Chapter 1

Introduction
1.1 Background of the study

Ecology is the subject of study of the relationships between living organisms and their physical environment. Basic motivation of ecology is to understand the vital connections between plants and animals and the world around them. Ecology provides information for the benefits of ecosystems suggesting the optimal use of Earth’s resources. The subject ecology covers the descriptions of different areas like development of ecosystem, interactions among different species, conservation of species, biodiversity and spread of disease among the living organisms. Studies in ecology are mainly classified as experimental or mathematical ecology. Experimental ecology is the scientific discipline devoted to the study of ecological systems with different experiments. Mathematical ecology uses the power of modelling and simulations to solve problems of ecology, and to interpret those solutions in the ecologically clearest possible manner.

Applications of dynamical systems theory in mathematical biology and ecological systems have attracted much attention of Biologists, Physicists, Engineers as well as Mathematicians. Modelling is a frequently evolving process to gain a deeper understanding of the mathematical aspects of natural problems and to yield non-trivial biological insights. Therefore, one must carefully construct biologically meaningful and mathematically tractable population models (Kuang, 2002). Modelling of predator-prey system has become an area of fundamental importance in mathematical ecology. In the last few decades, interest has been growing steadily in designing and studying of mathematical models of population interactions. Predator-prey models are the building blocks of the ecosystems. Depending on the motivations of modelling, one can select the interactions as resource-consumer, plant-herbivore, parasite-host, infection-immune system, susceptible-infectious interactions, etc. Lotka-Volterra model (Lotka, 1925, 1932; Volterra, 1926) is the pioneer predator prey model which focused on interactions of one prey and one predator species. There are so many fundamental predator-prey models (Rosenzweig-MacArthur predator-prey model (Rosenzweig and MacArthur, 1963), Gause-type predator-prey model (Freedman, 1980), Leslie-Gower predator-prey models (Leslie, 1945, 1948)) are proposed to investigate the various ecological phenomena in different ecosystem.

Epidemiology is the science which investigates the patterns, causes, and effects of disease in defined populations. It is the study of the frequency, distribution, and factors responsible for disease in a system. It help us to develop methodology used in clinical research, public health studies for disease control. Modelling and parametrization of infectious diseases is necessarily required to study in mathematical ecology. Epidemiological models have a long history in infectious disease ecology starting from Bernoulli’s (Bernoulli, 1766) modelling of smallpox and including Ross’s (Ross, 1911) analysis of malaria. The earliest attempt to provide a quantitative understanding of the dynamics of malaria transmission was that of Ross
Ross models consisted of a few differential equations to describe changes in densities of susceptible and infected people, and susceptible and infected mosquitoes. Based on his modelling, Ross introduced the concept of a threshold density and concluded that ‘in order to counteract malaria anywhere we need not banish Anopheles there entirely we need only to reduce their numbers below a certain figure’ (Ruan et al., 2008). Classical papers of mathematical modelling of infectious disease was by Kermack and McKendrick (1927, 1932, 1933). These papers had a major influence on the development of mathematical models for disease spread and are still relevant in many epidemic situations.

Some field experiments reported that predator removal is one of the most important cause of disease spreading in prey populations (Cote and Sutherland, 1997; Sih et al., 1985). Sih et al. (1985) documented predator-removal experiments and found 54 out of 135 systems in which predator removal reduced the prey population. Similarly, Cote and Sutherland (1997) found that predator removal reduced prey populations in 3 of 11 laboratory studies. If predators eliminate the most infectious individuals from the prey population, they will have an outcome equivalent to quarantine—whereby infectious individuals are removed from the healthy population and thereby prevented from spreading disease (Packer et al., 2003). Paramecium, which also proved useful in test-tube studies of competition, was placed in culture with a predaceous protozoan. These laboratory studies found that cycles were short-lived, and the system soon collapsed. However, if one added more paramecium every few days, the expected cycle was observed. There is increasing emphasis on multi species food chain systems involving more complex interactions (Hastings and Powell, 1991; Huxel and McCann, 1998; Huxel et al., 2002; Hale and Waltman, 1989).

Chaos theory has applications in several disciplines including meteorology, physics, engineering, economics, biology, and philosophy. It is well known that chaotic systems are unpredictable in nature. It is known from many theoretical studies that ecological chaos may have numerous significant impacts on the population and community dynamics. Therefore, identification of the factors potentially enhancing or suppressing chaos is a challenging problem. May’s works (May, 1974, 1975, 1976) on discrete time models in population ecology was one of the first studies of chaos in any discipline (Hastings et al., 1993). These works (May, 1974, 1975, 1976) attracted a tremendous attention among theoreticians as well as experimentalists in chaos in other disciplines, but chaos as an ecological phenomenon was overlooked by most researchers for a number of years. However, Schaffer and Kot (1985) emphasized the possibility of chaos in ecology. In the case of transient chaos, the dynamical variables of the system behave chaotically for a period of time and then switch their behaviour suddenly, to a fixed value or to some regular patterns including zero population density. An ecosystem exhibits transient chaos means that the population size of some species can behave chaotically for a long period of time.
and then decreases to zero in a relatively short period of time. It was shown by McCann and Yodzis that such a transient chaotic behaviour, which is responsible for species extinction, can indeed occur in a simple three-species food chain model which incorporates biologically reasonable assumptions about species interactions. In most cases such systems are capable of displaying chaotic behaviour (El-Gohary and Al-Ruzaiza, 2007; Gakkhar and Naji, 2003; Guckenheimer and Holmes, 1983; Hastings and Powell, 1991; Kendall, 2001; Schaffer, 1985; Wang and Zeng, 2007). The population dynamics of a large class of ecosystems can be effectively modelled by deterministic chaotic systems (Hastings and Powell, 1991; Holt and McPeek, 1996; May, 1976). The control of chaotic population dynamics is one of the main objectives of mathematical modelling in fishery management and species conservation.

Harvesting is an important ecological factor of population biology. The removal of a constant number of populations during each time period is known as harvesting. In fact, ever since primitive humans began hunting and fishing thousands of years ago, there has been a need to know how harvesting a certain number of animals will affect the population after long time. The fact that there are over 750 plants and animals on the endangered species list indicates that humans are not always cognizant of how their actions will affect plants and animals. Management of harvesting involves determining how harvesting impacts population growth, composition and density, then setting harvest goals and practices accordingly. Many harvest models (Dai and Tang, 1998; Zhang et al., 2000; Martin and Ruan, 2001; Zeng et al., 2008; Liu et al., 2010) are proposed based on extension of single-species, density-dependent, population-dynamics models, such as the logistic growth model, to two-species models, especially predator-prey models. A common goal of harvesting management is to harvest prey or predator species from a population without putting that population in danger.

In reality, predator-prey systems are complex; they often involve multiple predators and multiple types of prey. Under ideal circumstances, an individual will encounter high-quality food items on a regular basis. These preferred foods provide the most nutritional benefit with the fewest costs. Costs for an organism may be handling time (e.g., time required to catch prey or remove a nut from its shell) or presence of chemicals, such as tannins, that reduce the nutritional quality of the food item. When preferred foods are scarce, organisms must switch to other, less-desirable alternatives. The point at which an organism should make this shift is not easy to predict. It depends upon many factors, including the relative abundance of each of the foods, the potential costs associated with each food, and other factors, such as the risk of exposure to predators while eating. It is well known that most of the predators do not feed only on a single prey, but also depend on diverse prey species and are therefore involved in a complex food web interactions. Many species are also migratory and their spatial scale much longer than the habitat occupied by
some of their prey. Therefore, for these types of species, alternative prey needs to be considered to develop a realistic predator-prey model. Therefore, the consequences of providing additional food to predator and corresponding effects on the predator-prey systems and its importance in biological control have been interesting topic of modern research. The mathematical model with additional/alternative food is like as one predator-two prey (one is different from existing prey in the ecosystem which could also be a non-reproducing prey or food source). It is also observed that additional food to predator increases predator population and also their effect on target prey (Holt and Lawton, 1994; Huxel and McCann, 1998; Huxel et al., 2002; Sabelis et al., 2006; Srinivasu et al., 2007; van Rijn et al., 2002). Harwood and Obrycki (2005) pointed out that provision of alternative food to a generalist predator is two fold: on one hand these nutritious food items improve the predator population by enhancing their fecundity and on the other hand presence of these alternative food, result in reduction of prey consumption per individual predator. The role of alternative prey in sustaining predator populations has been widely reported in laboratory studies and field trials examining the fecundity, feeding behaviour, extinction risk and growth rates of species subjected to diets of varying quality. Huxel and McCann (1998) reported the role of allochthonous resources (alternative resources) to investigate the impact of allochthonous inputs on the stability of a food web model. Actually, allochthonous resources (energetic resources) frequently enter into a food chain to increase local productivity and influence community structure and their stabilities. Allochthonous resources can alter consumption rates of predator species in the recipient food systems, thereby influencing food web stability. In nature, fluxes across habitats often bring both nutrient and energetic resources into areas of low productivity from areas of higher productivity. Therefore, the availability of suitable additional food (non-prey food, energetic resource etc.) in a predator-prey system can have a significant impact on the dynamics of the system. Thus the dynamics of complex food web critically depend on an accurate understanding of the relative importance of direct and indirect effects of additional food in multi-species prey-predator system (Huxel and McCann, 1998; Kar and Ghosh, 2012; Sahoo, 2014). Thus studying the effects of additional food on a predator-prey systems where the predators are provided with additional food is important and which is the focus of this thesis.

1.2 Basic concepts

1.2.1 Dynamical system

A system evolves with time in the way that the states of the system at time \( t \) depends upon the states of the system at earlier times is called dynamical system. In other words, a dynamical system consists of a set of possible states, together with
a rule that determines the present states in terms of past states. According to the character of the time variable a dynamical system can be classified as a discrete dynamical system or as a continuous dynamical system.

### 1.2.2 Continuous dynamical system

The dynamical system is called continuous dynamical system if time is a continuous variable. A continuous dynamical system is represented by a differential equation

\[
\frac{dX}{dt} = f(X, t) ; X \in U \subseteq \mathbb{R}^n, t \in \mathbb{R},
\]

which possesses a unique solution \( X(t, t_0) = X(t) \) satisfying the condition \( X(t_0) = X_0 \).

### 1.2.3 Autonomous and non-autonomous systems

If \( f(X, t) = f(X) \), then the system is called autonomous system. Consider the following dynamical system

\[
\begin{align*}
\dot{x} &= f_1(x, y), \\
\dot{y} &= f_2(x, y).
\end{align*}
\]

The functions \( f_1(x, y) \) and \( f_2(x, y) \) does not depend on time \( t \) explicitly. Therefore, the above type of dynamical system is called autonomous dynamical system. For example

\[
\begin{align*}
\dot{x} &= 2x - xy, \\
\dot{y} &= xy,
\end{align*}
\]

is a autonomous dynamical system.

On the other hand, if \( f(X, t) \) explicitly depends on \( t \), then the dynamical system is called non-autonomous system.

For example

\[
\begin{align*}
\dot{x} &= 3(1 + \cos \omega t)x - xy, \\
\dot{y} &= xy,
\end{align*}
\]

is a non-autonomous dynamical system.

### 1.2.4 Equilibrium point

An equilibrium point \( X^* \) of a continuous dynamical system is the point in the phase space where phase space velocity is zero. That is, \( \left[ \frac{dX}{dt} \right]_{X=X^*} = 0 \Rightarrow F(X^*) = 0 \).

For example, \( \frac{dx}{dt} = x(1 - x) \) has equilibrium points 0 and 1.
1.2.5 Stability of equilibrium points

Local stability

We consider a system
\[ \dot{x} = f(x, y), \]
\[ \dot{y} = g(x, y). \]

Let \((x^*, y^*)\) be the equilibrium point of the system. Then \(f(x^*, y^*) = 0, g(x^*, y^*) = 0\). Taylor series expansion in the neighborhood of \((x^*, y^*)\) can be written as
\[ f(x, y) \approx f(x^*, y^*) + (x - x^*)(\frac{\partial f}{\partial x})_{(x^*, y^*)} + (y - y^*)(\frac{\partial f}{\partial y})_{(x^*, y^*)} + \text{higher order terms}, \]
\[ g(x, y) \approx g(x^*, y^*) + (x - x^*)(\frac{\partial g}{\partial x})_{(x^*, y^*)} + (y - y^*)(\frac{\partial g}{\partial y})_{(x^*, y^*)}, \]

Therefore, by using very small disturbances in the neighborhood of equilibrium point will follow the linear equations
\[ \dot{x} = ax + by, \]
\[ \dot{y} = cx + dy, \]

i.e.
\[ \dot{X} = AX, \]
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}. \]

A is a non-zero matrix. The equilibrium point \((x^*, y^*)\) is linearly stable if all eigenvalues of \(A\) have negative real parts. The equilibrium point \((x^*, y^*)\) is unstable if at least one eigenvalue of \(A\) have positive real part. If \(A\) has any 0 eigenvalue then linear stability analysis fails and nonlinear analysis is necessary.

Global stability

Lyapunov functions in dynamical system theory are those functions which can be used to prove the stability of equilibrium points in dynamical system. Lyapunov functions are important to determine the stability and control theory. There is no general method to construct or find a Lyapunov function which proves the stability of an equilibrium point, and the inability to find a Lyapunov function is inconclusive with respect to stability. For dynamical systems (e.g., physical systems) conservation laws can often be used to construct a Lyapunov function. Global stability analysis of an equilibrium point can be done by Lyapunov function (without solving differential
A function $W(x_1, x_2, ..., x_n)$ is called Lyapunov function for the system with differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(x_1, x_2, x_3, \ldots, x_n), \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, x_3, \ldots, x_n), \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, x_3, \ldots, x_n),
\end{align*}
\]

if (a) $W(x_1, x_2, ..., x_n)$ is positive definite.

(b) $\frac{dW}{dt}$ is negative definite.

Origin is a globally stable equilibrium point of a system if there exist a Lyapunov function for the system.

Let $W : \mathbb{R}^n \to \mathbb{R}$ be a scalar function. $W$ is a Lyapunov function if it is a locally positive-definite function, i.e. $W(0) = 0, W(x) > 0, \forall x \in U \setminus \{0\}$, with $U$ being a neighborhood region around $x = 0$.

If the Lyapunov function $W$ is globally positive definite, radially unbounded and the time derivative of the Lyapunov function is globally negative definite: $\dot{W}(x) < 0, \forall x \in \mathbb{R}^n \setminus \{0\}$, then the equilibrium is proven to be globally asymptotically stable. The Lyapunov function $W(x)$ is radially unbounded if $\|x\| \to \infty \Rightarrow W(x) \to \infty$.

For example,

\[
\begin{align*}
\dot{x} &= -x, \\
\dot{y} &= -y.
\end{align*}
\]

Let $W(x, y) = (x^2 + y^2)/2$ is positive definite.

Therefore, $\frac{dW}{dt} = x\dot{x} + y\dot{y} = -x^2 - y^2$, is negative definite. Hence, $(0, 0)$ is a globally asymptotically stable equilibrium point for the system. Note that $W(0) = 0$ is required. Otherwise choosing $W(x) = 1/(1 + |x|)$, one can prove that $\dot{x}(t) = x$ is locally stable.

1.2.6 Mathematical model

Predator and prey

A predator is an organism that eats another organism. The prey is the organism which the predator eats. Predator and prey evolve together. The prey is part of the predator’s environment, and likewise the predator is also the part of prey’s environment. The predator dies if it does not get food, so it evolves whatever is necessary in order to eat the prey. The interactions like Lion-zebra, bear-fish and fox-rabbit are some examples of predator-prey.
Functional responses

A functional response is described as a predator’s instantaneous per capita feeding rate as a function of prey abundance (Holling, 1959). This means that the consumption rate of an individual predator depends on the prey density.

According to Abrams and Ginzburg (2000), functional responses are generally classified as: (i) prey dependent, when prey density alone determines the response; (ii) predator dependent, when both predator and prey populations affect the response; and (iii) multi species dependent, when species other than the focal predator and its prey species influence the functional response. Until recently, predation theory was dominated by prey-dependent models and Holling (1959) categorized the prey-dependent responses into three types:

(a) **Holling Type I**: Type I functional response assumes a linear increase in intake rate with food density, either for all food densities, or only for specified food densities up to a maximum level, beyond which the intake rate is constant. This functional response which is the standard mass action or linear response \( f(N) = aN \), where \( a > 0 \) is the attack rate of the predator and \( N \) denotes food (or resource) density. This linear increase assumes that the time needed by the consumer to process a food item is negligible, or that consuming food does not interfere with searching for food.

(b) **Holling Type II**: Type II functional response is characterized by a decelerating intake rate, which follows from the assumption that the consumer is limited by its capacity to process food. Type II functional response is often modeled by a rectangular hyperbola, which assumes that processing of food and searching for food are mutually exclusive behaviors. The equation is \( f(N) = \frac{aN}{1+ahN} \), where \( f \) denotes intake rate and \( N \) denotes food density. The rate at which the consumer encounters food items per unit of food density is called the attack rate, \( a \). The average time spent on processing a food item is called the handling time, \( h \). For example, small mammals destroy most of gypsy moth pupae in sparse populations of gypsy moth. However, in high-density defoliating populations, small mammals kill a negligible proportion of pupae.

(c) **Holling Type III**: Type III functional response is similar to type II response in which at high levels of prey density, saturation occurs. Type III functional response is represented by the equation \( f(N) = \frac{aN^\alpha}{b+N^\alpha} \), where \( \alpha > 1 \) is the encounter rate between predator and prey before the predator reaches maximum efficiency. Holling Type III functional response occurs in predators which increase their search activity with increasing prey density. For example, many predators respond to kairomones (chemicals emitted by prey) and increase their activity. Polyphagous vertebrate predators (e.g., birds) can switch to the most abundant prey species by learning to recognize it visually. Mortality first increases with prey increasing density, and then declines. If predator density is constant (e.g., birds, small mammals)
then they can regulate prey density only if they have a Holling Type III functional response because this is the only type of functional response for which prey mortality can increase with increasing prey density.

### 1.2.7 Mathematical model of predator-prey

An application of the nonlinear system of differential equations in mathematical biology / ecology is to model the predator-prey relationship of a simple eco-system. In a closed eco-system there are only two types of animals: the predator and the prey. They form simple food chain where the predator hunts the prey species, while the prey grazes vegetation. The size of the two populations can be described by a simple system of two nonlinear first order differential equations.

Let $x(t)$ denotes the population of the prey species, and $y(t)$ denotes the population of the predator species. Then a general model of interacting populations is written in terms of two autonomous differential equations

$$\begin{align*}
\dot{x} &= xf(x, y), \\
\dot{y} &= yg(x, y).
\end{align*}$$

The functions $f$ and $g$ denote the respective per capita growth rates of the two species. It is assumed that $\frac{\partial f(x, y)}{\partial y} < 0$ and $\frac{\partial g(x, y)}{\partial x} > 0$. This general model is often called Kolmogorov’s predator-prey model (Freedman, 1980).

As for example, we consider a predator-prey model with Holling Type I functional response in the following form (Lotka, 1932):

$$\begin{align*}
\dot{x} &= x(a - by), \\
\dot{y} &= y(cx - d),
\end{align*}$$

where $a$, $b$, $c$ and $d$ are positive constants. Note that in the absence of the predators (when $y = 0$), the prey population would grow exponentially. If the preys are absence (when $x = 0$), the predator population would decay exponentially to zero due to starvation.

Another example of predator-prey model with Holling Type II functional response is of the form

$$\begin{align*}
\dot{x} &= ax - \frac{bxy}{c + x}, \\
\dot{y} &= \frac{bxy}{c + x} - dy,
\end{align*}$$

where $x$ and $y$ are represented the density of prey and predator respectively. The parameter $a$ denotes the birth rate of prey; $b$ denotes food intake rate of predator; $c$ be the half saturation constant, and $d$ denotes the natural death rate of predator.
A predator-prey model with Holling Type III is of the form

\[
\dot{x} = ax - \frac{bx^\alpha y}{c + x^\alpha},
\]

\[
\dot{y} = \frac{bx^\alpha y}{c + x^\alpha} - dy,
\]

where \(x\) and \(y\) be the number of prey and predator species respectively. The parameter \(a\) denotes the birth rate of prey; \(b\) denotes food intake rate of predator; \(c\) be the half saturation constant, \(d\) denotes the natural death rate of predator and \(\alpha > 1\).

### 1.2.8 Additional/alternative food

It is well-known that the non-prey food sources has long been recognized and attempts have been made to manipulate these non-prey sources (viz., nectar, pollen etc.,) in agricultural lands. Many experiments (Wäckers and van Rijn, 2005; Wäckers et al., 2007; Wäckers et al., 2008; Wade et al., 2008) reported the benefits of using additional or alternative food supplements in biological control programs. Wäckers and van Rijn (2005) pointed out that additional food in plant-herbivore-carnivore interactions is not only an important topic in basic ecology, but also directly applied for biological pest control. The availability of suitable additional food (non-prey food) in a predator-prey system can have significant impact on the dynamics of the system. The consequences of providing additional food to predator and the corresponding effects on the predator-prey dynamics and its utility in biological control (such as species conservation and pest management) have been the topic of great attention for many researchers. It is assumed that the additional food is not dynamic but maintained at a specific constant level either by the nature or by an external agency. It has been suggested that other, “additional foods” serve only to maintain the predator when essential foods are not available, but little research of has evaluated the significance a mixed diet of essential and alternative foods for predator growth or reproduction.

### 1.2.9 Chaos and chaos control

The simplest and most intuitive definition of chaos is extreme sensitivity to initial conditions. If a system has chaotic dynamics, then the difference between the trajectories of two populations that have slightly different initial conditions grows until this difference is essentially as large as the variation in either trajectory. The necessary requirements for a deterministic continuous dynamical system to be chaotic are that the system must be nonlinear and be at least three dimensional. At least one Lyapunov exponent must be positive for chaotic systems. For a hyper-chaotic systems at least two Lyapunov exponents are positive. The idea of chaos control was developed at the University of Maryland by Ott, Grebogi and Yorke in the year
1990. The important fact is that because of chaos there exist infinite number of unstable periodic orbits and applying suitable small perturbation (disturbance) one can stabilizes any one of these orbits. Chaotic dynamics, then, consists of a motion where the system state moves in the neighborhood of one of these orbits for a while, then falls close to a different unstable, periodic orbit where it remains for a limited time, and so forth.

Control of chaos is the stabilization, by means of small system perturbations, of one of these unstable periodic orbits. The result is to render an otherwise chaotic motion more stable and predictable, which is often an advantage. The control of chaotic behavior has another important application, namely, the synchronization of chaotic systems. If one consider two identical chaotic systems starting from different initial conditions, then the critical sensitivity to initial conditions implies that their difference grows exponentially in time, and that they will evolve in an unsynchronized manner. The feeding of the right signal from one system to another can, however, reduce to zero such difference, and push the two systems into a synchronized manifold, wherein the chaotic motion is now developed so as the system are in step during the course of time.

1.3 Objectives and aspects

Mathematical models are formulated in understanding predator-prey interactions. The predator-prey models are constructed, including disease factors, applying harvesting strategies etc. A large number of predator-prey models are controlled using different mechanisms and ecological factors. Predator-prey dynamics are controlled using chemical as well as non chemical methods. Non chemical methods has great popularities due its eco-friendly nature. In this thesis, applications of non-chemical techniques are studied in different predator-prey systems. The main objectives of this study is to investigate the effects of additional food/alternative food in different predator-prey systems. Through out the whole thesis, the effects of additional food on predator-prey systems are investigated in the following aspects:

(i) Disease control aspect,
(ii) Chaos control aspect, and
(iii) Harvesting aspect.

1.4 Scope of the study

Dynamics of various predator-prey models are investigated in presence of additional food to predator. Formulating different mathematical models to study the dynamics of the population densities of predator and prey supplying additional food to predators. The disease control, chaos control and harvesting strategy in different
food chain models are studied in presence of additional/alternative food.

1.5 Structure of our presentation

This dissertation is presented in seven chapters. Chapter 1 gives the background to the study, Basic concepts, objectives of the study, scope of the study, significance of the study. Chapters 2-3 describe the role of additional food as a disease controller in predator-prey systems. Chapters 4-5 devote the chaos control scenarios in presence of additional food. Chapters 6-7 contain the effects of additional food to predator on harvesting in predator-prey systems. Summery of the work and possible future developments are pointed out in Chapter 8.