 0.018, \ 0.015, \ -0.158, \ -0.122], \]
\[ S_i = \sqrt{2}s_i \]
\[ S_i = [1.228, \ 1.086, \ 0.919, \ 0.795, \ 0.701, \ 0.627, \ 0.483, \ 0.368], \]

we get new approximation as,
\[ S = \begin{bmatrix} 1.228, & 1.086, & 0.919, & 0.795, & 0.701, & 0.627, & 0.483, & 0.368, \\
                         & 0.049, & 0.037, & 0.029, & 0.023, & 0.018, & 0.015, & -0.158, & -0.122 \end{bmatrix}_{16 	imes 1}. \]

**Reconstruction:**
Then apply the BWLS reconstruction procedure as,
\[ s_i = \frac{1}{\sqrt{2}}S_i, \]
\[ s_i = [0.868, \ 0.767, \ 0.649, \ 0.562, \ 0.496, \ 0.443, \ 0.341, \ 0.260], \]
\[ d_i = \sqrt{2}D, \]
\[
\begin{align*}
\mathbf{d}_i &= [0.070, 0.053, 0.041, 0.033, 0.026, 0.022, -0.224, -0.1, 72] \\
S(1:2:15) &= s_i - \frac{1}{4}\left([d_i(8) + d_i(1:7)] + [d_i]\right), \\
S(2:2:16) &= d_i - \frac{1}{2}\left(\left[S(1:2:15)\right] + \left[S(3:8, S(1:2))\right]\right), \\
S &= [0.894, 0.808, 0.737, 0.677, 0.626, 0.582, 0.544, 0.510, 0.481, 0.455, 0.431, 0.410, 0.392, 0.375, 0.360, 0.346]^{T}. 
\end{align*}
\]

Which is the required BWLS solution of the given equation. The results are presented in figure 8.1 and the maximum error \(E_{\max} = \max|u_e - u_a|\), where \(u_e\) and \(u_a\) are exact and approximate solution respectively, with CPU time versus grid points is given in table 8.1.

**Test Problem 8.5.2** Now consider the elasto-hydrodynamic lubrication with line contact problem [21],

\[
\frac{\partial}{\partial x} \left( \varepsilon(P) \frac{\partial P}{\partial x} \right) - \frac{\partial}{\partial x} \left( \rho(P)H(P) \right) = 0, \quad (8.15)
\]

where \(\varepsilon(P) = \frac{\rho(P)H(P)^3}{\lambda \eta(P)}\), \(P(x)\) and \(H(x)\) are pressure and film thickness respectively to be determine, \(\lambda = \frac{12 p_0 VR^2}{b^2 p_h}\) is a dimensionless speed parameter. The boundary conditions are

\[
P(x_a) = P(x_b) = \frac{\partial P(x_a)}{\partial x} = 0. \quad (8.16)
\]

The domain of the problem is from the inlet \(x_a\) to the cavitation point \(x_c\). Boundary conditions of zero pressure are imposed at \(x_a\) and \(x_b\). The nondimensional film thickness equation is

\[
H(x) = H_{\infty} + \frac{x^2}{2} - \frac{1}{\pi} \int_{x_a}^{x_b} \log|x-x'|P(x')dx'. \quad (8.17)
\]

where \(H_{\infty}\) is the central offset film thickness, the term \(\frac{x^2}{2}\) defines the undeformed contact shape and the integral term represents the elastic deformation of the contact.

The nondimensional force balance equation,
represents the balance between the applied load and the total internal pressure in the lubricant. The nondimensional form for viscosity $\eta(P)$, which was established by Roelands [22] and density $\rho(P)$, which was presented by Dowson and Higginson [1] and are as follows

$$\eta(P) = \exp\left(\frac{\alpha p_o}{z}\left[-1 + \left(1 + \frac{pp}{p_o}\right)^z\right]\right), \quad \rho(P) = \frac{0.59e+09 + 1.34pp}{0.59e+09 + pp}$$

(8.19)

where $z = 0.6$ is the viscosity index, $\alpha = 2.165e-08$ is the pressure viscosity index, $p_o$ is the ambient pressure and $pp = 1.84e+09$ is the maximum Hertzian pressure. The three non-dimensional physical parameters that characterize the line contact problem are velocity ($U$), load force ($W$) and elasticity ($G$). The Eqs. (8.15)-(8.18) are discretized using finite differences with uniform grid of $N$ points $x_j$, $1 \leq j \leq N$, and the domain of interest is $[x_a, x_b] = [-5, 2.5]$, $x_c$ is the cavitation point, to be determined in the solution process, is an internal point near exit. The discretized form of Reynolds equation as,

$$\left(\frac{\varepsilon_j + \varepsilon_{j+1}}{2}\right)\left(\frac{p_{j+1} - p_j}{\Delta x}\right) - \left(\frac{\varepsilon_j + \varepsilon_{j-1}}{2}\right)\left(\frac{p_j - p_{j-1}}{\Delta x}\right) - \rho_j H_j - \rho_{j-1} H_{j-1} = 0$$

(8.20)

where $\varepsilon_j = \frac{\rho_j H_j^3}{\lambda \eta_j}$

The film thickness equation approximated at $x_j$ on the regular grid is given by

$$H(x_j) = H_0 + \frac{x_j^2}{\pi} - \frac{1}{\pi} \sum_{i=0}^{N} K_y P(x_i)$$

(8.21)

where

$$K_y = \left(x_i - x_j + \frac{\Delta x}{2}\right)\left(\log\left|x_i - x_j + \frac{\Delta x}{2}\right| - 1\right) - \left(x_i - x_j - \frac{\Delta x}{2}\right)\left(\log\left|x_i - x_j - \frac{\Delta x}{2}\right| - 1\right)$$

(8.22)

for $i = 0, 1, 2, \ldots, N$ and $j = 0, 1, 2, \ldots, N$, the force balance equation in discretized from is

$$h \sum_{j=0}^{N} \left(\frac{p_j + p_{j+1}}{2}\right) - \frac{\pi}{2} = 0.$$  

(8.23)
In Eqns. (8.20) and (8.21), a piecewise constant form of \( P(x) \) is assumed. The boundary conditions are

\[
P(x_c) = 0, \quad \frac{dP}{dx} \geq 0 \text{ at } x = x_c.
\] (8.24)

Using these boundary conditions we consider Eqns. (8.20)-(8.23) together in the form of \( A(P_j) = f_j \). To solve this equation as explained in section 8.4, we get the numerical solutions are presented in figure 8.2. Comparison of residual vs iterations is given in figure 8.3 and table 8.2 for different \( N \). And also, comparison of CPU time for different schemes is given in table 8.4.

**Test Problem 8.5.3** Finally, we consider the dimensionless Reynolds equation of elasto-hydrodynamic lubrication point contact problem [21],

\[
\frac{\partial}{\partial X} \left( \epsilon \frac{\partial P}{\partial X} \right) + \alpha^2 \frac{\partial}{\partial Y} \left( \epsilon \frac{\partial P}{\partial Y} \right) - \frac{\partial}{\partial X} (\bar{\rho}H) = 0,
\] (8.25)

where \( \epsilon = \frac{\bar{p}H^3}{\lambda \bar{\eta}} \), \( P(X, Y) \) and \( H(X, Y) \) are unknown pressure and film thickness, \( X \) is the dimensionless coordinate \( X = \frac{x}{a} \), \( a \) is the half length of the elliptic contact area in the \( x \) direction, \( Y \) is the dimensionless coordinate \( Y = \frac{y}{b} \), \( b \) is the half length of the elliptic contact area in the \( y \) direction, \( \lambda \) a dimensionless speed parameter,

\[
\lambda = \frac{12 \eta U R^2}{a^3 \rho_h K_{ex}}, \quad \bar{\eta} \text{ is dimensionless lubricant viscosity (} \bar{\eta} = \frac{\eta}{\eta_0} \text{)}, \quad \bar{\rho} = \frac{\rho}{\rho_0} \text{ is dimensionless lubricant density (} \bar{\rho} = \frac{\rho}{\rho_0} \text{)}, \quad K_{ex} \text{ is the elliptic coefficient of the surface in the } x \text{ direction, } \alpha \text{ is a ratio parameter (} \alpha = \frac{a}{b} \text{)}, \quad P \text{ is dimensionless pressure (} P = \frac{p}{p_h} \text{), } p_h \text{ is the maximum Hertzian contact stress and } H \text{ is the dimensionless film thickness (} H = \frac{hR}{a^2} \text{). The boundary conditions of (8.25) are,}

Inlet boundary condition \( P(X_a, Y) = 0 \), Outlet boundary conditions \( P(X_b, Y) = 0 \) and \( \frac{\partial P(X_c, Y)}{\partial X} = 0 \), Side boundary conditions \( P(X, -Y_a) = P(X, Y_a) = 0 \)

(8.26)
where \( X_a \) and \( X_b \) are the dimensionless coordinates of the inlet and outlet, \( X_a \) is given but \( X_b \) should be determined by the outlet boundary conditions and \( Y = \pm 1 \) are the two sides of the contact region. The film thickness equation is given, in integral form as

\[
H(X, Y) = H_{00} + \frac{X^2 + Y^2}{2} + \frac{2}{\pi^2} \int_{x_a}^{x_b} \int_{y_a}^{y_b} \frac{P(S, T) dS dT}{\sqrt{(X - S)^2 + (Y - T)^2}}
\]

(8.27)

where \( H_{00} \) is the central offset film thickness, the term \( \frac{X^2 + Y^2}{2} \) defines the undeformed contact shape and the integral term represents the elastic deformation of the contact.

The dimensionless force balance equation, given by

\[
\int_{x_a}^{x_b} \int_{y_a}^{y_b} P(X, Y) dX dY = \frac{2\pi}{3}
\]

(8.28)

represents the balance between the applied load and the total internal pressure in the lubricant. The dimensionless form for viscosity \( \eta \), which was established by Roelands [22] and density \( \rho \), which was presented by Dowson and Higginson [9] and are given by

\[
\eta = \exp\left\{ \ln(\eta_0) + 9.67 \left[ -1 + \left( 1 + 5.1e^{-0.09P \cdot p_h} \right)^7 \right] \right\}
\]

(8.29)

and

\[
\rho = \left( 1 + \frac{0.6P}{1 + 1.7P} \right)
\]

(8.30)

where \( z = 0.68 \) is the viscosity index, \( \eta_0 = 1.98e + 08 \) is the ambient pressure and \( p_h = 1.84e + 09 \) is the maximum Hertzian pressure, \( \alpha \) is the ratio of \( a \) and \( b \). \( K_{ex} \) is the relative curvature in the \( x \) direction. For the equivalent curvature of the point contact, \( K_{ex} = 1 \). The following discussions are based on \( \alpha = 1 \) and \( K_{ex} = 1 \). The finite difference discretization of Eqn. (8.25) can be written as
The film thickness equation becomes

\[
\Delta X^2 \left\{ \left( \frac{e_{i,j} + e_{i-1,j}}{2} \right) P_{i-1,j} + \left( \frac{e_{i,j} + e_{i+1,j}}{2} \right) P_{i+1,j} - \left( 2e_{i,j} + e_{i-1,j} + e_{i+1,j} \right) P_{i,j} \right\} + \\
\Delta Y^2 \left\{ \left( \frac{e_{i,j} + e_{i,j-1}}{2} \right) P_{i,j-1} + \left( \frac{e_{i,j} + e_{i,j+1}}{2} \right) P_{i,j+1} - \left( 2e_{i,j} + e_{i,j-1} + e_{i,j+1} \right) P_{i,j} \right\} - \left\{ \frac{\bar{P}_{i,j} H_{i,j} - \bar{P}_{i-1,j} H_{i-1,j}}{\Delta X} \right\} = 0
\]  

(8.31)

The film thickness equation becomes

\[ H_{i,j} = H_\infty + \frac{X_i^2 + Y_j^2}{2} + \frac{2}{\pi} \sum_{i=1}^{N} \sum_{j=1}^{N} D_{ij} P_{ij} \]  

(8.32)

where \( D_{ij} \) is the stiffness coefficient of the elastic deformation

\[ D_{ij} = \left( X_i - X_j + \frac{h}{2} \right) \left( \log \left| X_i - X_j + \frac{h}{2} \right| - 1 \right) - \left( X_i - X_j - \frac{h}{2} \right) \left( \log \left| X_i - X_j - \frac{h}{2} \right| - 1 \right) \]  

(8.33)

for \( i = 0, 1, 2, \ldots, N \) and \( j = 0, 1, 2, \ldots, N \), the force balance equation which as

\[ \Delta X \Delta Y \sum_{i=1}^{N} \sum_{j=1}^{N} P_{ij} = \frac{2\pi}{3} \]  

(8.34)

three non-dimensional physical parameters that characterize the point contact problem are velocity \( (U) \), load force \( (W) \) and elasticity \( (G) \). To solve Eqns. (8.31)-(8.34) as explained in section 8.4, we get the numerical solutions are presented in figure 8.4 and figure 8.5, pressure and film thickness respectively. Comparison of residual with iterations for different \( N \) is presented in table 8.3. And also, comparison of CPU time for different schemes is given in table 8.4.
Table 8.1 The maximum error with CPU time (in seconds) versus grid points of Test Problem 8.5.1.

<table>
<thead>
<tr>
<th>N</th>
<th>Method</th>
<th>$E_{\text{max}}$</th>
<th>Setup time</th>
<th>Running time</th>
<th>Total time</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>HWLS</td>
<td>1.2792e-01</td>
<td>2.5480e-03</td>
<td>4.5733e-03</td>
<td>7.1213e-03</td>
</tr>
<tr>
<td></td>
<td>WFAS</td>
<td>2.5194e-03</td>
<td>1.1129e-02</td>
<td>1.0320e-03</td>
<td>1.2161e-02</td>
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<tr>
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<td>DWLS</td>
<td>2.5194e-03</td>
<td>2.2486e-03</td>
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<td>8.5925e-02</td>
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<td>1.3175e-02</td>
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<td>1.4365e-02</td>
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<td>4.5964e-03</td>
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<tr>
<td>16</td>
<td>HWLS</td>
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<td>2.8328e-03</td>
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<tr>
<td></td>
<td>WFAS</td>
<td>7.2752e-04</td>
<td>1.2149e-02</td>
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<td>5.8888e-04</td>
<td>5.0348e-03</td>
<td>5.6237e-03</td>
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<tr>
<td>32</td>
<td>HWLS</td>
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<td>2.0245e-03</td>
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<td>1.9089e-02</td>
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<td>1.1954e-03</td>
<td>2.1140e-02</td>
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<td>8.7549e-03</td>
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<td>1.1937e-03</td>
<td>5.0698e-02</td>
</tr>
<tr>
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<td>BWLS</td>
<td>4.6112e-05</td>
<td>6.2993e-04</td>
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<td>5.7729e-03</td>
</tr>
<tr>
<td>128</td>
<td>HWLS</td>
<td>1.4809e-02</td>
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<td>1.2036e-02</td>
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<td>5.9401e-04</td>
<td>5.0805e-03</td>
<td>5.6745e-03</td>
</tr>
</tbody>
</table>
Figure 8.1 Comparison of numerical solutions with exact solution for $N = 16$ of Test Problem 8.5.1.
Figure 8.2 Comparison of numerical solutions for $N = 256$ of Test Problem 8.5.2.

Figure 8.3 Comparison of residual v/s iterations for $N = 128$ of Test Problem 8.5.2.
Table 8.2 Comparison of residual with iterations of different schemes of the Test Problem 8.5.2.

<table>
<thead>
<tr>
<th>N</th>
<th>FDM</th>
<th>HWLS</th>
<th>DWLS</th>
<th>BWLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>2.88e-07(32)</td>
<td>6.06e-08(28)</td>
<td>1.56e-08(28)</td>
<td>8.97e-09(25)</td>
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<tr>
<td>128</td>
<td>8.49e-08(41)</td>
<td>7.81e-09(35)</td>
<td>4.53e-09(34)</td>
<td>8.35e-10(31)</td>
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<tr>
<td>256</td>
<td>1.01e-09(68)</td>
<td>2.51e-10(69)</td>
<td>1.28e-10(65)</td>
<td>9.69e-11(59)</td>
</tr>
</tbody>
</table>

Figure 8.4 Pressure numerical solutions comparison for $N = 4096$ of the Test Problem 8.5.3.
Figure 8.5 Film thickness numerical solution comparison for $N = 4096$ of the Test Problem 8.5.3.

Table 8.3 Comparison of residual with iterations of different schemes of the Test Problem 8.5.3.

<table>
<thead>
<tr>
<th>N</th>
<th>Residual (Iterations)</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
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<td>64</td>
<td>4.62e-04(61)</td>
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<tr>
<td>256</td>
<td>1.98e-05(211)</td>
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<td>1024</td>
<td>7.19e-06(312)</td>
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</tbody>
</table>
Table 8.4 The comparison of CPU time (in seconds) for $N = 256$ of the different methods.

<table>
<thead>
<tr>
<th>Test Problems</th>
<th>Method</th>
<th>Setup time</th>
<th>Running time</th>
<th>Total time</th>
</tr>
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<td>HWLS</td>
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<td>3.57</td>
<td>7.92</td>
</tr>
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<td>DWLS</td>
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</tr>
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<td>BWFAS</td>
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<td>3.98</td>
<td>11.22</td>
</tr>
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<td>BWLS</td>
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<td>5.79</td>
</tr>
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<td>9.67</td>
</tr>
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<td>7.10</td>
<td>10.70</td>
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</tbody>
</table>

8.6 Conclusion

In this chapter 8 introduced an efficient wavelet based lifting technique for the numerical solution of EHL line and point contact problems. The numerical results are shown in the figures and tables, from which the proposed schemes are very effective and convenient. The observations from the table 8.1, conclude that BWLS has advantage over HWLS and DWLS in terms of maximum error and CPU time. However the CPU time of HWLS is lower than others but BWLS shows the super convergence. From the last table the proposed scheme shows supper convergence over the existing ones, hence the scheme has wide range of applications to real world problems (for example, EHL problems) arising in fluid dynamics.