REPRINTS
Disease control in a food chain model supplying alternative food
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Abstract
Necessity to find a non-chemical method of disease control is being increasingly felt due to its eco-friendly nature. In this paper the role of alternative food as a disease controller in a disease induced predator–prey system is studied. Stability criteria and the persistence conditions for the system are derived. Bifurcation analysis is done with respect to rate of infection. The main goal of this study is to show the non-trivial consequences of providing alternative food in a disease induced predator–prey system. Numerical simulation results illustrate that there exists a critical infection rate above which disease free system cannot be reached in absence of alternative food whereas supply of suitable alternative food makes the system disease free up to certain infection level. We have computed the disease free regions in various parametric planes. This study is aimed to introduce a new non-chemical method for controlling disease in a predator–prey system.

1. Introduction
Mathematical models are increasingly used to guide public health policy decisions and for controlling infectious disease. Epidemic dynamics is an important method of studying the spread of infectious disease qualitatively and quantitatively. It is based on the specific property of population growth, the spread rules of infectious diseases, and the related social factors to construct mathematical models reflecting the dynamic properties of infectious diseases, to analyse the dynamical behaviour and to do some simulations. The research results are helpful to predict the developing tendency of the infectious disease, to determine the key factors of the spread of infectious disease and to seek the optimum strategies for preventing and controlling the spread of infectious diseases. Mathematical models have a long history in infectious disease ecology starting with Bernoulli’s modelling of smallpox and including Ross’s analysis of malaria. The earliest attempt to provide a quantitative understanding of the dynamics of malaria transmission was that of Ross (1911). Ross models consisted of a few differential equations to describe changes in densities of susceptible and infected people, and susceptible and infected mosquitoes. Based on his modelling, Ross introduced the concept of a threshold density and concluded that in order to control malaria anywhere we need not banish Anopheles there entirely we need only to reduce their numbers below a certain figure [3]. Classical papers of mathematical modelling of infectious disease was by Kermack and McKendrick [1927 [4], 1932 [5], and 1933 [6]]. These papers had a major influence on the development of mathematical models for disease spread and are still relevant in many epidemic situations. Aim of ecological modelling is to understand the prevalence and distribution of a species, together with the factors that determine incidence, spread, and persistence (Anderson and May [7]; May and Anderson [8]; Bascompte and Rodriguez-Trelles [9]). We now have models for many of the most important human emerging infectious diseases e.g., HIV (May and Anderson [10]), malaria (Aron and May [11]; Macdonald [12]), SARS-coronavirus (Anderson et al. [13]), rabies (Murray and Seward [14]), and influenza (Ferguson and Anderson [15]). Mathematical models...
are also being used to explore wildlife disease dynamics (Grenfell and Dobson [16]; Hudson et al. [17]) and possible routes of zoonotic disease emergence. Understanding disease dynamics across hosts is an essential first step in understanding and articulating those conditions under which new diseases can emerge from wildlife reservoirs [18]. A predator–prey system with infected prey in polluted environment is proposed by Sinha et al. [19]. Anderson and May [20] were probably the first who considered the disease factor in a predator–prey dynamics and found that the pathogen tends to destabilize the predator–prey interaction. In Rosenzweig predator–prey model, Hadeler and Freedman [21] determined a threshold above which an infected equilibrium or an infected periodic solution appear. Chattopadhyay and Arino [22] considered a three species ecoepidemiological model and studied local stability of equilibrium points, extinction criteria of species and found condition for Hopf-bifurcation in an equivalent two-dimensional model. Haque and Chattopadhyay [23] studied the role of transmissible diseases in a prey dependent predator–prey system with prey infection. Bhattacharyya et al. [24] proposed a epidemiological model with nonlinear infection incidence. In 2012, scientists [25,26] investigated infectious disease model [27–29] were done for population control in a predator–prey model providing alternative food to predator. Disease controlling aspects of predator–prey model providing alternative food to predator is unexplored. In this paper we formulate a diseased predator–prey model in presence of alternative food for predator. This is a totally new research area which has applications in many important practical fields. This paper aims to analyse the role of alternative food for controlling disease in the system.

To model the disease induced predator–prey system with alternative food to predator we do the following assumptions:

(a) In the presence of disease, the prey population consists of two sub classes, namely, the susceptible prey \( S(t) \) and infected prey \( I(t) \) and the density of the predator is denoted by \( P(t) \) at time \( t \).
(b) In presence of disease, the susceptible prey population grows logistically with intrinsic growth rate \( R_0 \) and environmental carrying capacity \( K_0 \).
(c) The susceptible prey population become infected at a rate of \( W \), following the law of mass action.
(d) Infected prey population is not in a state of reproduction and does not compete for the resource.
(e) The interaction between predator and susceptible prey is of Holling type-II and that between predator and infected prey is of Holling type-I (mass action law) in presence of alternative food. This combination of functional forms are taken because the capturing of infected prey is easier than the susceptible prey.
(f) Predators are provided with alternative food of constant biomass \( A \) which is distributed uniformly in the habitat. The constant biomass assumption is valid for many arthropod predators because they can feed on plant-provided alternative food sources such as pollen or nectar which approximately remains constant.
(g) The number of encounters per predator with the alternative food is proportional to the density of the alternative food.
(h) The proportionality constant characterizes the ability of the predator to identify the alternative food.

With the above assumptions, we formulate the following model:

\[
\begin{align*}
\frac{dS}{dt} &= R_0 S \left( 1 - \frac{S}{K_0} \right) - WSI - A_1 \frac{SP}{B_1 + \mu A + S}, \\
\frac{dI}{dt} &= WSI - A_2 \frac{IP}{B_1 + \mu A} - D_1 I, \\
\frac{dP}{dt} &= A_3 \frac{(S + \mu A)P}{B_1 + \mu A + S} + C_2 A_2 \frac{IP}{B_1 + \mu A} - D_2 P.
\end{align*}
\]

(1)

The constants \( A_1, A_2 \) are maximal predation rate of predator for susceptible prey and infected prey, respectively. The terms \( C_1, C_2 \) are conversion rates of susceptible prey and infected prey to predator, respectively; \( B_1 \) is the half saturation constant for predator; \( D_1 \) and \( D_2 \) are constant death rates for \( I \) and \( P \), respectively. If \( h_1 \) represents the handling time of the predator per prey and \( h_3 \) represents the handling time of the predator per unit quantity of alternative food, then \( x = h_2/h_1 \). If the constant \( e_1, e_2 \) represent ability of the predator to detect the prey item and to detect alternative food, respectively, then \( \mu = e_2/e_1 \). The term \( \mu A \) represents effectual alternative food for the predator.

We nondimensionalize the system (1) using \( s = \frac{S}{S^*}, i = \frac{I}{I^*}, p = \frac{P}{P^*} \) and \( t = R_0 T \) and obtain the following system

\[
\begin{align*}
\frac{ds}{dt} &= s(1 - s) - \gamma is - \frac{asp}{1 + \omega s + bs}, \\
\frac{di}{dt} &= \gamma is - \frac{ip}{1 + \omega i} - di, \\
\frac{dp}{dt} &= \epsilon(s + \omega i)p + \eta ip \left( \frac{1}{1 + \omega i} - ep. \right)
\end{align*}
\]

(2)
where \( \gamma = \frac{Wy}{R_0}, a = \frac{d}{d + a}, \xi = \frac{d}{d + a}, b = \frac{d}{d + a}, \beta = \frac{d}{d + a}, \epsilon = \frac{d}{d + a}, \eta = \frac{d}{d + a}, d = \frac{d}{d + a} \) and \( e = \frac{d}{d + a} \). We analyse the system dynamics assuming \( s(t) \geq 0, i(t) \geq 0 \) and \( p(t) \geq 0 \).

The constant \( \gamma \) represents the infection rate of the prey. The terms \( x \) and \( \xi \) are the parameters which characterize the alternative food. From the relation \( x = h_2/h_1, \) it can be inferred that \( x \) is directly proportional to the handling time \( h_1 \) of the alternative food. Hence the parameter \( x \) is proportionally related to the “quality” of the alternative food. If the relation \( h_2 < h_1 \) holds, then the predator can easily capture alternative food \( x \) prey species which implies that the alternative food is of high quality. Therefore for high quality of alternative food \( x \) is less than 1. Again, from the relation \( \xi = \frac{\omega}{x}, \) it can be inferred that \( \xi \) is directly proportional to the biomass of the alternative food \( (A) \) and thus \( \xi \) is a representative of the “quantity” of the alternative food that is supplied to predator [28].

This paper is organized as follows: in Section 2, the existence condition for equilibrium points of the system are derived. The stability analysis of equilibrium points are done in Section 3. Persistence conditions of the system are derived in Section 4. We have analysed the dynamics of this model through bifurcation analysis with respect to the rate of infection and quantity of alternative food in Section 5. Finally, we draw conclusions in Section 6.

2. Theoretical study

In this section, positivity and boundedness for the system (2) are established. Since the state variables \( s, i \) and \( p \) represent population size, positivity implies that they never become negative. The boundedness may be interpreted as a natural restriction to growth as a consequence of limited resources.

2.1. Positive invariance

The system (2) can be written as \( \dot{X} = F(X) \) with \( X(0) = X_0 \in \mathbb{R}^3, \) where \( X = (s, i, p)^T \in \mathbb{R}^3 \) and \( F(X) \) is given by

\[
F = F(X) = \left( \begin{array}{c}
s(1 - s) - \gamma is - \frac{dp}{s + c} \\
\gamma is - \frac{dp}{s + c} - di \\
\epsilon s (i + c) + \frac{ep}{s + c}
\end{array} \right),
\]

where \( F : \mathbb{R} \to \mathbb{R}^3 \) and \( F \in C^\omega(\mathbb{R}^3). \)

It can be shown that whenever \( X(0) \in \mathbb{R}^3, \) such that \( X_i = 0 \) then \( F_i(X)|_{X_i=0} \geq 0 \) (for \( i = 1, 2, 3 \)). Now any solution of \( \dot{X} = F(X) \) with \( X_0 \in \mathbb{R}^3, \) say \( X(t) = X(t, X_0), \) is such that \( X(t) \in \mathbb{R}^3 \) for all \( t > 0 \) (Nagumo [30]).

2.2. Boundedness

**Theorem 1.** All solutions of the system (2) which initiated in \( \mathbb{R}^3, \) are uniformly bounded.

**Proof.** Let \( (s(t), i(t), p(t)) \) be any solution of the system (2) with positive initial condition.

Let us consider that, \( w = s + i + p \)

\[
\frac{dw}{dt} = \frac{ds}{dt} + \frac{di}{dt} + \frac{dp}{dt}.
\]

Using Eq. (2), we have

\[
\frac{dw}{dt} = s(1 - s) - \frac{(a - c)p}{1 + \alpha c} - \frac{p}{s + c} - \frac{(\beta - \eta)p}{1 + \alpha c} - di - ep.
\]

Therefore, \( \frac{dw}{dt} \leq -(s + i + p) - (1 - s)^2 + 1, \) where \( \theta = \min \{1, d, e - \epsilon \xi \}. \)

Thus, \( \frac{dw}{dt} + \theta w \leq 1 - (1 - s)^2, \)

That is, \( \frac{dw}{dt} + \theta w \leq 1, \) since \( (1 - s)^2 \geq 0. \)

Applying the theory of differential inequality we obtain

\[
0 < w < \frac{1 - e^{-\theta}}{\theta} + w(s(0), i(0), p(0))e^{-\theta}.
\]

For \( t \to \infty, \) we have \( 0 < w < \frac{1}{\theta}. \)

Hence all the solutions of (2) that initiate in \( \mathbb{R}^3, \) are confined in the region.

\[
B = \{ (s, i, p) \in \mathbb{R}^3 : w = \frac{1}{\theta} + t, \text{ for any } t > 0 \}.
\]

This proves the theorem.
2.3. Extinction criterion of $s$ and $i$

Lemma 1. If $1 \leq \gamma i(t)$, then $\lim_{t \to \infty} s(t) = 0$. If $\gamma s(t) \leq \frac{\beta p(t)}{1 + \alpha n}$, then $\lim_{t \to \infty} i(t) = 0$.

Proof. \[ \frac{ds}{dt} = s(1 - s) - \gamma is - \frac{\alpha p}{1 + \alpha n} \leq s(1 - \gamma i). \]
Therefore, \( s(t) \leq s(t_0)\exp\left(\int_{t_0}^{t} (1 - \gamma i(t))\,dt\right). \)
Hence \( \lim_{t \to \infty} s(t) = 0 \), provided \( 1 \leq \gamma i(t). \)
\[ \frac{di}{dt} = \gamma is - \frac{\beta p}{1 + \chi n} - di \leq i(\gamma s - \frac{\beta p}{1 + \chi n}). \]
Thus, \( i(t) \leq i(t_0)\exp\left(\int_{t_0}^{t} (\gamma s(t) - \frac{\beta p(t)}{1 + \chi n})\,dt\right). \)
Hence \( \lim_{t \to \infty} i(t) = 0 \), provided \( \gamma s(t) \leq \frac{\beta p(t)}{1 + \chi n} \), i.e., \( 0 \leq \gamma \leq \frac{\beta p(t)}{\alpha n}. \)

2.4. Existence of equilibrium points

The system (2) possesses the following equilibrium points.

(i) The trivial equilibrium point $E_T = (0, 0, 0)$.

(ii) The axial equilibrium point $E_A = (1, 0, 0)$.

(iii) The disease free boundary equilibrium point is $E_D = (\hat{s}, 0, \hat{p})$, where $\hat{s} = \frac{1 - \gamma i}{\gamma s - \frac{\beta p}{1 + \chi n}}$, \( \hat{p} = \frac{1 - \gamma i}{\gamma s - \frac{\beta p}{1 + \chi n}}(1 + \alpha \xi - \beta \zeta) \). $E_D$ exists if $\gamma > \frac{\beta p}{1 + \chi n}$ and $\gamma \leq \frac{\beta p}{\alpha n} - \beta e > \frac{\beta p}{1 + \chi n}$. Therefore, the existence criteria of disease free boundary equilibrium point is $\frac{\beta p}{1 + \chi n} < \frac{\beta p}{\alpha n} - \beta e$.

(iv) The predator free equilibrium point $E_P = (\hat{s}, \hat{i}, 0)$, where $\hat{s} = \frac{1}{\gamma}$ and $\hat{i} = \gamma - \hat{d}$. The predator free equilibrium point $E_P$ exists if $\gamma > \frac{\gamma i}{\gamma s - \frac{\beta p}{1 + \chi n}}$, i.e., if infection rate is greater than the death rate of infected prey.

(v) The steady state of coexistence $E^* = (s^*, i^*, p^*)$, where $p^* = \frac{1 - \gamma i}{\beta p - \alpha n}$, $i^* = \frac{1 - \gamma i}{\beta p - \alpha n} - \beta e$. The existence condition of $E^*$ is $\frac{\beta p}{1 + \chi n} > \frac{\beta p}{\alpha n} - \beta e$.

The sufficient conditions of the existence of $E^*$ in the interior of the first octant are easily obtained as follows: $T_2^2 - 4T_1 T_3 \geq 0, d > \gamma s$ and $e > \frac{1 - \gamma i}{\beta p - \alpha n}$.

Therefore, the values of $E_T, E_A$ and $E^*$ depend on the quality ($\alpha$) and quantity ($\xi$) of alternative food. The existence conditions of $E_D, E_P$ and $E^*$ also depend on $\alpha$ and $\xi$.

3. Local stability analysis

Theorem 2. The trivial equilibrium point $E_T$ is always unstable. The axial equilibrium point $E_A$ is unstable if $\gamma > \frac{\gamma i}{\gamma s - \frac{\beta p}{1 + \chi n}}$.

Proof. The Jacobian matrix $J(E_T)$ at $E_T$ is given by
\[
J(E_T) = \begin{pmatrix}
1 & 0 & 0 \\
0 & -d & 0 \\
0 & \alpha \xi - \beta e
\end{pmatrix}
\]
We observe that $J(E_T)$ has one positive eigenvalue 1, therefore $E_T$ is always unstable.

The Jacobian matrix $J(E_A)$ at $E_A$ is given by
\[
J(E_A) = \begin{pmatrix}
-1 & -\gamma & \frac{\gamma i}{\gamma s - \frac{\beta p}{1 + \chi n}} \\
0 & \gamma - \hat{d} & 0 \\
0 & \frac{\alpha \xi}{\gamma s - \frac{\beta p}{1 + \chi n}} - \beta e
\end{pmatrix}
\]
From the Jacobian matrix $J(E_0)$, it is observed that it has one negative eigenvalue $-1$ and two positive eigenvalues if $\gamma > d$ and $\epsilon < \frac{(\gamma - d)}{\gamma - d}$. That means it has stable and unstable manifold in the neighborhood of $E_0$. Hence the axial equilibrium point is unstable for $\gamma > d$ or $\epsilon < \frac{(\gamma - d)}{\gamma - d}$.

**Theorem 3.** The disease free equilibrium point $E_0$ for the system (2) is locally stable if the following conditions hold: $\Theta_1 > 0, \Theta_3 > 0$ and $\Theta_1 \Theta_2 - \Theta_3 > 0$, where $\Theta$'s are given in the proof of the theorem.

**Proof.** The Jacobian matrix $J(E_0)$ at disease free equilibrium point $E_0$ is given by

$$J(E_0) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where, $a_{11} = 1 - \gamma s - \frac{\alpha(1+\theta_1)}{1+\alpha+c}, a_{12} = -\gamma s, a_{13} = \frac{\alpha}{1+\alpha+c}, a_{21} = 0, a_{22} = \gamma s - \frac{\alpha}{1+\alpha+c} - d, a_{31} = 0, a_{32} = \frac{\alpha}{1+\alpha+c} - d, a_{33} = \frac{(\gamma - d)}{\gamma - d} - \epsilon.$

The characteristic equation of the Jacobian matrix $J(E_0)$ is given by

$$\lambda^3 + \Theta_1 \lambda^2 + \Theta_2 \lambda + \Theta_3 = 0,$$

where,

$$\Theta_1 = -(a_{11} + a_{22} + a_{33}),$$

$$\Theta_2 = (a_{23} a_{31} - a_{21} a_{33} - a_{13} a_{11} + a_{11} a_{22}),$$

and $\Theta_3 = a_{21} (a_{33} - a_{13} a_{11}).$

If $a_{11} + a_{22} < 0, a_{33} < 0$ and $a_{23} a_{31} < a_{13} a_{11}$ then it is easy to see that $\Theta_1 > 0, \Theta_3 > 0$ and $\Theta_1 \Theta_2 - \Theta_3 > 0$. Using the Routh–Hurwitz criteria [31] we observe that the system (2) is stable at the positive equilibrium point $E_0$ if the conditions $\Theta_1 > 0, \Theta_3 > 0$ and $\Theta_1 \Theta_2 - \Theta_3 > 0$ hold. Hence the disease free system is locally stable under these conditions. But, the disease free equilibrium $E_0$ is unstable if at least one of these conditions is violated.

**Theorem 4.** The predator free equilibrium point $E_p$ is locally stable if $\frac{(\gamma - d)}{\gamma - d} + \frac{\alpha}{1+\alpha+c} < \epsilon$ and $(\gamma - 2)\hat{s} < \gamma \hat{i} + d - 1.$

**Proof.** The Jacobian matrix $J(E_p)$ at $E_p(\hat{s}, \hat{i}, 0)$ is given by

$$J(E_p) = \begin{pmatrix} 1 - 2\hat{s} - \hat{i} & -\hat{i} & -\frac{\alpha}{1+\alpha+c} \\ \hat{i} & \hat{i} & \gamma s - d - \frac{\alpha}{1+\alpha+c} \\ 0 & 0 & \frac{(\gamma - d)}{\gamma - d} + \frac{\alpha}{1+\alpha+c} - \epsilon \end{pmatrix}.$$

The characteristic roots of the Jacobian matrix $J(E_p)$ are $\frac{(\gamma - d)}{\gamma - d} + \frac{\alpha}{1+\alpha+c} - \epsilon$ and the roots of the equation $\lambda^3 - (\gamma - 2)\hat{s} - \gamma \hat{i} + d + 1|\lambda + (\gamma s - d)(1 - 2\hat{s} - \hat{i}) = 0.$

Hence $E_p$ is locally stable if the conditions given in the theorem are satisfied.

**Theorem 5.** The interior equilibrium point $E^*$ for the system (2) is locally stable if the conditions $\Omega_1 > 0, \Omega_3 > 0$ and $\Omega_1 \Omega_2 - \Omega_3 > 0$ hold, where $\Omega$'s are given in the proof of the theorem.

**Proof.** The Jacobian matrix $J(E^*)$ at the interior point $E^*(v, \hat{f}, \hat{p})$ is

$$J(E^*) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

where, $A_{11} = 1 - 2v - \hat{f} - \frac{\alpha(1+\theta_1)}{1+\alpha+c}, A_{12} = -\gamma s, A_{13} = \frac{\alpha}{1+\alpha+c}, A_{21} = 0, A_{22} = \gamma s - \frac{\alpha}{1+\alpha+c} - d, A_{31} = 0, A_{32} = \frac{\alpha}{1+\alpha+c} - d, A_{33} = \frac{(\gamma - d)}{\gamma - d} - \epsilon.$

The characteristic equation of the Jacobian matrix $J(E^*)$ is given by

$$\lambda^3 + \Omega_1 \lambda^2 + \Omega_2 \lambda + \Omega_3 = 0.$$
where,
\[ \Omega_1 = -(A_{11} + A_{22} + A_{33}), \]
\[ \Omega_2 = (A_{22}A_{33} - A_{23}A_{32}) + (A_{11}A_{33} - A_{22}A_{31}) + (A_{11}A_{22} - A_{12}A_{21}), \]
and \[ \Omega_3 = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{22}A_{31} - A_{23}A_{32}) + A_{13}(A_{22}A_{32} - A_{23}A_{31}). \]

Using the Routh–Hurwitz criteria [31] we observe that the system (2) is locally stable at the equilibrium point \( E^1 \) if the conditions \( \Omega_1 > 0, \Omega_2 > 0 \) and \( \Omega_1 \Omega_2 - \Omega_3 > 0 \) hold.

Therefore, we observe that the stability conditions for all the equilibrium points depend on the parameters \( \xi \) and \( z \).

4. Persistence of the system

Butler et al. [32], Freedman and Waltman [31,33] developed the following definition of persistence:

If a population \( N(t) \) is such that \( N(t) > 0 \), then

(i) \( N(t) \) is said to be weakly persistent if \( \lim_{t \to \infty} \text{sup} N(t) > 0 \).

(ii) \( N(t) \) is said to be (strongly) persistent if \( \lim_{t \to \infty} \text{inf} N(t) > 0 \).

Further, if \( N(t) \in C \), where \( C \) is a certain class of function, then

(iii) \( N(t) \) is said to be weakly uniformly persistent if there exists \( \delta > 0 \) such that \( \lim_{t \to \infty} \text{sup} N(t) > \delta \) for all \( N(t) \in C \).

(iv) \( N(t) \) is said to be uniformly persistent if there exists \( \delta > 0 \) such that \( \lim_{t \to \infty} \text{inf} N(t) > \delta \) for all \( N(t) \in C \).

From biological point of view, persistence of a system means the long term survival of all populations of the system, no matter what the initial populations are. Mathematically, persistence of a system means that strictly positive solutions do not have omega (Ω) limit points on the boundary of the non-negative cone.

Theorem 6. Let the following conditions are satisfied

(i) \( \gamma > d \),

(ii) \( \epsilon < \frac{\gamma^{1+r}}{d^{1+r}} \),

(iii) \( \frac{\gamma^{1+r}}{d^{1+r}} + \frac{r}{1+r} > \epsilon \),

and if there exists a finite number (say, \( n \)) of periodic solutions \( s = \phi_i(t), p = \psi_j(t), r = 1, 2, 3, \ldots, n \), in the \( s - p \) plane, then system (2) is uniformly persistent provided for each periodic solutions of period \( T_{s,p} = -d + \frac{1}{d} \left( \gamma \phi_i(t) - \frac{\phi_i(t)}{T_{s,p}} \right) dt > 0, r = 1, 2, \ldots, n \).

Proof. Let \( X = (s, i, p) \) be a point in the positive quadrant and \( O(X) \) be orbit through \( X \) and \( \Omega \) be the omega limit set of the orbit through \( X \). Note that \( \Omega(X) \) is bounded.

We claim that \( E_1 \notin \Omega(X) \). If \( E_1 \in \Omega(X) \) then by the Butler-McGehee lemma [31] there exist a point \( P \) in \( \Omega(X) \cap W^s(E_1) \) where \( W^s(E_1) \) denotes the stable manifold of \( E_1 \). Since \( O(P) \) lies in \( \Omega(X) \) and \( W^s(E_1) \) is the \( i - p \) plane, we conclude that \( O(P) \) is unbounded, which is contrary to the boundedness of the system.

Next \( E_2 \notin \Omega(X) \), for otherwise, since \( E_2 \) is a saddle point which follows from the conditions \( \gamma > d \) and \( \epsilon < \frac{\gamma^{1+r}}{d^{1+r}} \), by the Butler-McGehee lemma [31] there exist a point \( P \) in \( \Omega(X) \cap W^u(E_2) \). Now \( W^u(E_2) \) is the \( s \)-axis implies that an unbounded orbit lies in \( \Omega(X) \), a contradiction.

The condition \( \frac{\gamma^{1+r}}{d^{1+r}} + \frac{r}{1+r} > \epsilon \) implies that \( E_2 \) is unstable and therefore \( E_2 \notin \Omega(X) \).

Lastly, we show that no periodic orbits in the \( s - p \) plane or \( E_0 \in \Omega(X) \). Let \( r_i, i = 1, 2, \ldots, n \) denote the closed orbit of the periodic solution \( (\phi_i(t), \psi_i(t)) \) in \( s - p \) plane such that \( r_i \) lies inside \( r_{i-1} \). Let the Jacobian matrix \( J_i(\phi_i(t), 0, \psi_i(t)) \) corresponding to \( r_i \) is given by

\[
J_i = \begin{pmatrix}
F_{1i}(\phi_i(t), 0, \psi_i(t)) & F_{2i}(\phi_i(t), 0, \psi_i(t)) & F_{3i}(\phi_i(t), 0, \psi_i(t)) \\
0 & F_{2i}(\phi_i(t), 0, \psi_i(t)) & F_{3i}(\phi_i(t), 0, \psi_i(t)) \\
F_{3i}(\phi_i(t), 0, \psi_i(t)) & F_{3i}(\phi_i(t), 0, \psi_i(t)) & F_{3i}(\phi_i(t), 0, \psi_i(t))
\end{pmatrix}.
\]

Here \( F_2 = -d + \gamma \phi_i(t) - \frac{\phi_i(t)}{T_{s,p}} \). Computing the fundamental matrix of the linear periodic system

\[
M' = J_i(t)M, M(0) = M_0,
\]

we find that its Floquet multiplier in the \( i \)-direction is \( e^{\gamma(i)} \). Then proceeding in an analogous manner like Kumar and Freedman [34], we conclude that no \( r_i \) lies on \( \Omega(X) \). Thus, \( \Omega(X) \) lies in the positive quadrant and system (2) is persistent. Finally,
since only the closed orbits and the equilibria from the omega limit set of the solutions on boundary of $R^3$, and the system (2) is dissipative. Now using a theorem of Butler et al. [32], we conclude that system (2) is uniformly persistent.

Note that the persistence condition of the system depends on infection rate, death rate of infected prey, death rate of predator, capturing rate of predator on infected prey and quality and quantity of alternative food.

5. Results and discussions

In this section, we perform numerical simulations with the help of parameter values taken from data of field works and experimental data (see, Table 1) which remains fixed for all numerical simulations. In our numerical simulations we have varied the infection rate $\gamma$, quality of alternative food $a$ and quantity of alternative food $\xi$.

The bifurcation diagram of the system (2) with respect to infection rate of prey in the range $0 \leq \gamma \leq 1.74$ in absence of alternative food (i.e., $a = 0, \xi = 0$) to predator is presented in Fig. 1. Fig. 1 shows that within $0 \leq \gamma \leq 0.76$, there is no infected prey species in the system and the susceptible prey and predator species have periodic behaviour. But, for $0.76 < \gamma \leq 1.21$, the susceptible prey, infected prey and predator species have limit cycle oscillations. All prey and predator species settle down to their respective steady states after $\gamma > 1.21$. Notice that the average density of the infected prey species becomes higher than the average density of susceptible prey for $\gamma > 0.76$. With the increase of infection rate susceptible prey becomes infected at higher rate and as a result the density of susceptible prey population decreases and the density of infected prey population increases.

The bifurcation diagram of the system (2) with respect to infection rate $\gamma$ in the range $0 \leq \gamma \leq 1.74$ in presence of alternative food ($a = 0.85, \xi = 0.7$) to predator is plotted in Fig. 2. It is evident from Fig. 2 that the system becomes disease free within $0 \leq \gamma \leq 0.9$ in presence of alternative food. Therefore, supply of alternative food to predator makes the system disease free in $0.76 \leq \gamma \leq 0.9$ which is impossible in absence of alternative food. The infected prey population oscillates within $0.9 < \gamma < 1.6$ and it reaches steady state after $\gamma > 1.6$. The susceptible prey and predator species have periodic behaviour for $0 \leq \gamma \leq 1.6$ and finally they settle down to their respective steady state after $\gamma > 1.6$.

We plot dynamical behaviours of the infected prey population in absence of alternative food as well as in presence of alternative food in Fig. 3. Time evolution of infected prey is plotted in Fig. 3(a) in absence of alternative food taking $\gamma = 0.8$, on the other hand Fig. 3(b) represents time evolution of the infected prey species for same infection rate in presence of alternative food ($a = 0.85, \xi = 0.6$). Fig. 3(c) depicts the time evolution of infected prey without alternative food with infection rate $\gamma = 1$ whereas Fig. 3(d) represents the disease free dynamics for ($a = 0.85, \xi = 0.7$) with same infection rate.

Table 1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Description</th>
<th>Range</th>
<th>Default value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Attack rate of susceptible prey</td>
<td>0.61–6.087 [35]</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>Attack rate of infected prey</td>
<td>0.05–0.33 [38]</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td>Conversion rate of susceptible prey</td>
<td>0.05–0.33 [38]</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>$d$</td>
<td>Conversion rate of infected prey</td>
<td>0.05–0.33 [38]</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>$e$</td>
<td>Conversion rate of additional food</td>
<td>0 &lt; $\gamma$ &lt; 1</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>$f$</td>
<td>Infection rate</td>
<td>0.04–0.16 [39,40]</td>
<td>0.08</td>
<td></td>
</tr>
<tr>
<td>$g$</td>
<td>Predator mortality rate</td>
<td>0.04–0.16 [39,40]</td>
<td>0.04</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 1. Bifurcation diagram of the system (2) with respect to infection rate of the prey in absence alternative food i.e., for $a = 0$ and $\xi = 0$. 

Fig. 2. Bifurcation diagram of the system (2) with respect to infection rate of the prey in presence alternative food.
From the Fig. 3, it is clear that the system becomes disease free under suitable supply of alternative food even if the infection rate is high which is impossible in absence of alternative food.

In Fig. 4, the plane is divided into infected and uninfected regions for fixed infection rate $c$. We have determined the regions for $c = 0.8$ (Fig. 4(a)) and for $c = 1$ (Fig. 4(b)) considering other parameters values are as in Table 1. Fig. 4(a) is plotted in the $\gamma$ plane parametric plane for $0 < a < 2.5$ and $0 < \xi < 6.3$ with constant infection rate $\gamma = 0.8$, whereas Fig. 4(b) divides the $\gamma$ plane into infected and uninfected regions for $0 < \gamma < 2.5$ and $0 < \xi < 7.7$ with constant infection rate $\gamma = 1$. We observe from Fig. 4 that it is impossible to make the system disease free in absence of alternative food for higher infection rate $\gamma (> 0.76)$ but disease free system can be achieved by suitable supply of alternative food. It is obvious that disease free solution exists for high quality of alternative food even if it is supplied in small quantity. On the other hand if alternative food is of low quality then high quantity of alternative food may not efficient to make the system disease free. Therefore high quality of alternative food supply is very efficient for disease control.

The $\gamma$ and $\gamma \times$-planes are divided into infected and uninfected regions which are presented in Fig. 5 for $a = 0.85$ and in Fig. 5(b) for $\xi = 0.62$, respectively. Fig. 5(a) shows that the system becomes disease free for small quantity of alternative food for lower infection rate $\gamma (< 0.5)$. But for higher infection rate, the supply level $\xi$ increases depending on the increase of infection rate. There exists a critical infection rate above which alternative food may not be useful to make the system disease free. It is clear from Fig. 5(b) that disease free state can be reached for higher infection rate ($\gamma > 0.76$) supplying high quality of alternative food. It is evident from Fig. 5(b) that for lower infection rate quality of alternative is not so important for obtaining disease free system.
6. Conclusions

We examine some nontrivial consequences of alternative food to predator in an infected predator–prey model. We have studied stability and persistence criterion of the system in presence of alternative food. One parameter bifurcation analysis is done with respect to infection rate using experimental and field data. Numerical simulation results predict that the system becomes disease free for lower infection rate \((0 < \gamma \leq 0.76)\) in absence of alternative food (Fig. 1). This is happening because infected prey is weaker than susceptible prey and is easily captured by predator. This will reduce the number of infected prey and ultimately the system may become disease free at certain stage for lower infection rate. However, for higher infection rate \((\gamma > 0.76)\), disease free system cannot be obtained without alternative food.

Fig. 4. Graph of \(a\) versus \(\xi\) for fixed (a) \(\gamma = 0.8\) and (b) \(\gamma = 1\). Parameters are as in Table 1. The \(a\xi\)-plane is divided into infected and uninfected regions. This figure indicates that for high quality of alternative food, small quantity of it is sufficient to make the system disease free.

Fig. 5. Graph of \(a\) versus \(\xi\) for fixed \(a = 0.85\) and (b) \(\gamma\) versus \(\xi\) for fixed \(\gamma = 0.62\). Parameters are as in Table 1. The \(\gamma\xi\) and \(\gamma\alpha\)-planes are divided into infected and uninfected regions. The figure (a) gives the minimum values of \(\xi\) for different infection rate \(\gamma\) and (b) depicts the values of \(a\) for different infection rate \(\gamma\) to make the system disease free.
(Fig. 3). Whereas for higher infection rate disease free state may be reached by supplying suitable alternative food (Figs. 4 and 5). It is evident from our study that high quality of alternative food ($\alpha < 1$) has the capability of making the system disease free for higher infection rate. Quantity of alternative food supply increases with the increase of infection rate for fixed quality to obtain a disease free system. Possible reason behind it is that with the supply of high quality or high quantity of alternative food, the predator catches the alternative food as well as infected prey at higher rate compare to susceptible prey. As a result, the growth rate of predator species will increase with the supply of high quality or high quantity of alternative food and it captures the infected prey population at faster rate than susceptible prey. Due to this, infected prey population become very very small at certain stage and consequently infection cannot spread at that stage and the system becomes disease free. Notice that there is a critical infection rate above which the alternative food may not be useful to make the system disease free. Results presented in the present paper provided a useful platform to understand the role of alternative food as disease controller in a diseased food chain model. A possible interesting aspects for future developments are to study the role of alternative food as a disease controller in different types of disease induced food chain models and also the study the effects of supplying alternative food to a network of infected food chain models. This new non-chemical method of disease control will be very useful for biological conservation of prey species in real world biological systems.

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References

Effects of additional food on an ecoepidemic model with time delay on infection

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ABSTRACT

We propose a predator–prey ecoepidemic model with parasitic infection in the prey. We assume infection time delay as the time of transmission of disease from susceptible to infectious prey. We examine the effects of supplying additional food to predator in the proposed model. The essential theoretical properties of the model such as local and global stability and in addition bifurcation analysis is done. The parameter thresholds at which the system admits a Hopf bifurcation are investigated in presence of additional food with non-zero time lag. The conditions for permanence of the system are also determined in this paper. Theoretical analysis results are verified through numerical simulations. By supplying additional food we can control predator population in the model. Most important observation is that we can control parasitic infection of prey species by supplying additional food to predator. Eliminating the most infectious individuals from the prey population, predator quarantine the infected prey and prevent the spreading of disease.

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1. Introduction

Over the last 50 years, disease control has relied heavily on the use of chemical fungicides, bactericides and soil fumigants. Use of chemical to control disease has proved to have long term side effects on ecosystem viz. infected individuals contaminate the environment, further they get absorbed and remain as residue in the biomass of coexisting healthy species in the ecosystem. Therefore, non-chemical methods of disease control become a topic of great attention for many scientists. It is well-known that the non-prey food sources has long been recognised and attempts have been made to manipulate these non-prey sources (viz., nectar, pollen etc.) in agricultural lands to enhance levels of biological disease control [1,2]. Several experiments [3,4] reported the benefits of using additional or alternative food supplements in biological control programs. Wackers and van Rijn [3] pointed out that additional food in plant–herbivore–carnivore interactions is not only an important topic in basic ecology, but also directly applied for biological pest control. It is observed that quality and quantity of additional food play a vital role in the controllability of the system. These findings agree with the observations made in the recent review [4] dealing with the effects of artificial food sprays on conservational biological control wherein it is emphasised that the success of biological control much depends on quantity and quality of food sprays.

Additional foods (to predator) help to increase predators [5]. Harwood and Obrzycki [6] pointed out that provision of alternative food to a generalist predator is twofold: on one hand these nutritious food items improve the predator population and on the other results reduction of prey consumption per individual predator. Thus studying the effects of additional food on a diseased predator–prey system when the predators are provided with additional food is important, which is the focus of the

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current study. In recent decades significant progress has occurred in the theory and application of epidemiology modelling of predator–prey systems [7–8] in absence of alternative food resource. The first eco-epidemic model with alternative food for the predator has been introduced by Haque and Greenhalgh [9]. Recently, Sahoo and Poria [10] reported the effects of additional food to predator in a diseased predator–prey system. They reported that increasing predator population by supplying additional food, one can remove infection from the system. But in the considered system [10] the infection process is instantaneous. Actually, in reality, an infected prey contacts a susceptible prey, the latter becomes infectious and therefore there is an infection time [11]. In the present paper, we investigate the effects of additional food on a epidemic predator–prey model with infection time delay.

Habitat complexity is the structural complexity of habitats. Habitat complexity can strongly mediate predator–prey interactions, affecting not only total predation rates, but also modifying selectivities for different prey species or size classes [12–14]. It is well established that habitat complexity reduces encounter rates of predators with prey [15,16]. Aquatic habitat becomes structurally complex in presence of submerged vegetation or aquatic weeds. It is observed that structural complexity of the habitat stabilizes the predator–prey interaction between piscivorous perch (predator) and juvenile perch and roach (prey) by reducing predator foraging efficiency. Therefore, it is important to incorporate the effect of habitat complexity when predator–prey interaction is studied by means of theoretical models.

Some field experiments reported that predator removal is one of the most important cause of disease spreading in prey populations [17,18]. Sih et al. [17] documented predator-removal experiments and found 54 out of 135 systems in which predator removal reduced the prey population. Similarly, Cote and Sutherland [18] found that predator removal reduced prey populations in 3 of 11 laboratory studies. If predators eliminate the most infectious individuals from the prey population, they will have an outcome equivalent to quarantine—whereby infectious individuals are removed from the healthy population and thereby prevented from spreading disease [19]. The above facts motivate us to investigate the effects of predator control on a diseased predator–prey model. Predator population is controlled by supplying additional food to predator in our model.

The paper is organised as follows: In the next section, an infected predator–prey model with habitat complexity in presence of additional food to predator is proposed. The effects of time delay in infection is incorporated. Section 3 contains the preliminary properties of the model. The conditions for the existence of various equilibria, local stability and conditions for existence of Hopf bifurcations are investigated in Section 4. The conditions for permanence of the system is presented in Section 5. Section 6 illustrates some of the key findings through numerical simulations. Section 7 highlights the consequence of providing additional food to predator for disease control. Finally, we draw conclusions in Section 8.

2. The model

The most commonly used functional response in a predator–prey model is the Holling Type II functional response [20]. The Holling type-II functional response is defined as

\[ f(X) = \frac{A_1X}{B_1 + X}, \]

where \( f(X) \) is the amount of food consumed, \( X \) is the amount of food offered, \( A_1 \) is a proportionality constant related to the attack rate. Since the existence of habitat complexity reduces the probability of capturing a prey by reducing the searching efficiency of predator and habitat complexity affects the attack coefficient [21]. Therefore, the attack coefficient \( A_1 \) has to be replaced by \( A_1(1 - c) \), where \( c \) \( (0 < c < 1) \) is a dimension less parameter that measures the strength of habitat complexity. We assume that the complexity is homogeneous throughout the habitat. Then following Kot [22], the total number of prey caught \( V \), is given by

\[ V = A_1(1 - c)T_sX, \]

where \( T_s = T - hV \).

Here \( T \) is the total time, \( T_s \) is the available searching time. Solving for \( V \) we get the modified Holling type-II response function as

\[ V = \frac{TA_1(1 - c)X}{B_1 + (1 - c)X}. \]

Since, predator’s functional response is defined as the number of prey caught by a predator at unit time, so the functional response in presence of habitat complexity is given by

\[ f(X) = \frac{A_1(1 - c)X}{B_1 + (1 - c)X}. \]

Notice that for \( c = 0 \), there is no complexity, we get the original Holling Type II response function. Assuming density-dependent logistic growth of prey with intrinsic growth rate \( K_0 \), a predator–prey model in presence of habitat complexity is given by:

\[
\begin{align*}
\frac{dX}{dT} &= R_0 X \left( 1 - \frac{X}{K_0} \right) - \frac{A_1(1 - c)XY}{B_1 + (1 - c)X}, \\
\frac{dY}{dT} &= cA_1(1 - c)XY - D_2Y. 
\end{align*}
\]  

(1)
where $K_0$ is the carrying capacity of the prey in ecosystem and $c$ is the conversion rate of prey into the predator, $D_2$ is the mortality rate of predator.

Now, we modify the model (1) by supplying “additional food” to predator. We make the following assumptions:

(a) The predator is provided with additional food of constant biomass $A$ which is assumed to be distributed uniformly in the habitat. The constant biomass assumption is valid for many arthropod predators because they can feed on plant-provided alternative food sources such as pollen or nectar which approximately remains constant.

(b) The number of encounters per predator with the additional food is proportional to the density of the additional food.

The proportionality constant characterizes the ability of the predator to identify the additional food [10].

Therefore, the functional response incorporating the effects of additional food is of the form:

$$f(X) = \frac{A_1(1-c)X}{B_1 + a\mu A + (1-c)X}.$$  

If $h_1$ and $e_1$ are two constants representing handling time of the predator $y$ per prey item and ability of the predator to detect the prey respectively. If $\beta$ represents the efficiency with which the food consumed by predator gets converted into the predator biomass, then the constant $c$ (the maximum growth rate of the predator) is given by $\beta/h_1$. Ultimately, the quality of additional food depends on handling time of the predator upon the food. So, in this case, the quality of additional food will be the ratio between predator’s handling time towards additional food and predator’s handling time per prey item. If $h_2$ and $e_2$ respectively, represent the handling time of the predator per unit quantity of additional food and ability for the predator to detect the additional food, then we have $\mu = e_2/e_1$, and $\sigma = h_2/h_1$ [23,24].

Therefore, supplying additional food to predator, the model (1) becomes

$$\begin{align*}
\frac{dX}{dt} &= R_0X \left(1 - \frac{X}{K_0}\right) \frac{A_1(1-c)XY}{B_1 + a\mu A + (1-c)X}, \\
\frac{dY}{dt} &= \epsilon A_1 \left(1 - \frac{X}{Y}\right) + \mu \frac{A_1(1-c)XY}{B_1 + a\mu A + (1-c)X} - D_2Y.
\end{align*}$$

To model the disease induced prey-predator system with additional food to predator we make the following assumptions:

(a) In the presence of disease, the prey population($X$) consists of two sub classes, namely, the susceptible prey $S(t)$ and infected prey $I(t)$ and the density of the predator is denoted by $P(t)$ at any time $t$.

(b) It is also assumed that the disease spread only among the prey population and the disease is not genetically inherited.

The infected population does not recover or become immune.

(c) The susceptible prey population become infected at a rate $W$, following the law of mass action.

(d) The interaction between predator and susceptible prey is of Holling type-II and that between predator and infected prey is of Holling type-I (mass action law) in presence of additional food. This combination of functional forms are taken assuming that the capturing of infected prey is easier than the susceptible prey. Biologically, it can be explained if predator prefers infected prey than susceptible prey [10].

With the above assumptions, the model (2) reduces to the form:

$$\begin{align*}
\frac{dS}{dt} &= R_0S \left(1 - \frac{S}{K_0}\right) - WSI - \frac{A_1(1-c)SP}{B_1 + a\mu A + (1-c)S}, \\
\frac{dI}{dt} &= WSI - \frac{A_1IP}{B_1 + a\mu A} - D_1I, \\
\frac{dP}{dt} &= A_1e(1-c)S + \mu A(1-c)S - \frac{epsilon A_1IP}{B_1 + a\mu A} - D_2P.
\end{align*}$$

The term $A_2$ denotes the attacking rate of predator to infected prey; $D_1$ represents the natural death rate of infected prey.

We nondimensionalize the system (3) using $s = \frac{s}{x_0}, i = \frac{i}{x_0}, p = \frac{p}{x_0}$ and $t = R_0T$ and obtain the following system

$$\begin{align*}
\frac{ds}{dt} &= s(1 - s - i) - \gamma is - \frac{a_1(1-c)sp}{1 + a\xi + b(1-c)s}, \\
\frac{di}{dt} &= \gamma is - \frac{a_2ip}{1 + a\xi} - d_i, \\
\frac{dp}{dt} &= \epsilon a_1(1-c)s + \epsilon c_1p - \frac{c_2ip}{1 + a\xi + b(1-c)s} - c_2p.
\end{align*}$$

where $\gamma = \frac{w_0}{x_0}, a_1 = \frac{aK_0}{x_0b}, \xi = \frac{aK_0}{x_0}, b = \frac{bK_0}{x_0}, a_2 = \frac{a_1K_0}{x_0b}, c_1 = \frac{c_1K_0}{x_0}, c_2 = \frac{c_2K_0}{x_0}, e = \frac{\epsilon}{x_0}, d_1 = \frac{d_1}{x_0}$ and $d_2 = \frac{d_2}{x_0}$.
In the model (4), it is assumed that the infection process is instantaneous. In other words, as soon as an infected prey contacts a susceptible prey, the latter becomes susceptible and the newly infected prey becomes infected. However, in reality, there is a time-delay between these two events, namely the first effective contact between susceptible and infected preys and the newly infected prey becomes productively infectious [11,25]. To incorporate this phenomenon, we rewrite the system (4) as

\[
\begin{align*}
\frac{ds}{dt} & = s(1 - s - i) - \gamma is - \frac{a_1(1 - c)sp}{1 + a\xi + b(1 - c)s}, \\
\frac{di}{dt} & = \gamma \int_{-\infty}^{t} i(u)s(u)(t - u) du - \frac{a_2ip}{1 + a\xi} - d_i i, \\
\frac{dp}{dt} & = \epsilon a_1(1 - c)s + \epsilon\xi p \\
& \quad - \frac{a_2ip}{1 + a\xi} - d_2 p.
\end{align*}
\]

(5)

Here, we have assumed that the number of actively infected prey at time \(t\) is arising from the contacts of actual population of susceptible and infected prey at time \((t - u)\), where \(u\) is distributed according to a probability distribution function \(F(u)\), known as delay kernel, defined by

\[F(u) = \frac{e^{-\alpha u}}{\alpha} e^{-\beta u},\]

where \(\alpha > 0\) is a constant and \(\beta \geq 0\) known as order of the delay, is an integer. According to MacDonald [26], the average delay is defined as

\[\tau = \int_{-\infty}^{\infty} uF(u)du = \frac{e^\alpha}{\alpha}.
\]

If the kernel takes the form of a delta function, as \(F(u) = \delta(u - \tau)\), where \(\tau \geq 0\) is a constant, then the system (5) becomes the following delay differential equations with discrete delay \(\tau\) [25]:

\[
\begin{align*}
\frac{ds}{dt} & = s(1 - s - i) - \gamma is - \frac{a_1(1 - c)sp}{1 + a\xi + b(1 - c)s}, \\
\frac{di}{dt} & = \gamma(t - \tau)s(t - \tau) - \frac{a_2ip}{1 + a\xi} - d_i i, \\
\frac{dp}{dt} & = \epsilon a_1(1 - c)s + \epsilon\xi p \\
& \quad - \frac{a_2ip}{1 + a\xi} - d_2 p.
\end{align*}
\]

(6)

The initial conditions for the system (6) are chosen as:

\[s(\theta) = \phi_1(\theta) \geq 0, i(\theta) = \phi_2(\theta) \geq 0, p(\theta) = \phi_3(\theta) \geq 0, \theta \in [-\tau, 0],\]

and \(\phi_i(0) > 0, i = 1, 2, 3\), where \(C_+ = (\phi_i(\theta) \in C([-\tau, 0], R_+^3) : s(\theta) = \phi_1(\theta), i(\theta) = \phi_2(\theta), p(\theta) = \phi_3(\theta))\) is the Banach space of continuous functions form the interval \([-\tau, 0]\) into \(R_+^3 = \{(s, i, p) : s(t) \geq 0, i(t) \geq 0, p(t) \geq 0\}\). Setting \(c = 0\) and \(\tau = 0\) in this model (5), we obtain the model proposed by Sahoo [27].

3. Preliminaries

3.1. Positive invariance

The system (6) can be written as \(\dot{X} = F(X)\) with \(X(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C_+, \phi_i(0) > 0\), for \(i = 1, 2, 3\), where \(X = (s, i, p)^T \in R_+^3\) and \(F(X)\) is given by

\[
F = F(X) = \begin{pmatrix}
\frac{s(1 - s - i) - \gamma is - \frac{a_1(1 - c)sp}{1 + a\xi + b(1 - c)s}}{1 + a\xi + b(1 - c)s} \\
\frac{\gamma(t - \tau)s(t - \tau) - \frac{a_2ip}{1 + a\xi} - d_i i}{1 + a\xi + b(1 - c)s} \\
\frac{\epsilon a_1(1 - c)s + \epsilon\xi p - \frac{a_2ip}{1 + a\xi} - d_2 p}{1 + a\xi + b(1 - c)s}
\end{pmatrix}
\]

where \(F : C_+ \rightarrow R^3\) and \(F \in C^\infty(R_+^3)\).

It is easy to show that whenever choosing \(\dot{X}(\theta) \in C_+\) such that \(X_0 = 0\), then \(F(X)_{X_0 \in X_0, C_+} \geq 0\), for \(i = 1, 2, 3\). Following [28] it can be shown that any solution of \(\dot{X} = F(X)\) with \(X(\theta) = X_0 \in C_+\), say \(\dot{X}(t) = \dot{X}(t, X(\theta))\), is such that \(X(t) \in R_+^3\) for all \(t > 0\).

3.2. Boundedness

Boundedness of solution is a necessary condition for the system (6) to be biologically realistic. The following propositions ensure the boundedness of the system (6) with non-zero time lag.

Proposition 3.1. The prey population is always bounded from above.

Proof. From the first equation of the system (6) it follows that \(\frac{d}{d\tau} (s(1 - s - i) \leq s(1 - s, \lim_{t \to \infty} \sup s(t) \leq 1. \]
Proposition 3.2. If the quantity of additional food $\zeta$ satisfies the condition $0 < \zeta < \frac{d_1}{\gamma}$, then the system (6) is bounded.

Proof. We first consider the function $\chi = s(t - \tau) + i(t) + \frac{1}{2}p$. Calculating the time derivative along the solution of the system (6), for $t > t_0 - \tau$ letting $M = \min\{1, 1, a_2 - a_3\zeta e^\xi, 0\}$, provided $d_2 > a_3\zeta$, we have

$$0 < \chi < \frac{d_1}{\gamma} + \chi(s(0), i(0), p(0))e^{-M(t - \tau)},$$

where we used Proposition 3.1. for the inequality $s(t - \tau) < 1$ for all $t > t_0 - \tau$.

Applying the theory of differential inequality we obtain

$$0 < \chi < \frac{d_1}{\gamma} + \chi(s(0), i(0), p(0))e^{-M(t - \tau)}.$$

Now, as $\tau \to \infty$, we have $0 < \chi < \frac{d_1}{\gamma}$.

Hence all the solutions of (6) that initiate in $R^+_{z_1}$ are confined in the region

$$\Omega = \{(s, i, p) \in R^+_{z_1} : \chi < \frac{d_1}{\gamma}\}$$

for all large $\tau$. $\square$

3.3. Infected prey’s extinction

Proposition 3.3. If $R_0 < 1$, then the infected prey will extinct.

Proof. From the second equation of the system (6), we have $0 < \frac{d_1}{\gamma} < (\gamma - d_1)i < i - d_1$, since $s(t) \leq 1$.

Therefore, $\lim_{t \to \infty} i(t) = 0$, provided $R_0 = \frac{d_1}{\gamma} < 1$.

Thus, the basic reproduction number [29] for the system (6) is $R_0 = \frac{d_1}{\gamma}$. $R_0$ denotes the expected number of secondary infectious cases generated by one typical primary case in an entirely susceptible and sufficiently large population. System (6) always has disease free equilibrium if $R_0 < 1$ and on the other hand, system admits endemic equilibrium for $R_0 > 1$. $\square$

4. Equilibria, local stability and Hopf bifurcation

Introduction of time delays do not affect equilibrium of the system. Therefore, the system (6) has the following non-negative equilibria, namely

(i) trivial equilibrium point $E_0 = (0, 0, 0)$,

(ii) axial equilibrium point $E_A = (1, 0, 0)$,

(iii) disease free equilibrium point $E_{DF} = (s, i, p)$,

(iv) predator free equilibrium point $E_{PF} = (s, i, 0)$, where $s = \frac{d_1 + \zeta(1 - \gamma)b}{a_2 + (1 - \gamma)b}$ and $i$ is the positive root of the equation.

$v^2 + T_1v + T_2 = 0$, where $T_1 = \frac{a_3}{\alpha d_2}$ and $T_2 = \frac{a_2(1 - \gamma)b}{\alpha} + \frac{a_3(1 - \gamma)b}{\alpha} + \frac{a_3(1 - \gamma)b}{\alpha} + \frac{a_3(1 - \gamma)b}{\alpha}$.

The disease free equilibrium point $E_{DF}$ exists if $s < 1$, $a_1 : d_2b$ and $0 < \zeta < \frac{d_1}{\gamma}$. The predator free equilibrium point $E_{PF}$ exists if $R_0 > 1$ and the sufficient conditions for the existence of $E'$ in the interior of the first octant are easily obtained as follows: $T_1 + T_2 > 0$, $s^* > \frac{d_1}{\gamma}$ and $d_2 > \frac{a_3(1 - \gamma)b}{\alpha}$.

Therefore, the existence of $E_{DF}$ and $E'$ depend on the quality $(a)$ and quantity $(\zeta)$ of additional food and infection rate $\gamma$. On the other hand $E_{PF}$ depends on infection rate of infected prey $(\gamma)$ only.

We now analyse the system behaviour with time lag. We study the system’s trajectories around the equilibrium points $E_{DF}, E_{PF}$ and $E'$.

4.1. Analysis of disease free equilibrium point $E_{DF}$

The Jacobian matrix of the system (6) at the disease free equilibrium point $E_{DF} = (s, i, p)$ is given by:

$$J(E_{DF}) = \begin{pmatrix}
    r_{11} & r_{12} & r_{13} \\
    0 & r_{22}e^{-\zeta} - \frac{a_3p}{a_2} - d_1 & 0 \\
    r_{31} & 0 & r_{32} 
\end{pmatrix},$$

where $r_{11} = 1 - 2\gamma - \frac{a_1(1 - \gamma)(1 - \zeta)}{(1 + \gamma)b(1 - \gamma)b}$, $r_{12} = -\frac{(1 - \gamma)b}{\gamma}$, $r_{13} = -\frac{a_3(1 - \gamma)b}{\alpha}$, $r_{22} = \gamma\frac{a_2}{\alpha}$, $r_{31} = \frac{a_3(1 - \gamma)b}{\alpha}$ and $r_{32} = \frac{a_3(1 - \gamma)b}{\alpha}$.

The characteristic equation of the system (6) is a transcendental equation of the following form:
\[ \Delta(\lambda, \tau) = \lambda^3 + U_1 \lambda^2 + U_2 \lambda + U_3 + (V_2 \lambda^2 + V_1 \lambda + V_0)e^{-\tau} = 0, \]  

(8)

where, \( U_1 = d_1 + \frac{\omega_0^3}{\omega_1^2 + \omega_2^2} - r_{11}, U_2 = r_{11}(d_1 + \frac{\omega_0^3}{\omega_1^2 + \omega_2^2}) - r_{12}r_{21}, U_3 = -r_{13}r_{21}(\frac{\omega_0^3}{\omega_1^2 + \omega_2^2} + d_1), V_2 = -r_{22}, V_1 = r_{11}r_{22}, V_0 = r_{13}r_{21}r_{22}. \)

For \( \tau = 0 \), the characteristic Eq. (8) becomes

\[ \lambda^3 + (U_1 + V_2)\lambda^2 + (U_2 + V_1)\lambda + (U_3 + V_0) = 0. \]

Routh–Hurwitz criterion implies that if the conditions \( (H1) \ (a) \ (U_1 + V_2) > 0, (b) \ (U_2 + V_1)(U_2 + V_1) > (U_3 + V_0) \), then all roots of Eq. (8) with \( \tau = 0 \) have negative real part. Therefore, the disease-free equilibrium point \( E_{DF} \) is locally asymptotically stable under the conditions \( H1 \) for \( \tau = 0 \).

Now, we can find the condition for the non-existence of a delay-induced \( (\tau \neq 0) \) instability by using the following theorem, given by Gopalaswamy [30].

**Theorem 4.1.** The necessary and sufficient conditions for \( E_{DF} = (3, 0, 0) \) to be locally asymptotically stable in the presence of a time delay \( \tau \) are

(i) the real parts of all the roots of \( \Delta(\lambda, \tau) = 0 \) are negative,

(ii) for all real \( \omega \) and for \( \tau > 0, \Delta(i\omega, \tau) \neq 0 \), where \( i = \sqrt{-1} \).

For some \( \tau > 0, \) if \( \lambda = \omega_0 \omega \) (\( \omega_0 > 0 \)) is a root of the Eq. (8), then \(-i\omega_0^3 - U_1\omega_0 + iU_2\omega_0 + U_3 + (-V_2\omega_0^3 + iV_1\omega_0 + V_0)e^{-i\omega\tau} = 0\). Separating the real and imaginary parts, we obtain

\[ (V_0 - V_2\omega_0^3)\cos(\omega_0^2) + \omega_0V_1\sin(\omega_0^2) = U_1\omega_0^3 - U_3, \]

\[ V_2\omega_0^3 \cos(\omega_0^2) - (V_0 - V_2\omega_0^3)\sin(\omega_0^2) = \omega_0^3 - V_2\omega_0. \]

(9), (10)

Squaring and adding the two Eqs. (9) and (10), we finally get

\[ \omega_0^6 + a_0\omega_0^4 + b_0\omega_0^2 + c_0 = 0. \]

(11)

where, \( a_0 = U_1 - 2U_2 - V_2, \)

\( b_0 = U_1^2 + 2V_0V_2 - 2V_1U_3 - V_1, \)

\( c_0 = U_1^2 - V_2^2. \)

Taking \( \omega_0^2 = z, \) Eq. (11) becomes

\[ h(z) = z^3 + a_0z^2 + b_0z + c_0 = 0. \]

(12)

Since, \( \lim_{z \to \infty} h(z) = +\infty, \) we can conclude that if \( c_0 < 0, \) then Eq. (12) has at least one positive root. We can introduce the following results stated in [31] about the distributions of the positive roots of Eq. (12).

**Lemma 4.1.** The polynomial Eq. (12) has the following properties.

(i) If \( c_0 < 0, \) then Eq. (12) has at least one positive root.

(ii) If \( c_0 > 0 \) and \( \Delta = a_0^3 - 3b_0 < 0, \) then Eq. (12) has no positive root.

(iii) If \( c_0 > 0, \) and \( \Delta = a_0^3 - 3b_0 > 0, \) then Eq. (12) has positive roots if and only if \( \tau_1 = \frac{1}{2}(a_0 + \sqrt{\Delta}) > 0 \) and \( h(\tau_1) < 0. \)

Suppose that Eq. (12) has positive roots. Without loss of generality, we assume that it has three positive roots \( \omega_k, k = 1, 2, 3. \) Hence, the characteristic Eq. (11) will have a pair of purely imaginary roots of the form \( \pm i\omega_k. \) From (9) and (10), we get that

\[ \cos(\omega_0^2) = \frac{(V_1 - V_2U_3)\omega_0^3 + (V_0U_2 + U_1U_3 - V_2U_3)\omega_0}{V_2\omega_0^3 + (V_1 - 2V_0V_2)\omega_0^3 + V_0^2}, \]

therefore, for every fixed \( \omega_k, \) the corresponding critical value of time delay \( \tau_k^c \) is

\[ \tau_k^c = \frac{1}{\omega_k} \cos^{-1} \left[ \frac{(V_1 - V_2U_3)\omega_0^3 + (V_0U_2 + U_1U_3 - V_2U_3)\omega_0}{V_2\omega_0^3 + (V_1 - 2V_0V_2)\omega_0^3 + V_0^2} \right] + \frac{2\pi}{\omega_k}, \]

(13)

where \( k = 1, 2, 3 \) and \( j = 0, 1, 2, \ldots \). Let \( \tau_0 = \tau_{\text{crit}} \) be the minimum \( \tau_0 \) such that \( \omega_k \) is a root of \( h(\tau_0) \) for \( j = 0 \) and \( k = 1, 2, 3. \) Let us now investigate whether the system (6) undergoes a Hopf bifurcation phenomenon at \( E_{DF}(3, 0, 0) \) as \( \tau \) increases through \( \tau_0. \) For this purpose, let us now compute the transversality condition in the following Lemma.

**Lemma 4.2.** Suppose that \( z_0 = \omega_0^2 \) and \( h'(z_0) \neq 0, \) where \( h(z) \) is defined by (11). Then \[ \frac{\partial [\text{Re}(\Omega)]}{\partial \tau} \neq 0 \text{ and the sign of } \frac{\partial [\text{Im}(\Omega)]}{\partial \tau} \text{ is consistent with that of } h'(z_0). \]

**Proof.** Differentiating both sides of (8) with respect to \( \tau, \) it follows that

\[ \frac{dV}{d\tau} = \frac{(3z^2 + 2U_1\lambda + V_2)e^{-\tau}}{\lambda(V_2\lambda^2 + V_1\lambda + V_0)} + \frac{2V_1\lambda + V_1}{\lambda(V_2\lambda^2 + V_1\lambda + V_0)} \frac{V}{\lambda}. \]

(14)
Now,
\[
(3\lambda^2 + 2U_1\lambda + U_2)e^{\lambda t} = \left[ (U_2 - 3a_0^2\cos(\omega_0t_0) - 2U_1\omega_0\sin(\omega_0t_0)) \right] + i\left[ (U_2 - 3a_0^2\sin(\omega_0t_0) + 2U_1\omega_0\cos(\omega_0t_0)) \right],
\]
where 
\[
\lambda = \sqrt{V_2 \lambda^2 + V_1 \lambda + V_0} = -V_1\omega_0 + i(V_0 - V_2\omega_0^2)\omega_0,
\]
and 
\[
2V_2\lambda + V_1 = 2V_2\omega_0 + V_1.
\]
Therefore, by Eq. (8) and (14)-(17), we obtain
\[
\begin{align*}
\left[ \frac{d}{d\tau}\left( \frac{\text{sign}(\Delta(t))}{M} \right) \right]_{t=t_0}^{-1} & = \frac{1}{M} \left[ (U_2 - 3a_0^2\omega_0 \{ (V_0 - V_2\omega_0^2)\cos(\omega_0t_0) - V_1\omega_0\sin(\omega_0t_0) \}) + 2U_1\omega_0^2 \{ (V_0 - V_2\omega_0^2)\cos(\omega_0t_0) \ight. \\
& \quad - V_1\omega_0\sin(\omega_0t_0) - ) - 2V_1\omega_0(V_0\omega_0 - V_2\omega_0^2)) \right] \\
& = \frac{1}{M} \left[ 3a_0^2 + (2U_1^2 - 2V_2^2)\omega_0^4 + U_1^2U_3 - 2V_2U_1 - V_1^2 + 2V_0V_2\omega_0^2 \right] \\
& = \frac{1}{M} \left[ 6a_0^2 + 2a_0\omega_0^2 + b_0\omega_0^2 \right] = \frac{1}{M} \left[ z_0(3\omega_0^2 + 2a_0\omega_0 + b_0) \right] = \frac{z_0h'(z_0)}{M},
\end{align*}
\]
where 
\[ M = V_1\omega_0^2 + (V_0 - V_2\omega_0^2)^2 \omega_0^2 > 0. \]
Hence, \[ \text{sign}\left( \frac{d}{d\tau}\left( \frac{\text{sign}(\Delta(t))}{M} \right) \right)_{t=t_0}^{-1} = \text{sign}\left( \frac{z_0h'(z_0)}{M} \right) \neq 0. \]
Notice that \( M, z_0 > 0 \), we conclude that the sign of \( \frac{d}{d\tau}\left( \frac{\text{sign}(\Delta(t))}{M} \right) \) is determined by that of \( h'(z_0) \).
Hence, the proof is completed. \( \square \)
By Lemmas 4.1 and 4.2, we have the following theorem:

**Theorem 4.2.** Suppose that the condition (H1) holds. Then, the following results hold.

(i) When \( c_0 \geq 0 \) and \( \Delta = a_0^2 - 3b_0 \leq 0 \), all roots of Eq. (8) have negative real parts for all \( \tau \geq 0 \), and the disease free equilibrium \( E_{DF} = (0, 0, \bar{p}) \) of the system (6) is asymptotically stable for all \( \tau \geq 0 \). (ii) If either \( c_0 < 0 \) or \( c_0 \geq 0 \), \( \Delta = a_0^2 - 3b_0 > 0 \), \( \tau_1 = \frac{1}{2}(-a_0 + \sqrt{\Delta}) > 0 \) and \( h'(z_0) \) hold, then \( h(x) \) has at least one positive root \( z_0 \). And all roots of Eq. (8) have negative real parts for \( \tau \in [0, \tau_0) \). The system (6) is asymptotically stable for \( \tau \in (0, \tau_0) \) and unstable for \( \tau > \tau_0 \).

(iii) If all the conditions as stated in (ii) and \( h'(z_0) \neq 0 \) hold, then system (6) undergoes a Hopf bifurcation at the disease free equilibrium point \( E_{DF} \) when \( \tau = \tau_0 \).

4.2. Analysis of predator free equilibrium point \( E_{PF} \)

The Jacobian matrix of the system (6) at the predator free equilibrium point \( E_{PF} = (\bar{s}, \bar{i}, 0) \) is given by:
\[
J(E_{PF}) = \begin{pmatrix}
\kappa_{11} & k_{12} & k_{13} \\
k_{21}e^{-\lambda t} & k_{22}e^{-\lambda t} - d_1 & k_{23} \\
0 & 0 & k_{33}
\end{pmatrix},
\]
where \( \kappa_{11} = 1 - 2\bar{s} - (1 + \gamma)i, k_{12} = -(1 + \gamma)i, k_{13} = -\frac{a_1(1 - \gamma i)}{1 + c(1 - \gamma i)}, k_{21} = \gamma i, k_{22} = \gamma i, k_{23} = \frac{a_2(1 - \gamma i)}{1 + c(1 - \gamma i)}, k_{33} = \frac{a_3(1 - \gamma i)}{1 + c(1 - \gamma i)} - d_2. \]

The characteristic equation of the system (6) is a transcendental equation of the following form:
\[
\Delta(\lambda, \tau) = \lambda^3 + \Theta_2\lambda^2 + (\Theta_2 + \Theta_3)\lambda + (\Sigma_2\lambda + \Sigma_3\lambda + \Sigma_4\lambda) = \theta_0
\]
where \( \Theta_1 = d_1 - k_{11}, \Theta_2 = k_{13} - k_{11}d_1 - k_{11}d_1, \Theta_3 = k_{11}k_{23}d_1, \Sigma_2 = -k_{21}, \Sigma_1 = k_{22}k_{33} + k_{11}k_{23} - k_{13}k_{21}, \Sigma_0 = k_{21}k_{12}k_{33} - k_{11}k_{22}k_{33}. \)

For \( \tau = 0 \), the characteristic Eq. (7) becomes
\[
\lambda^3 + (\Theta_2 + \Theta_3)\lambda^2 + (\Theta_2 + \Theta_3)\lambda + (\Theta_2 + \Theta_3) = 0.
\]
Routh-Hurwitz criterion implies that if the conditions (H1) \( a) (\Theta_2 + \Theta_3) > 0, (b) (\Theta_2 + \Theta_3)(\Theta_2 + \Theta_3) > (\Theta_2 + \Theta_3), \) then all roots of Eq. (19) with \( \tau = 0 \) have negative real part. Therefore, the predator free equilibrium point \( E_{PF} \) is locally asymptotically stable under the conditions H1 for 0 < \( \tau \).

Now, we can find the condition for the non-existence of a delay-induced (\( \tau \neq 0 \)) instability by using the following theorem, given by Gopalswamy [30].

**Theorem 4.3.** The necessary and sufficient conditions for \( E_{PF} = (\bar{s}, \bar{i}, 0) \) to be locally asymptotically stable in the presence of a time delay \( \tau \) are

(i) the real parts of all the roots of \( \Delta(i, \tau) = 0 \) are negative.
(ii) for all real \( \omega \) and for \( \tau > 0 \), \( \Delta(i\omega, \tau) \neq 0 \), where \( i = \sqrt{-1} \).

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For some $\tau > 0$, if $z = \rho e^{i\theta}$ is a root of the Eq. (19), then $-\rho^3 - \Theta_1 \rho^3 + i\rho \Theta_2 \rho + \Theta_3 + (-\Sigma_2 \rho^2 + i\Sigma_1 \rho + \Sigma_0) e^{-i\omega \tau} = 0$. Separating the real and imaginary parts, we obtain

\begin{align}
\left( \Sigma_0 - \Sigma_2 \rho \right) \cos(\omega \tau) + \rho \Sigma_1 \sin(\omega \tau) &= \Theta_1 \rho^3 - \Theta_3, \\
\Sigma_0 \cos(\omega \tau) - (\Sigma_0 - \Sigma_2 \rho \rho) \sin(\omega \tau) &= \rho^3 - \Sigma_0 \rho.
\end{align}

(20)

(21)

Squaring and adding the two Eqs. (20) and (21), we finally get

$$\rho^4 + d_0 \rho^4 + e_0 \rho^2 + f_0 = 0.$$  

(22)

where, $d_0 = \Theta_1^2 - 2\Theta_2 - \Sigma_0^2$, $e_0 = \Theta_1^2 + 2\Sigma_2 \Sigma_0 - 2\Sigma_1 \Theta_2 - \Sigma_0^2$, $f_0 = \Theta_3^2 - \Sigma_0^2$.

Taking $\omega \tau = z$, Eq. (22) becomes

$$h(z) = z^4 + d_0 z^2 + e_0 z + f_0 = 0.$$  

(23)

Since, $\lim_{\tau \to +\infty} h(z) = +\infty$, we can conclude that if $f_0 < 0$, then Eq. (23) has at least one positive root. We can introduce the following results stated in [31] about the distributions of the positive roots of Eq. (23).

**Lemma 4.3.** For the polynomial Eq. (23), we have the following results.

(i) If $f_0 < 0$, then Eq. (23) has at least one positive root.
(ii) If $f_0 > 0$ and $\lambda = d_0^2 - 2e_0 < 0$, then Eq. (23) has no positive root.
(iii) If $f_0 > 0$ and $\lambda = d_0^2 - 2e_0 > 0$, then Eq. (23) has positive roots if and only if $z^* = \frac{1}{2}(-d_0 + \sqrt{\lambda}) > 0$ and $h(z^*) < 0$.

Suppose that Eq. (23) has positive roots. Without loss of generality, we assume that it has three positive roots $\omega_k$, $k = 1, 2, 3$. Hence, the characteristic Eq. (22) will have a pair of purely imaginary roots of the form $\pm i\omega_k$. From (20) and (21), we get

$$\cos(\omega \tau) = \frac{\Sigma_1 - \Sigma_2 \Theta_1 \rho^4 + (\Sigma_0 \Theta_1 + \Sigma_1 \Theta_2 - \Sigma_0 \Theta_3) \rho^2 - \Sigma_0 \Theta_2}{\Sigma_2 \rho^3 + (\Sigma_1 - 2\Sigma_0 \rho) \rho^2 + \Sigma_0 \rho},$$

Therefore, for every fixed $\omega_k$, the corresponding critical value of time delay $\tau_k^*$ is

$$\tau_k^* = \frac{1}{\omega_k} \cos^{-1} \left[ \frac{\Sigma_1 - \Sigma_2 \Theta_1 \rho^4 + (\Sigma_0 \Theta_1 + \Sigma_1 \Theta_2 - \Sigma_0 \Theta_3) \rho^2 - \Sigma_0 \Theta_2}{\Sigma_2 \rho^3 + (\Sigma_1 - 2\Sigma_0 \rho) \rho^2 + \Sigma_0 \rho} \right] + \frac{2\pi}{\omega_k},$$  

(24)

where $k = 1, 2, 3$ and $j = 0, 1, 2, \ldots$

Let $\tau_k = \tau_k^*$, $\omega_0 = \omega_k$ for $j = 0$ and $k = 1, 2, 3$.

Let us now investigate whether the system (6) undergoes a Hopf bifurcation phenomenon at $E_{eq}(z, \hat{1}, 0)$ when $\tau$ increases through $\tau_k$. For this purpose, let us now compute the transversality condition in the following Lemma.

**Lemma 4.4.** Suppose that $z_0 = \rho_0 e^{i\theta}$ and $h'(z_0) \neq 0$, where $h(z)$ is defined by (23). Then $\frac{dRe(\lambda)}{d\tau} \neq 0$ at $\tau = \tau_k$ and the sign of $\frac{dRe(\lambda)}{d\tau}$ is consistent with that of $h'(z_0)$.

**Proof.** Differentiating both sides of (19) with respect to $\tau$, it follows that

$$\left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{3\lambda^2 + 2\Theta_1 \lambda + \Theta_2 \lambda^2 e^{i\omega \tau}}{\lambda (\Sigma_2 \rho^2 - \Sigma_1 \rho + \Sigma_0) + \frac{2\Sigma_2 \lambda + \Sigma_1}{\lambda}}.$$  

(25)

Now,

$$[3\lambda^2 + 2\Theta_1 \lambda + \Theta_2 \lambda^2 e^{i\omega \tau}]_{\tau = \tau_{k-1}, \omega_k \rightarrow \omega_0} = [\Theta_2 - 3\omega_0^2] \left[ \cos(\omega_0 \tau_{k-1}) - 2\Theta_1 \omega_0 \sin(\omega_0 \tau_{k-1}) \right] + i[\Theta_2 - 3\omega_0^2] \sin(\omega_0 \tau_{k-1}) + 2\Theta_1 \omega_0 \cos(\omega_0 \tau_{k-1}),$$  

(26)

$$\lambda (\Sigma_2 \rho^2 + \Sigma_1 \rho + \Sigma_0)_{\tau = \tau_{k-1}, \omega_k \rightarrow \omega_0} = -\Sigma_0 \omega_0^2 + i(\Sigma_0 - \Sigma_2 \omega_0^2) \omega_0,$$  

(27)

and

$$\Sigma_2 \rho^2 + \Sigma_1 \rho + \Sigma_0 = 2\Sigma_2 \omega_0^2 + \Sigma_1.$$  

(28)

Therefore, by Eq. (19) and (25)–(28), we obtain

$$\left( \frac{d(Re(\lambda))}{d\tau} \right)^{-1} = \frac{1}{M} \left[ \Theta_2 - 3\omega_0^2 \right] \left[ \Sigma_0 \cos(\omega_0 \tau_{k-1}) - \Sigma_1 \omega_0 \cos(\omega_0 \tau_{k-1}) \right] + 2\Theta_1 \omega_0 \left[ (\Sigma_0 - \Sigma_2 \omega_0^2) \cos(\omega_0 \tau_{k-1}) - \Sigma_1 \omega_0 \sin(\omega_0 \tau_{k-1}) \right] - \Sigma_0 \omega_0^2 + 2\Sigma_2 \omega_0 \left( \Sigma_0 \omega_0 - \Sigma_2 \omega_0^2 \right),$$  

$$= \frac{1}{M} \left[ 3\omega_0^2 + \left( 2\Theta_1^2 - 4\Theta_2 - 2\Sigma_0^2 \right) \omega_0^2 + \left( \Theta_2^2 - 2\Theta_1 \Theta_3 - \Sigma_0^2 - 2\Sigma_2 \Sigma_0 \omega_0 \right) \omega_0^2 \right],$$  

$$= \frac{1}{M} \left[ 3\omega_0^2 + 2d_0 \omega_0^2 + e_0 \omega_0^2 \right], = \frac{1}{M} \left[ 2\omega_0^2 + 2d_0 \omega_0 + e_0 \right], = \frac{2}{M} \left[ 2\omega_0^2 + 2d_0 \omega_0 + e_0 \right], = \frac{2}{M} \left[ 2\omega_0^2 + 2d_0 \omega_0 + e_0 \right].$$  

(29)

Therefore, the transversality condition is satisfied, and $\omega_k \rightarrow \omega_0$ when $\tau \rightarrow \tau_k$. This completes the proof.
where 

\[ M = \sum \lambda_i^2 + (\Sigma \lambda_0 - \Sigma \lambda_i) \lambda_0 > 0. \]

Hence, 

\[ \text{sign} \left( \frac{\partial \text{Re}(s(\tau))}{\partial \tau} \right)_{\tau = \tau_1} = \text{sign} \left( \frac{\partial \text{Re}(s(\tau))}{\partial \tau} \right)_{\tau = \tau_1} = \text{sign} \left( \frac{\Delta(\tau)}{M} \right) \neq 0. \]

Notice that \( M, \lambda_0 > 0 \), we conclude that the sign of 

\[ \frac{\partial \text{Re}(s(\tau))}{\partial \tau} \mid_{\tau = \tau_1} \]

is determined by that of \( h'(\lambda_0) \). Hence, the proof is completed. \( \square \)

By Lemmas 4.3 and 4.4, we have the following theorem:

**Theorem 4.4.** Suppose that the condition (H1) holds. Then, the following results hold.

(i) When \( f_0 > 0 \) and \( \Delta - d_1 - 3\lambda_0 < 0, \) all roots of Eq. (19) have negative real parts for all \( \tau > 0, \) and the predator free equilibrium \( E_{PF} = (\bar{\lambda}, \bar{\tilde{\lambda}}, 0) \) of the system (6) is asymptotically stable for all \( \tau > 0. \)

(ii) If either \( f_0 < 0 \) or \( f_0 > 0, \) \( \Delta - d_1 - 3\lambda_0 > 0, \) \( \tau_1 = \frac{1}{1 - d_0 - \sqrt{\Delta}} > 0 \) and \( h(\lambda_0) \leq 0 \) hold, then \( h(x) \) has at least one positive root \( \lambda_0, \) and all roots of Eq. (19) have negative real parts for \( \tau < 0, \) \( \tau_1, \) the system (6) is asymptotically stable for \( \tau \in (0, \tau_1), \) and unstable for \( \tau > \tau_1. \)

(iii) If all the conditions as stated in (ii) and \( h'(\lambda_0) \neq 0 \) hold, then system (6) undergoes a Hopf bifurcation at the disease free equilibrium \( E_{PF} \) when \( \tau = \tau_1. \)

4.3. Analysis of endemic equilibrium point \( E^* \)

We shall now investigate the dynamics of delay system (6) at endemic equilibrium point \( E^*. \) The Jacobian matrix of the system (6) at \( E^*(\bar{s}, \bar{\hat{s}}, \bar{p}) \) is found from the following variational matrix:

\[ J(E^*) = \begin{pmatrix}
\begin{array}{c}
\bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} \\
\bar{a}_{21} e^{-\bar{a}_{11} \tau} & \bar{a}_{22} e^{-\bar{a}_{11} \tau} & \bar{a}_{23} \\
\bar{a}_{31} & \bar{a}_{32} & 0
\end{array}
\end{pmatrix}, \quad (29)
\]

where 

\[ \bar{a}_{11} = 1 - 2\bar{s} - (1 + \gamma)\bar{s} + \frac{\bar{a}_{12}(1 - \gamma)\bar{s} + \bar{a}_{13}}{1 + \bar{a}_{13}(1 - \gamma)\bar{s}}, \]

\[ \bar{a}_{12} = -(1 + \gamma)\bar{s}, \quad \bar{a}_{13} = -\frac{\bar{a}_{12}(1 - \gamma)\bar{s}}{1 + \bar{a}_{13}(1 - \gamma)\bar{s}}, \quad \bar{a}_{21} = \gamma \bar{s}, \quad \bar{a}_{22} = \gamma \bar{s}, \quad \bar{a}_{23} = \frac{\bar{a}_{12}}{1 + \bar{a}_{13}} \bar{s}, \quad \bar{a}_{31} = \frac{\bar{a}_{12}}{1 + \bar{a}_{13}} \bar{s}. \]

The characteristic equation of the system (6) is a transcendental equation of the following form:

\[ \Delta(\lambda, \tau) = \lambda^2 + A_1 \lambda^2 + A_2 \lambda + A_3 + (B_2 \lambda^2 + B_1 \lambda + B_0) e^{-\lambda \tau} = 0, \quad (30) \]

where

\[ A_1 = A_1 \quad \frac{\partial \lambda}{\partial \tau} + A_2 = A_2 \quad \frac{\partial \lambda}{\partial \tau} - a_{11} a_{11} - a_{12} a_{22} - a_{13} a_{32} - a_{13} a_{32} - a_{13} a_{32} + A_3, \]

\[ A_3 = a_{12} a_{11} a_{22} - a_{12} a_{22} - a_{12} a_{32} - A_3. \]

For \( \tau > 0, \) the characteristic Eq. (30) becomes \( \lambda^2 + (A_1 + B_2) \lambda^2 + (A_2 + B_1) \lambda + (A_3 + B_0) = 0. \)

Routh–Hurwitz criterion implies that if the conditions (H2) \( a_{12} A_2 + B_2 > 0, \) \( a_{12} A_2 + B_2 > 0, \) then all roots of Eq. (30) with \( \tau = 0 \) have negative real part. Therefore, the disease-free equilibrium point \( E^* \) is locally asymptotically stable under the conditions H2 for \( \tau = 0. \)

Now, we can find the condition for the non-existence of a delay-induced \((\tau \neq 0)\) instability by using the following theorem, given by Gopalswamy [30].

**Theorem 4.5.** The necessary and sufficient conditions for \( E^* = (\bar{s}, \bar{\hat{s}}, \bar{p}) \) to be locally asymptotically stable in the presence of a time delay \( \tau \) are

(i) the real parts of all the roots of \( \Delta(\lambda, \tau) = 0 \) are negative.

(ii) for all real \( \omega \) and for \( \tau > 0, \) \( \Delta(i \omega, \tau) \neq 0, \) where \( i = \sqrt{-1}. \)

For some \( \tau > 0, \) if \( \lambda = i \omega, (\omega > 0) \) is a root of the Eq. (30), then

\[ -i \omega^2 - A_1 \omega^2 + i \omega B_1 + A_3 + (B_2 \omega^2 + i \omega B_1 + B_0) e^{-i \omega \tau} = 0. \]

Separating the real and imaginary parts, we obtain

\[ (R_0 - B_2 \omega^2) \cos(\omega \tau) + \omega B_1 \sin(\omega \tau) = A_1 \omega^2 - A_3, \]

\[ B_1 \cos(\omega \tau) - (R_0 - B_2 \omega^2) \sin(\omega \tau) = \omega^2 - A_2 \omega. \]

Squaring and adding the two Eqs. (31) and (32), we finally get

\[ \omega^2 + p \omega^4 + q \omega^2 + R_0 = 0. \quad (33) \]

where

\[ p = A_1 A_1 - 2A_2 - B_2, \]

\[ q = A_1^2 + 2B_2 A_2 - 2A_1 A_3 - B_2. \]
Letting \( w^2 = z \) Eq. (21) deduces to
\[ h(z) = z^5 + p_0 z^4 + q_0 z^2 + r_0 = 0. \] (34)
Since, \( \lim_{z \to \infty} h(z) = +\infty \), we can conclude that if \( r_0 < 0 \), then Eq. (34) has at least one positive root. We can introduce the following results stated in [31] about the distributions of the positive roots of Eq. (34).

**Lemma 4.5.** For the polynomial Eq. (34), we have the following results.

(i) If \( r_0 < 0 \), then Eq. (34) has at least one positive root.

(ii) If \( r_0 > 0 \) and \( \Delta = p_0^2 - 4q_0 < 0 \), then Eq. (34) has no positive root.

(iii) If \( r_0 > 0 \) and \( \Delta = p_0^2 - 4q_0 > 0 \), then Eq. (34) has positive roots if and only if \( \gamma_1 > \frac{1}{2}(-p_0 + \sqrt{\Delta}) > 0 \) and \( h(x_\gamma) < 0 \).

Suppose that Eq. (34) has positive roots. Without loss of generality, we assume that it has three positive roots \( \omega_k \), \( k = 1, 2, 3 \). Hence, the characteristic Eq. (33) will have a pair of purely imaginary roots of the form \( \pm i\omega_k \). Calculating from (31) and (32), we get
\[
\cos(\omega_k \tau) = \frac{\Re\{e^{\omega_k \tau}\}}{\omega_k}
\]

Therefore, for every fixed \( \omega_k \), the corresponding critical value of time delay \( \tau_k \) is
\[
\tau_k = \frac{1}{\omega_k} \cos^{-1}
\left\[ \frac{(B_1 - A_0 \omega_k) \omega^4 + (B_0 A_1 + B_2 A_3 - A_1 B_1) \omega^2 - R_0}{B_1^2 \omega^4 + (B_1^2 - 2B_0 B_2) \omega^2 + B_0^2} \right\] + \frac{2\pi}{\omega_k}
\] (35)
where \( k = 1, 2, 3 \) and \( j = 0, 1, 2, \ldots \).

Let \( \tau_j = \tau_0 = \tau_k^0 = \min\{\tau_j^k\} \), \( \omega_j = \omega_{k_0} \) for \( j = 0 \) and \( k = 1, 2, 3 \).

Let us now investigate whether the system (6) undergoes a Hopf bifurcation phenomenon at \( E^*(\tau^*, \rho^*, \rho^*) \) when \( \tau \) increases through \( \tau_k \). For this purpose, let us now compute the transversality condition in the following Lemma.

**Lemma 4.6.** Suppose that \( z_0 = \omega_j^0 \) and \( h'(z_0) \neq 0 \), where \( h(z) \) is defined by (34). Then \( \left\{ \frac{d(Re(z(\tau)))}{d\tau} \right\}_{\tau = \tau_k^0} \neq 0 \) and the sign of \( \left\{ \frac{d(Re(z(\tau)))}{d\tau} \right\}_{\tau = \tau_k^0} \) is consistent with that of \( h'(z_0) \).

**Proof.** Differentiating both sides of (30) with respect to \( \tau \), it follows that
\[
\left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{(3\lambda^2 + 2A_1 \lambda + A_2)e^{i\theta - i\omega_k \tau}}{\lambda(B_2 \lambda^2 + B_1 \lambda + B_0)} + \frac{2B_2 \lambda + B_1}{\lambda(B_2 \lambda^2 + B_1 \lambda + B_0)} \tau.
\] (36)

Now,
\[
\left\{ [(3\lambda^2 + 2A_1 \lambda + A_2)e^{i\theta - i\omega_k \tau}]_{\tau = \tau_k^0} \right\} = \left\{ [(A_2 - 3\omega_k^2) \cos(\omega_k \tau_k^0) - 2A_1 \omega_k \sin(\omega_k \tau_k^0)] + i[(A_2 - 3\omega_k^2) \sin(\omega_k \tau_k^0) + 2A_1 \omega_k \cos(\omega_k \tau_k^0)] \right\},
\]
\[
\lambda(B_2 \lambda^2 + B_1 \lambda + B_0) = -B_1 \omega_k^2 + i(B_0 - B_2 \omega_k^2) \omega_k,
\]
and
\[
2B_2 \lambda + B_1 = 2B_2 \omega_k \omega_k + B_1.
\] (38) (39)

Therefore, by Eq. (30) and (36)–(39), we obtain
\[
\left\{ \frac{d(Re(z(\tau)))}{d\tau} \right\}_{\tau = \tau_k^0} = \frac{1}{N} \left\{ (A_2 - 3\omega_k^2) \omega_k \{ (B_0 - B_2 \omega_k^2) \sin(\omega_k \tau_k^0) - B_1 \omega_k \cos(\omega_k \tau_k^0) \} - B_1^2 \omega_k^2 + 2B_2 \omega_k \omega_k \omega_k - B_0 \omega_k \omega_k \omega_k \right\},
\]
\[
= \frac{1}{N} \left\{ (3\omega_k^2 + 2A_1 \omega_k^2) \cos(\omega_k \tau_k^0) - B_1 \omega_k \sin(\omega_k \tau_k^0) \right\} - B_1^2 \omega_k^2 + 2B_2 \omega_k \omega_k \omega_k - B_0 \omega_k \omega_k \omega_k \right\},
\]
\[
= \frac{1}{N} \left\{ (3\omega_k^2 + 2p_0 \omega_k^2 + q_0 \omega_k^2) \right\}, = \frac{1}{N} \left\{ z_0 (3z_0^2 + 2p_0 z_0 + q_0) \right\}, = \frac{z_0 h'(z_0)}{N},
\]
where \( N = B_1^2 \omega_k^4 + (B_0 - B_2 \omega_k^2)^2 \omega_k^2 > 0 \).

Hence, \( \text{sign} \left\{ \frac{d(Re(z(\tau)))}{d\tau} \right\}_{\tau = \tau_k^0} = \text{sign}(\frac{d(Re(z(\tau)))}{d\tau})^{-1}_{\tau = \tau_k^0} = \text{sign}(h'(z_0)) \neq 0 \).

Noting that \( N, z_0 > 0 \), we conclude that the sign of \( \left\{ \frac{d(Re(z(\tau)))}{d\tau} \right\}_{\tau = \tau_k^0} \) is determined by that of \( h'(z_0) \).

Thus, the proof is completed. \( \Box \)

By Lemmas 4.5 and 4.6, we have the following theorem:
Theorem 4.6. Suppose that the condition (H2) holds. Then, the following results hold.

(i) When \( r_0 > 0 \) and \( \Delta - p_2^2 - 3q_2 \leq 0 \), all roots of Eq. (30) have negative real parts for all \( \tau \geq 0 \), and the positive endemic equilibrium \( E^* = (S^*, I^*, P^*) \) of the system (6) is asymptotically stable for all \( \tau \geq 0 \).

(ii) If either \( r_0 < 0 \) or \( r_0 \geq 0, \Delta = p_2^2 - 3q_2 > 0, \tau = \frac{1}{2}(-p_2 + \sqrt{\Delta}) > 0 \) and \( h(\tau) \leq 0 \) hold, then \( h'(\tau) \) has at least one positive root \( \tau_0 \), and all roots of Eq. (30) have negative real part for \( \tau \in [0, \tau_0) \), the system (6) is asymptotically stable for \( \tau \in [0, \tau_0] \) and unstable for \( \tau > \tau_0 \).

(iii) If all the conditions as stated in (ii) and \( h'(\tau_0) \neq 0 \) hold, then system (6) undergoes a Hopf bifurcation at the positive endemic equilibrium \( E^* \) when \( \tau = \tau_0 \).

5. Permanence

The permanence plays a vital role in ecology since the criterion of permanence for ecological systems is a condition ensuring the long-term survival of all species. Haque [32, 33] discussed the permanence result of a system by using the boundedness and dissipativeness. Now, we shall prove that the instability of \( E_1, E_{3v} \) and \( E_{3e} \) imply that system (6) is permanent. Before starting our theorem, we give some important definitions:

Definition 5.1. The system (6) is said to be uniformly persistent if there is an \( \Delta > 0 \) (independent of the initial data) such that every solution \( (s(t), i(t), p(t)) \) with initial condition of system (6) satisfies

\[
\lim_{t \to +\infty} \inf s(t) \geq \Delta, \quad \lim_{t \to +\infty} \inf i(t) \geq \Delta, \quad \lim_{t \to +\infty} \inf p(t) \geq \Delta.
\]

Definition 5.2. The system (6) is said to be permanent if there exists a compact region \( \Omega_0 \subset \text{int} \Omega \) such that every solution of the system (6) with initial condition will eventually enter and remain in region \( \Omega_0 \).

From the above definitions, it is clear that for a dissipative system uniform persistence is equivalent to permanence.

To prove the permanence of the system (6) with non zero time lag, we need the following Lemma:

Lemma 5.1. The following differential equation \( \dot{w} = aw(t - \nu) - bw(t), \) where \( a, b, \nu > 0, w(t) > 0 \) for \( -\nu \leq t \leq 0 \) has three ultimate behaviour \( \lim_{t \to -\infty} w(t) = 0 \) if \( a < b \) and \( \lim_{t \to -\infty} w(t) = +\infty \) if \( a > b \).

Proof. Proof is given in Appendix A in [34]. □

Lemma 5.2. For any positive solution of the system (6), the relation \( s(t - \tau) > s(t)e^{-\tau} \) is always satisfied.

Proof. From the first equation of the system (6) we have \( \frac{ds}{dt} < s. \) Integrating it between the limits \( t - \tau \) and \( t \), we find

\[
s(t) < s(t - \tau)e^\tau \text{ from which } s(t - \tau) > s(t)e^{-\tau}. \quad \Box
\]

Theorem 5.1. The system (6) is permanent provided that for \( \epsilon_i, i = 1, 2, \) being sufficiently small positive numbers,

\[
e_1[(1 - c)(5 - e_1) + e_1] > d_2[1 + 5\epsilon + b(1 - c)(5 - e_2)], \quad \gamma(5 - e_2) > \frac{r_{3v} + r_{3e} + d_1}{1 + 5\epsilon}.
\]

In order to prove Theorem 5.1, we present the uniform persistence theory for infinite dimensional systems from [35].

Let \( X \) be a complete metric space. Suppose that \( X^0 \) is open, dense in \( X \) and \( X^0 \cup X_0 = X, X_0 \cap X^0 = \emptyset \). Assume that \( T(t) \) is a \( C^0 \) semi group on \( X \) satisfying

\[
T(t) : X^0 \to X^0,
\]

\[
T(t) : X_0 \to X_0.
\]

Let \( T_0 = T(t)_{|X_0} \) and let \( A_0 \) be the global attractor for \( T_0(t) \).

Lemma 5.3. Let us assume that \( T(t) \) satisfies (40) and we have the following [35]:

(i) there is a \( t_0 \geq 0 \) such that \( T(t) \) is compact for \( t > t_0 \);
(ii) \( T(t) \) is a point dissipative in \( X \);
(iii) \( A_0 = \bigcup_{\alpha \in \omega(x)} A_0(x) \) is isolated and has an acyclic covering \( M, \) where \( M = \{M_1, M_2, \ldots, M_n\}. \)
(iv) \( W^s(M_i) \cap X^0 = \emptyset \) for \( i = 1, 2, \ldots, n. \)

Then \( X_0 \) is a uniform repeller with respect to \( X^0, \) i.e., there is an \( \epsilon > 0 \) such that for any \( x \in X^0, \lim_{t \to +\infty} d(T^t)x, X_0 \geq \epsilon, \) where \( d \) is the distance of \( T(t)x \) from \( X_0. \) We are now able to state the Proof of the Theorem 5.1.
Proof of Theorem 5.1. We begin by showing that the boundary planes of $\mathbb{R}_+^3$ repel the positive solutions of system (6) uniformly. Let us define

$$C_1 = \{ (\phi_1, \phi_2, \phi_3) \in C([-\tau, 0], \mathbb{R}_+^3) : \phi_1(0) \neq 0, \phi_2(0) = 0, \phi_3(0) \neq 0, \theta \in [-\tau, 0] \},$$

$$C_2 = \{ (\phi_1, \phi_2, \phi_3) \in C([-\tau, 0], \mathbb{R}_+^3) : \phi_2(0) = 0, \phi_1(0) = 0, \phi_3(0) \neq 0, \theta \in [-\tau, 0] \},$$

$$C_3 = \{ (\phi_1, \phi_2, \phi_3) \in C([-\tau, 0], \mathbb{R}_+^3) : \phi_3(0) = 0, \phi_1(0) = 0, \phi_2(0) \neq 0, \theta \in [-\tau, 0] \}.$$ 

If $C_0 = C_1 \cup C_2 \cup C_3$ and $C^0 = \text{int}C([-\tau, 0], \mathbb{R}_+^3)$, suffices to show that there exists an $\epsilon_0 > 0$ such that for any solution $u_t$ of system (6) initiating from $C^0$, $\lim_{t \to \infty} \inf d(u_t, C_0) \geq \epsilon_0$. To this end, we verify below that the conditions of Lemma 5.3 are satisfied. It is easy to see that $C^0$ and $C_0$ are positively invariant. Moreover, conditions (i) and (ii) of Lemma 5.3 are clearly satisfied. Thus we only need to verify the conditions (iii) and (iv). There are three constant solutions $E_{0a}, E_{0b}$ and $E_{0c}$ in $C_0$, corresponding, respectively, to $(s(t) = 1, i(t) = 0, p(t) = 0)$, $(s(t) = \bar{s}, i(t) = \bar{i}, p(t) = \bar{p})$ and $(s(t) = \bar{s}, i(t) = \bar{i}, p(t) = \bar{p})$. If $(s(t), i(t), p(t))$ is a solution of the system (6) initiating from $C_3$ with $\phi_3(0) > 0$, it follows that $s(t) \to 1, i(t) \to 0, p(t) \to 0$ as $t \to +\infty$. If $(s(t), i(t), p(t))$ is a solution of the system (6) initiating from $C_2$ with $\phi_1(0) > 0$, $j = 1, 2$, it follows that $s(t) \to \bar{s}, i(t) \to \bar{i}, p(t) \to 0$ as $t \to +\infty$. If $(s(t), i(t), p(t))$ is a solution of the system (6) initiating from $C_1$ with $\phi_1(0) = 0, j = 1, 2$, it follows that $s(t) \to \bar{s}, i(t) \to \bar{i}, p(t) \to 0$ as $t \to +\infty$. If $(s(t), i(t), p(t))$ is a solution of the system (6) initiating from $C_0$, it follows that $s(t) \to \bar{s}, i(t) \to \bar{i}, p(t) \to 0$ as $t \to +\infty$. This shows that if the invariant sets $E_{0a}, E_{0b}$ and $E_{0c}$ are isolated invariant, $(E_{0a}, E_{0b}, E_{0c})$ is isolated and is an acyclic covering. It is obvious that $E_{0a}$ is isolated invariant. The isolated invariance of $E_{0a}$ and $E_{0c}$ will follow from the following proof.

We now show that $W^s(E_{0a}) \cap C^0 = \phi$, $W^s(E_{0b}) \cap C^0 = \phi$, and $W^s(E_{0c}) \cap C^0 = \phi$. We restrict our attention to the second and third equations, since the proof of the first is simple. Assuming the contrary, i.e., $W^s(E_{0a}) \cap C^0 = \phi$, then there exists a positive solution $(s(t), i(t), p(t))$ of the system (6) such that $(s(t), i(t), p(t)) \to (\bar{s}, \bar{i}, 0)$ as $t \to +\infty$. We can choose $\epsilon_0 > 0$ small enough such that $\bar{s} - \epsilon_1 < s(t) < \bar{s} + \epsilon_1$ for some large $t > \tau$. Then, from the third equation of the system (6) for $t > \tau$, we have

$$\frac{dp}{dt} \geq e_{01}(1 - c\bar{s} + e_{02}\bar{p}) \geq e_{01}(1 - c\bar{s} + e_{02}\bar{p}) - d_2p.$$ 

Let us consider that

$$w = \frac{e_{01}(1 - c\bar{s} + e_{02}\bar{p}) - d_2p}{1 + e_{01} + b(1 - c\bar{s} + e_{02}\bar{p})}.$$ 

Let $u_t$ and $n > 0$ be small enough such that $nu_t < p(t)$. If $w_1$ is a solution of the system (41) satisfying $w_1(t) = nu_t$, we know from the comparison theorem in [34], $p(t) > w_1(t)$ for all $t > \tau$. Using condition of Lemma 5.1, we have $w_1(t) \to +\infty$. Therefore, $p(t) \to +\infty$ as $t \to +\infty$. This contradicts the boundedness of the system (6), given by Proposition 3.2. Hence $W^s(E_{0b}) \cap C^0 = \phi$.

Again assuming the contrary, i.e., $W^s(E_{0c}) \cap C^0 = \phi$, then there exists a positive solution $(s(t), i(t), p(t))$ of the system (6) such that $(s(t), i(t), p(t)) \to (\bar{s}, 0, 0)$ as $t \to +\infty$. Let us choose $\epsilon_0 > 0$ small enough such that $\bar{s} - \epsilon_2 < s(t) < \bar{s} + \epsilon_2$ and $\bar{p} - \epsilon_2 < p(t) < \bar{p} + \epsilon_2$ for some large $t > \tau$. Then, from the second equation of the system (6) for $t > \tau$, we have using Lemma 5.2

$$\frac{di}{dt} = \gamma i(t - \tau)s(t - \tau) - \frac{a_{21}i}{1 + x_i} - d_1i \geq \gamma i(t - \tau)s(t - \tau)e^{-\gamma\tau} - \frac{a_{21}i}{1 + x_i} - d_1i \geq \gamma i(t - \tau)(\bar{s} - \epsilon_2)e^{-\gamma\tau} - \frac{a_{21}(\bar{p} + \epsilon_2)}{1 + x_i} - d_1i.$$ 

We consider that

$$Z = \gamma\bar{Z}(\bar{s} - \epsilon_2)e^{-\gamma\tau} - \frac{a_{21}(\bar{p} + \epsilon_2)}{1 + x_i} - d_1i.$$ 

Let $u_2$ and $n > 0$ be small enough such that $nu_2 < p(t)$. If $Z_1$ is a solution of the system (42) satisfying $Z_1(t) = nu_2$, we know from the comparison theorem in [34], $i(t) > Z_1(t)$ for all $t > \tau$. Using condition of Lemma 5.1, we have $Z_1(t) \to +\infty$. Therefore, $i(t) \to +\infty$ as $t \to +\infty$. This contradicts the boundedness of the system (6), given by Proposition 3.2. Thus $W^s(E_{0c}) \cap C^0 = \phi$. Therefore, we conclude from Lemma 5.3 that $C_0$ repels the positive solutions of system (6) uniformly.

6. Numerical results

In this section, we present numerical experiments on the system (6) to verify our analytical findings. Few numerical results are listed in the Table 1.

Fig. 1(a) is the graphical representation of basic reproduction number $R_0$ with respect to infection rate $\gamma$ for fixed $d_1 = 0.08$. From Fig. 1(b), it is clear that the system (6) is disease free if $R_0 < 1$ and infection exists in prey for $R_0 > 1$ for $\tau = 0.5$. Similar behaviour is observed from Fig. 1(c) for $\tau = 3$. Therefore, system (6) has disease free equilibrium if $R_0 < 1$ and the system (6) admits endemic equilibrium if $R_0 > 1$, which is consistent with our theoretical analysis.

Figs. 2–6 represents the nature of different equilibrium points for different parameter values (taken from Table 1). Fig. 2 shows the stability of disease free equilibrium point $E_{0F}$ for all discrete delay $\tau \geq 0$. For $\tau = 0.5 < \tau_1 = 0.755$, $E_{0F}$ is globally...
Table 1
The set of parameter values and their corresponding figures.

<table>
<thead>
<tr>
<th>Fixed parameters</th>
<th>$\tau$</th>
<th>Nature/results of equilibrium points</th>
<th>Figures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1 = 1.25, a_2 = 0.12, b = 3, d_1 = 0.08, d_2 = 0.04, e = 0.25, c = 0.1, e = 0.3, \gamma = 1, \alpha = 0.8, \zeta = 0.195$</td>
<td>$\tau &gt; 0$</td>
<td>GAS around $E_{qu}$</td>
<td>Fig. 2</td>
</tr>
<tr>
<td>$a_1 = 2, a_2 = 0.12, b = 3, d_1 = 0.08, d_2 = 0.04, e = 0.25, c = 0.3, e = 0.3, \gamma = 1.5, \alpha = 0, \zeta = 0$</td>
<td>0.5</td>
<td>GAS around $E_Y$</td>
<td>Fig. 3</td>
</tr>
<tr>
<td>$a_1 = 2, a_2 = 0.12, b = 3, d_1 = 0.08, d_2 = 0.04, e = 0.25, c = 0.3, e = 0.3, \gamma = 1.5, \alpha = 0, \zeta = 0$</td>
<td>0.8</td>
<td>Existence of limit cycle around $E_Y$</td>
<td>Fig. 4</td>
</tr>
<tr>
<td>$a_1 = 1.25, a_2 = 0.12, b = 3, d_1 = 0.08, d_2 = 0.04, e = 0.25, c = 0.3, e = 0.3, \gamma = 1.5, \alpha = 0, \zeta = 0$</td>
<td>0.5</td>
<td>GAS around $E^*$</td>
<td>Fig. 5</td>
</tr>
<tr>
<td>$a_1 = 1.25, a_2 = 0.12, b = 3, d_1 = 0.08, d_2 = 0.04, e = 0.25, c = 0.3, e = 0.3, \gamma = 1.5, \alpha = 0, \zeta = 0.3$</td>
<td>0.75</td>
<td>Existence of limit cycle around $E^*$</td>
<td>Fig. 6</td>
</tr>
</tbody>
</table>

![Graphical representation of Basic reproduction number $R_0$](a)  
![Infected prey $I$](b)  
![Infection rate $\gamma$](c)

Fig. 1. Graphical representation of Basic reproduction number $R_0$. Figure depicts that the system has disease free equilibrium if $R_0 < 1$ and the system admits endemic equilibrium for $R_0 > 1$.

![Global stability of the system](a)  
![3D view in the phase space](b)

Fig. 2. Global stability of the system (6) around disease free equilibrium point $E_{qu}$ and its 3D view in the phase space. Parameter values are given in the Table 1.
Fig. 3. Global stability of the system (6) around predator free equilibrium point $E_R$ for time delay $\tau = 0.5 < \tau_1 = 0.755$ and its 3D view in the phase space. Parameter values are given in the Table 1.

Fig. 4. Limit cycle behaviour of the dynamical system (6) around predator free equilibrium point $E_R$ with discrete time delay $\tau = 0.8 > \tau_1 = 0.755$ and its 3D view in the phase space. Parameter values are given in the Table 1.

stable, illustrated in Fig. 3. When $\tau$ is closer to $\tau_1$, its oscillatory behaviour increases. Ultimately $E_R$ loses its stability when $\tau = 0.8 > \tau_1$, shown in Fig. 4. Figs. 5 and 6 illustrate the dynamical behaviour of endemic equilibrium point $E^*$. Fig. 5 shows the global stability of endemic equilibrium point $E^*$ for $\tau = 0.5 < \tau_2 = 0.71$; when $\tau = 0.75 > \tau_2$, the existence of limit cycle around $E^*$ is observed in Fig. 6. It is also noted that the value of $\tau_2$ decreases monotonically with the increase of either quality or quantity of additional food. Therefore, suitable additional food helps to obtain oscillatory coexistence of species.

7. Impacts of additional food on disease control

In this section, we have discussed the possible impacts of additional food to predator in the presence of infection in prey through numerical simulations. For this convenience, we perform numerical simulations with the help of parameter values taken from data of field works and experimental data (see, Table 2) which remains fixed for all numerical simulations. In our numerical simulations we have varied the infection rate $\gamma$, quality of additional food $x$ and quantity of additional food $\xi$.

We draw a bifurcation diagram of infected prey in absence of additional food ($x = 0, \xi = 0$) in Fig. 7. From Fig. 7(a), we observe that the system is disease free within $0 \leq \gamma \leq 1.7$ for $\tau = 0.1$. For $\tau = 0.5$, Fig. 7(b) shows that the system is disease free within $0 \leq \gamma \leq 1.9$. The system is disease free within $0 \leq \gamma \leq 2.05$ for $\tau = 1$ (Fig. 7(c)) and for $\tau = 2$, Fig. 7(d) depicts a disease free system within $0 \leq \gamma \leq 2.35$. Therefore, from Fig. 7, it is clear that the system (8) becomes disease free for low
Fig. 5. Global stability of the system (6) around endemic equilibrium point $E^\ast$ for time delay $\tau = 0.5 < \tau_\ast = 0.71$ and its 3D view in the phase space. Parameter values are given in the Table 1.

Fig. 6. Limit cycle behaviour of the dynamical system (6) around endemic equilibrium point $E^\ast$ with discrete time delay $\tau = 0.75 > \tau_\ast = 0.71$ and its 3D view in the phase space. Parameter values are given in the Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Range</th>
<th>Default value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>Attack rate of susceptible prey</td>
<td>0.61–5.087 [36]</td>
<td>2.0</td>
</tr>
<tr>
<td>$a_2$</td>
<td>Attack rate of infected prey</td>
<td>0–0.5 [37]</td>
<td>0.12</td>
</tr>
<tr>
<td>$b$</td>
<td>Half saturation constant</td>
<td>2–6.2 [36]</td>
<td>3.0</td>
</tr>
<tr>
<td>$c$</td>
<td>Conversion rate of prey</td>
<td>0.05–0.33 [39]</td>
<td>0.25</td>
</tr>
<tr>
<td>$e$</td>
<td>Conversion rate of additional food</td>
<td>$0 &lt; e &lt; 1$ [10]</td>
<td>0.3</td>
</tr>
<tr>
<td>$d_1$</td>
<td>Infected prey mortality rate</td>
<td>0.04–0.16 [40]</td>
<td>0.08</td>
</tr>
<tr>
<td>$d_2$</td>
<td>Predator mortality rate</td>
<td>0.04–0.16 [40]</td>
<td>0.04</td>
</tr>
<tr>
<td>$c$</td>
<td>Habitat complexity</td>
<td>$0 &lt; c &lt; 1$ [41]</td>
<td>0.3</td>
</tr>
</tbody>
</table>
Fig. 7. Bifurcation of infected prey with respect to infection rate $\gamma$ in absence of additional food ($\alpha = 0$, $\xi = 0$). Parameter values are given in the Table 2.

Fig. 8. Bifurcation of infected prey with respect to quantity of additional food $\xi$ for $\gamma = 0.1$. Parameter values are given in the Table 2.

Fig. 9. Bifurcation of infected prey with respect to quantity of additional food $\xi$ for $\gamma = 0.5$. Parameter values are given in the Table 2.
Fig. 10. Bifurcation of infected prey with respect to quantity of additional food ξ for τ = 1. Parameter values are given in the Table 2.

Fig. 11. Bifurcation of infected prey with respect to quantity of additional food ξ for τ = 2. Parameter values are given in the Table 2.

Fig. 12. Infected and uninfected regions in the plane (τ, ξ) fixing γ = 3 and x = 0.8. Other parameter values are taken from the Table 2.
infection rate depending on delay effects. But, disease persists in the system in absence of additional food for higher infection rate.

Figs. 8–11, illustrate the bifurcation diagrams of infected prey with respect to quantity of additional food with fixed high quality of additional food ($\alpha = 0.8 < 1$) for higher infection rate. We have also investigated the effects of infection time delay in Figs. 8–11. The parameter values are taken from Table 2. From Figs. 8–11, it is evident that for higher infection rate, suitable quantity of additional food to predator is helpful to remove disease of prey. It is also observed that in case of low delay, larger quantity of additional food is required compare to higher delay.

The $\tau$ plane is divided into infected and uninfected regions, which is presented in Fig. 12 for fixed infection rate $\gamma = 3$ and for quality of additional food $\alpha = 0.8$. We observe, from Fig. 12, that infected region decreases when time delay increases. On the other hand, small quantity of additional food is sufficient to obtain infection free system in higher time delay. The regions help to choose suitable quantity of additional food for disease free system.

8. Conclusions

In presence of additional food to predator a delayed ecoepidemic model is formulated in which disease spreads among prey individuals. The proposed model is investigated both analytically and numerically incorporating the effects of time delay as well as the effects of additional food to predator. We derive that the infected prey will extinct if $R_0 < 1$ (Proposition 3.3) and which is consistent with our simulation results presented in Fig. 1. We examine the role of delay in our model at disease free, predator free and endemic equilibrium points using parameter values of Table 1. For the system with time lag, a threshold time delay $\tau_0$ has been determined such that below which the disease free equilibrium $E_{DF}$ is stable, above which oscillations arise (Theorem 4.2). The numerical stability behaviour of the system at $E_{DF}$ is presented in Fig. 2. Second threshold time delay $\tau_1$ has been identified as the Hopf bifurcation point of predator free equilibrium $E_{PF}$ (Theorem 4.4) and the corresponding numerical simulation results are plotted in Figs. 3 and 4. The threshold $\tau_2$ has been found leading to the Hopf bifurcation at the endemic equilibrium point $E^*$ (Theorem 4.6) and which matches very well with the simulation results presented in Figs. 5 and 6.

The impact of additional food is examined in our model for fixed delay and fixed infection rate. We find the number of infected prey population in absence of additional food by varying infection rate (Fig. 7) using parameter values taken from experimental and field data (Table 2). It is observed that the system automatically becomes disease free for lower infection rate, but for higher infection rate the infected prey exists in the system (Fig. 7). We observe from Figs. 8–11 that the system becomes disease free in presence of suitable additional food. Additional food increases predator population which helps to remove disease of prey. It is also pointed out that large quantity of additional food is required for higher infection rate (for small delay) to make the system disease free. On the other hand, for lower infection rate (low delay), low quantity of additional is sufficient. Actually, low predator population will increase the incidence of prey infection and as a result it reduces the number of susceptible prey and ultimately overall prey population decreases. In the present paper, we introduce the habitat complexity effects on attack rate on prey species as well as the time-delay effects on the conversion rate of infection from susceptible prey to infected prey. These two effects are not considered by Sahoo [27] in his model. We observe that infection time–delay in the model is helpful to obtain disease free system for higher infection rate. At higher infection rate disease control is possible if the infection time delay is large but it is impossible without time delay [27] or very low infection time delay.

Sih et al. [17] examined predator–removal experiments and found that 54 of 135 systems in which prey populations subsequently declined. Similarly, Cote and Sutherland [18] reported that predator removal reduced prey populations in 3 out of 11 controlled studies. Therefore, predator removal or reduction of predator population is more likely to be harmful when prey is virulent [19]. However, the supply of additional food will increase the number of predator population and then it takes infected prey at higher rate and as a result, after certain time the system becomes disease free. It is also predicted by Packer et al. [19] that if predators primarily capture common prey species and reduce their level of infection, the predators may reduce the potential impact of infection to rare prey species. They also remarked (by removing infectious individuals from the wildlife population) that predators not only reduce the force of infection in wildlife by removal predator infection, but indirectly reduce the impact of disease on the domesticated species. Therefore, our results are consistent with the experimental results in different population system and we can also conclude that additional food to predator is very essential for conservation of species. Results presented in this paper provided a useful platform to understand the role of additional food as disease controller in an ecoepidemic food chain model. A possible interesting aspect for future development is to study the effects of additional food on coupled infected food chain models. Another important aspect is to investigate the role of additional food in disease control for different form of infection transfer. A non-chemical method of disease control was shown first time in the study of [9]. We have introduced another non-chemical method of disease control will be very useful for biological conservation of species in real world biological systems.

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References

[27] B. Sahoo, Disease control through provision of alternative food to predator: a model based study, Int. J. Dyn. Control (2014), <http://dx.doi.org/10.1007/s40441-014-0095-0>.
[31] Y. Song, M. Han, J. Wei, Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays, Physica D 205 (2005) 185–204.
The chaos and control of a food chain model supplying additional food to top-predator

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A R T I C L E  I N F O

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A B S T R A C T

The control and management of chaotic population is one of the main objectives for constructing mathematical model in ecology today. In this paper, we apply a technique of controlling chaotic predator–prey population dynamics by supplying additional food to top-predator. We formulate a three species predator–prey model supplying additional food to top-predator. Existence conditions and local stability criteria of equilibrium points are determined analytically. Persistence conditions for the system are derived. Global stability conditions of interior equilibrium point is calculated. Theoretical results are verified through numerical simulations. Phase diagram is presented for various quality and quantity of additional food. One parameter bifurcation analysis is done with respect to quality and quantity of additional food separately keeping one of them fixed. Using MATCONT package, we derive the bifurcation scenarios when both the parameters quality and quantity of additional food vary together. We predict the existence of Hopf point (H), limit point (LP) and branch point (BP) in the model for suitable supply of additional food. We have computed the regions of different dynamical behaviour in the quantity–quality parametric plane. From our study we conclude that chaotic population dynamics of predator prey system can be controlled to obtain regular population dynamics only by supplying additional food to top predator. This study is aimed to introduce a new non-chemical chaos control mechanism in a predator–prey system with the applications in fishery management and biological conservation of prey predator species.

1. Introduction

The dynamic behaviour of predator–prey model has been continuing as one of the most dominant themes in mathematical ecology due to its universal existence and importance. A variety of ecological systems are well modelled by nonlinear systems. In most cases such systems are capable of displaying chaotic behaviour [1–7]. The population dynamics of a large class of ecosystems can be effectively modelled by deterministic chaotic systems [1,8–10]. It is well known that chaotic systems are unpredictable in nature. The role of chaos in population dynamics involves severe implications for the management of species. Chaotic oscillations may lead to small densities making possible population extinction as well as to high densities making the population susceptible for disease outbreaks or over-crowding [11]. Schaffer et al. [12] argued that chaos may be a more important phenomenon and control of chaos is a challenging matter. In the case of transient chaos, the dynamical variables of the system behave chaotically for a period of time and then switch their behaviour suddenly, say, to a fixed value or to some regular patterns including zero population density. An ecosystem exhibits transient chaos means that the population size of some species can behave chaotically for a long period of time and then decreases to zero in a relatively short period of time. It was shown by McCann and Yodzis that such a transient chaotic behaviour, which is responsible for species extinction, can indeed occur in a simple three-species food chain model which incorporates biologically...
reasonable assumptions about species interactions. The control and the management of chaotic population dynamics is one of the main objectives of mathematical modelling in ecology today.

After the work of HP [1], many researchers reported various chaos control methods in this model by including various ecological factors. Eisenberg and Massle [13] observed that gradual addition of refugia provide a stabilizing influence for which the chaotic dynamics collapsed to stable limit cycle oscillation. Deterministic models of populations with chaos control are well known in the literature [14,15]. Doebeli and Ruxton [16] proposed two methods to control chaos in an ecological metapopulation with long-range dispersal. McCann and Hastings [17] stabilized the food web by eliminating chaotic dynamics and obtained limit cycle behaviour of the HP model by introducing omnivore on top-predator. Chattopadhyay et al. [18] obtained limit cycle behaviour of the HP model by introducing long-range dispersal. McCann and Hastings [17] stabilized the food web by eliminating chaotic dynamics and obtained limit cycle behaviour of the HP model by introducing an extra mortality term in zooplankton population to investigate the reduction of propensity of chaotic dynamics as described by Hastings and Powell. The dynamics of a prey–predator fishery with predator self limitation has been proposed and analysed by Kar et al. [19]. Maity et al. [20] focuses on the method of insect pest control by the Sterile Insect Release Method. Recently, Gakkhar et al. [21] introduced the additional predator in HP model for controlling chaos and obtained limit cycle oscillations and Huang et al. [22] reported the results of impulsive control for a predator–prey Comperczt system with stage structure.

The traditional approach of modelling prey predator interaction is often based on an organism’s primary function within a food web (e.g., prey and predator). Additional foods are an important component of most predators (e.g., coccinellid) diet, and although they receive less attention than basal prey in the scientific literature, these foods fundamentally shape the life histories of many predator species. The availability of suitable additional food (non-prey food) in a predator prey system can have significant impact on the dynamics of the system. The consequences of providing additional food to predator and the corresponding effects on the predator prey dynamics and its utility in biological control (such as species conservation and pest management) have been the topic of great attention for many scientists [23]. In recent years, many biologist, experimentalists, and theoreticians investigated the consequences of providing additional food to predators in a predator–prey systems [23–28]. Almost all predators will attempt to switch to another prey when the preferred prey is in low numbers and they may also resort to scavenging or a herbivorous diet if possible. An example of a predator–prey system with alternative food source is the hare–caribou–lynx relationship in Newfoundland [29].

Although many aspects of additional food was investigated, the chaos controllability aspects by supplying additional food was unexplored. In this paper, we apply a non-chemical method supplying additional food to top-predator for controlling chaos. This is a new technique and it has no pollutants factor or infection risk of the population system. So, it has an advantage to apply in the food chain model. In this paper, we investigate the effects of supplying additional food to top-predator in Hastings and Powell’s (HP) [1] three species chaotic food chain model. The section-wise split of this paper is as follows: In Section 2, the model representing the dynamics of a three species food chain in the presence of additional food to top-predator is introduced. In Section 3, the local stability analysis of equilibrium points, global dynamics and persistence criteria of the proposed model are determined. Section 4 illustrates some of the key findings through numerical simulations. In that section, we have studied the stability zones in parameter plane and the different routes of continuation of the associated bifurcations are investigated with the help of the software package MATCONT [30–32]. Finally, we draw a conclusion in Section 5.

2. Model formulation

The model we consider in this paper describes a tri-trophic food chain composed of a logistic prey (X), a Holling type II intermediate predator (Y), and a Holling type II top-predator (Z). In particular, we consider the HP model [1]:

\[
\begin{align*}
\frac{dX}{dT} &= R_0X \left( 1 - \frac{X}{K_0} \right) - C_1A_1 YZ, \\
\frac{dY}{dT} &= A_1 YZ - A_2 YZ - D_1 Y, \\
\frac{dZ}{dT} &= C_2A_2 YZ - D_2 Z,
\end{align*}
\]

where \( T \) is time. The constant \( R_0 \) is the “intrinsic growth rate" and the constant \( K_0 \) is the “ carrying capacity" of the species X. The constants \( C_1 \) and \( C_2 \) are conversion rates of prey to predators for species Y and Z respectively; \( D_1 \) and \( D_2 \) are constant death rates for species Y and Z, respectively. The constants \( A_1 \) and \( A_2 \) for \( i = 1, 2 \) are maximal predation rate and half saturation constants for Y and Z, respectively. The model is biologically meaningful only when the parameters are strictly positive. We modify the model (1) by supplying “additional food" to top-predator population. The additional food is assumed to be either a non-reproducing prey or some food source. We do not make any distinction regarding the additional food like complementary, essential or alternative. It is assumed that the supply of additional food is not dynamic, but maintained at a specific constant level. This simplification is justified for many arthropod predators because they can feed on plant-provided alternative food sources such as pollen or nectar, the availability of which is unlikely to be influenced by predators consumption [33].

We consider the following assumptions:

(a) Top-predators are provided with additional food of constant biomass \( A \) which is distributed uniformly in the habitat. The constant supply level is maintained either by nature or by external agency.

(b) The number of encounters per top-predator with the additional food is proportional to the density of the additional food. Therefore, there exists a proportionality constant for the system.
The term \( z \) which the food consumed by top-predator gets converted to food. If the relation is proportionally related to the “quality” of the additional food \([28,34]\). Therefore, for high quality of additional food than source food (intermediate predator).

The functions of the right hand sides of system (3) are continuous and have continuous partial derivatives on the state space \( \mathbb{R}^3_+ = \{ (x(t), y(t), z(t)) : x(t) \geq 0, y(t) \geq 0, z(t) \geq 0 \} \). Thus, they are Lipschitzian on \( \mathbb{R}^3_+ \) and therefore the solution of system (3) with non-negative initial condition exists and is unique. Moreover, from Cao et al. [35], it is easy to show that the state space \( \mathbb{R}^3_+ \) is an invariant domain of system (3).

3.3. Positive invariance

System (3) can be written in the matrix form as

\[
X = F(X), \quad X(0) = X_0 \in \mathbb{R}^3_+, \quad \text{where } X = (x, y, z)^T \in \mathbb{R}^3_+ \quad \text{and } F(X) \text{ is given by}
\]

\[
F = F(X) = \begin{pmatrix}
\frac{x(1-x)}{1+x} & a_{11} & a_{12} \\
1 + b_1 & a_{21} & a_{22} \\
1 + c_1 + b_2 & a_{31} & a_{32}
\end{pmatrix},
\]

where \( F : \mathbb{R}^3_+ \rightarrow \mathbb{R}^3 \) and \( F \in C^\infty(\mathbb{R}^3_+) \).

It can be shown that whenever \( X(0) \in \mathbb{R}^3_+ \) such that \( X_i = 0 \) then \( F(X(t)) |_{t \geq 0} \geq 0 \) (for \( i = 1, 2, 3 \)). Now any solution of \( F(X) \) with \( X_0 \in \mathbb{R}^3_+ \), say \( X(t) = X(t, X_0) \), is such that \( \dot{X}(t, X_0) \in \mathbb{R}^3_+ \) for all \( t > 0 \) [36].

3.2. Boundedness

**Theorem 1.** All solutions of system (3) which start in \( \mathbb{R}^3_+ \) are uniformly bounded.

**Proof.** Let \( (x(t), y(t), z(t)) \) be any solution of system (3) with positive initial conditions.

Let us consider that, \( w = x + y + z \), i.e.,

\[
\frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}.
\]

Using equations of (3), we have

\[
\frac{dw}{dt} = x(1-x) - a_{11} - a_{12} + a_{21} + a_{22} + a_{31} + a_{32} - d_1 y - d_2 z,
\]

i.e.,

\[
\frac{dw}{dt} = x(1-x) + \frac{a_{21} + a_{31} + a_{22} + a_{32} - d_1 y - d_2 z}{1 + c_1 + b_2}.
\]

Therefore,

\[
\frac{dw}{dt} \leq x - x^2 + a_{21} + a_{31} + a_{22} + a_{32} - d_1 y - d_2 z.
\]
i.e. \( \frac{dw}{dt} \leq -\theta(x + y + z) - (x - 1)^2 + 1 \).

where \( \theta = \min(1, d_1, d_2, a_\sigma c_{\xi}) \), provided \( d_2 > a_\sigma c_{\xi} \).

i.e. \( \frac{dw}{dt} + \theta w \leq 1 - (1 - \chi)^2 \).

Therefore, \( \frac{dw}{dt} + \theta w \leq 1 \).

Applying the theory of differential inequality we obtain

\[ 0 < w < \frac{1 - \chi}{\theta} + w(x(0), y(0), z(0))e^{-\theta t}. \]

Therefore, for \( t \to \infty \), we have \( 0 < w < \frac{1}{\theta} \). Hence, all the solutions of system (3) that initiate in \( \mathbb{R}^3 \) are confined in the region \( S = \{(x, y, z) \in \mathbb{R}^3 : w = \frac{1}{\theta} + \eta \text{ for any } \eta > 0 \} \).

This proves the theorem. \( \square \)

3.3. Existence and local stability criteria of equilibrium points

System (3) possesses the following equilibrium states:

(a) The trivial equilibrium state \( E_1 \equiv (0, 0, 0) \). Since 1 is an eigenvalue of the Jacobian matrix at \( E_1 \), therefore \( E_1 \) is an unstable equilibrium point.

(b) The axial equilibrium state \( E_2 \equiv (1, 0, 0) \). The Jacobian matrix at the equilibrium point \( E_2 \) is

\[
J(E_2) = \begin{pmatrix}
-1 & -\frac{a_1}{r_1^2} & 0 \\
0 & \frac{a_1}{r_1^2} & -d_1 \\
0 & 0 & -\frac{a_2}{r_2^2} - d_2
\end{pmatrix}
\]

The axial equilibrium point \( E_2 \) is stable if \( \frac{a_1}{r_1^2} < d_1 \) and \( \frac{a_2}{r_2^2} < d_2 \). The equilibrium point \( E_2 \) is a saddle point when either \( \frac{a_1}{r_1^2} < d_1 \) and \( \frac{a_2}{r_2^2} > d_2 \) or \( \frac{a_1}{r_1^2} > d_1 \) and \( \frac{a_2}{r_2^2} < d_2 \).

(c) The boundary equilibrium state \( E_3 \equiv (x, y, 0) \), where \( x = \frac{a_2}{r_2^2} + \frac{a_1}{r_1^2} + y \equiv \frac{a_2}{r_2^2} + b_1 \). The equilibrium point \( E_3 \) exists if \( a_1 > (b_1 + 1)d_1 \). The Jacobian matrix at \( E_3 \) is given by

\[
J(E_3) = \begin{pmatrix}
1 - 2x - \frac{a_1/y}{(1 + a_1/y)} & -\frac{a_1}{r_1^2} & 0 \\
\frac{a_1}{r_1^2} & \frac{a_1}{r_1^2} - d_1 & -\frac{a_2}{r_2^2} \\
0 & 0 & -\frac{a_2}{r_2^2} - d_2
\end{pmatrix}
\]

The characteristic roots of the Jacobian matrix \( E_3 \) are \( \frac{a_2/y}{(1 + a_1/y)} - d_2 \) and the roots of the equation \( \lambda^2 + \sigma_1 \lambda + \sigma_2 = 0 \), where \( \sigma_1 = -[1 - 2x - \frac{a_1/y}{(1 + a_1/y)} + \frac{a_1}{r_1^2} - d_1] \) and \( \sigma_2 = \left| (1 - 2x - \frac{a_1/y}{(1 + a_1/y)})(\frac{a_1}{r_1^2} - d_1) + \frac{a_1}{r_1^2} \right| \).

The boundary equilibrium point \( E_3(x, y, 0) \) is stable if \( \frac{a_2/y}{(1 + a_1/y)} < d_2, \sigma_1 > 0 \) and \( \sigma_2 > 0 \).

(d) The interior equilibrium state \( E \equiv (x^*, y^*, z^*) \), where \( y^* = \frac{a_2}{r_2^2} + \frac{a_1}{r_1^2} - z^* = (1 + a_1/y^*) \frac{a_1}{r_1^2} - d_1 \) and \( x^* \) is the positive root of the equation \( px^2 + Qx^* + R = 0 \), where \( P = b_1 \), \( Q = 1 - b_1 \) and \( R = \frac{a_2}{r_2^2} - \frac{a_1}{r_1^2} - 1 \). The positive interior equilibrium point \( E \) is stable if \( d_2(1 + a_2^2) > a_2^2 c_\zeta^2, a_2 > b_2 d_2, \frac{a_1}{r_1^2} > d_1 \) and \( Q^2 > 4PR, Q < 0 \).

The Jacobian matrix at \( E \) is given by

\[
J(E) = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{pmatrix}
\]

where, \( A_{11} = 1 - 2x - \frac{a_1/y}{(1 + a_1/y)}, A_{12} = -\frac{a_1}{r_1^2}, A_{13} = 0 \), \( A_{21} = \frac{a_1}{r_1^2} - \frac{a_1}{r_1^2} - d_1, A_{23} = -\frac{a_2}{r_2^2} \), \( A_{31} = 0, A_{32} = \frac{a_2}{r_2^2} + b_2 d_2, A_{33} = 0 \).

The characteristic equation of the Jacobian matrix \( J(E) \) is given by

\[
\lambda^3 + \Omega_1 \lambda^2 + \Omega_2 \lambda + \Omega_3 = 0
\]

where, \( \Omega_1 = -[A_{11} + A_{32}], \Omega_2 = [A_{11}A_{22} - A_{12}A_{21} - A_{13}A_{23}], \Omega_3 = A_{11}A_{32}A_{23} \).

Obviously, we can determine that \( A_{11} < 0, A_{32} < 0, A_{13} > 0, A_{23} < 0 \) and \( A_{12} > 0 \). Here \( \Omega_3 = -[A_{11}A_{32}] > 0 \) if \( A_{12} < 0 \) and obviously \( \Omega_3 = [A_{11}A_{23}] > 0 \).

Now \( \Omega_2 = -A_{12}A_{32} - (A_{11} + A_{23})(A_{11}A_{22} - A_{12}A_{21}) \).

If \( A_{23} < 0 \), then \( A_{12}A_{32} > 0 \), \( (A_{11} + A_{23}) < 0 \) and \( (A_{11}A_{22} - A_{12}A_{21}) > 0 \). Thus \( \Omega_1, \Omega_2 > 0 \).

Therefore the system (3) is locally stable at the interior equilibrium state \( E \) if \( \frac{a_1}{r_1^2} - \frac{a_2}{r_2^2} > d_1 \).

3.4. Persistence

Butler et al. [37] and Freedman and Waltman [38,39] developed the following definitions of persistence:

If a population \( N(t) \) is such that \( N(t) > 0 \), then

(i) \( N(t) \) is said to be weakly persistent if \( \lim_{t \to \infty} \sup N(t) > 0 \).

(ii) \( N(t) \) is said to be (strongly) persistent if \( \lim_{t \to \infty} \inf N(t) > 0 \).

Further, if \( N(t) \in C, \) where \( C \) is a certain class of function, then.

(iii) \( N(t) \) is said to be weakly uniformly persistent if there exists \( \delta > 0 \) such that \( \lim_{t \to \infty} \sup N(t) > \delta \) for all \( N(t) \in C \).

(iv) \( N(t) \) is said to be uniformly persistent if there exists \( \delta > 0 \) such that \( \lim_{t \to \infty} \inf N(t) > \delta \) for all \( N(t) \in C \).

From biological point of view, persistence of a system means the long term survival of all populations of the system, no matter what the initial populations are. Mathematically, persistence of a system means that strictly positive solutions do not have omega \( (\Omega) \) limit points on the boundary of the non-negative cone.

Theorem 2. Let the following conditions are satisfied

(i) \( \frac{a_1}{r_1^2} > d_1 \),
(ii) \( \frac{a_1}{r_1^2} < d_2 \),
(iii) \( 1 + \frac{a_2}{r_2^2} < d_1 + 2x + \frac{a_1}{r_1^2} \),
(iv) \( |(1 - 2x - \frac{a_1/y}{(1 + a_1/y)})(\frac{a_1}{r_1^2} - d_1) + \frac{a_1}{r_1^2}| > 0 \).
and if there exists a finite number \((n)\) of periodic solutions \(x = \phi(t), y = \psi(t), r = 1, 2, 3, \ldots, n\) in the \(xy\) plane, then system (3) is uniformly persistent provided for each periodic solutions of period \(T\).

\[ r = -d_2 + \frac{\alpha x y}{b + x y}, \quad r = 1, 2, 3, \ldots, n. \]

Proof. Let \(X = (x, y, z)\) be a point in the positive quadrant and \(O(x)\) be orbit through \(X\) and \(\Omega\) be the omega limit set of the orbit through \(X\). Note that \(\Omega(X)\) is bounded.

We claim that \(E_T \not\in \Omega(X)\), but if \(E_T \in \Omega(X)\) then by the Butler-McGehee lemma [38] there exist a point \(P\) in \(\Omega(X) \cap W^s(E_T)\) where \(W^s(E_T)\) denotes the stable manifold of \(E_T\). Since \(O(P)\) lies in \(\Omega(X)\) and \(W^s(E_T)\) is the \(yz\) plane, we conclude that \(O(P)\) is unbounded, which is contrary to the boundedness of the system.

Next \(E_T \not\in \Omega(X)\), for otherwise, since \(E_T \in \Omega(X)\), Conditions (iii) and (iv) implies that the eigenvalues of \(J(E_T)\) have negative real parts. Let \(\gamma_1, \gamma_2, \ldots, \gamma_n\) denote the closed orbit of the periodic solution \((\phi(t), \psi(t))\) in \(xy\) plane such that \(\gamma_i\) lies inside \(\gamma_{i-1}\). Let the Jacobian matrix \(J_i(\phi(t), \psi(t), 0)\) corresponding to \(\gamma_i\) is given by

\[
J_i = \begin{pmatrix}
F_x(\phi(t), \psi(t), 0) & 0 & 0 \\
F_y(\phi(t), \psi(t), 0) & F_z(\phi(t), \psi(t), 0) & 0 \\
F_w(\phi(t), \psi(t), 0) & F_y(\phi(t), \psi(t), 0) & F_z(\phi(t), \psi(t), 0)
\end{pmatrix}
\]

Here \(F_w = -d_2 + \frac{a_1 x y}{b + x y} \frac{a_2 y z}{1 + a_1 + a_2 y z}\). Computing the fundamental matrix of the linear periodic system \(M = J_i(t)M\), \(M(0) = M_0\), we find that its Floquet multiplier in the \(z\)-direction is \(e^{\nu(t)}\). Then proceeding in an analogous manner like Kumar and Freedman [40], we conclude that no \(\gamma_i\) lies on \(\Omega(X)\). Thus, \(\Omega(X)\) lies in the positive quadrant and system (3) is persistent. Finally, since only the closed orbits and the equilibria from the omega limit set of the solutions on boundary of \(R^3_+\) and system (3) is dissipative. Now using a theorem of Butler et al. [38], we conclude that system (3) is uniformly persistent.

3.5. Global stability of interior equilibrium point

Theorem 4. The positive equilibrium point \(E(x', y', z')\) is globally asymptotically stable if

\[
M = \begin{pmatrix}
a_1 x' + \frac{a_2 y' c_1}{1 + x' + a_2 z'} + d_1 y' + d_2 z' + \frac{(1 + x')^2}{2} \\
M_0 y' + a_2 z' (M + c_1) \frac{a_2 y' + d_1 y' + d_2 z' + \frac{(1 + x')^2}{2}}{1 + b_1 M} + x'
\end{pmatrix}
\]

where \(M = \frac{1}{2}, \theta = \min\{d_1, d_2 - a_2 c_1\} > 0\).

Proof. The proof can be reached by constructing a Lyapunov function. Now, we consider a positive definite function as

\[
W(x, y, z) = W_1 + W_2 + W_3,
\]

where, \(W_1(x, y, z) = x - x' \ln(x')\), \(W_2(x, y, z) = y - y' \ln(y')\), \(W_3(x, y, z) = z - z' \ln(z')\).

Therefore,

\[
\frac{dW}{dt} = \begin{pmatrix}
-x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt}
\end{pmatrix}
\]

Using theorem (1), without loss of generality, we may assume that there exist a constant \(\theta = \frac{1}{2}\) satisfying \(x(t), y(t), z(t) < M\), where \(\theta = \min\{d_1, d_2 - a_2 c_1\} > 0\) and after algebraic calculation we have

\[
\frac{dW}{dt} \leq \begin{pmatrix}
(x - x')^2 + (y - y')^2 + (z - z')^2
\end{pmatrix}
\]

i.e.,

\[
\frac{dW}{dt} \leq - \frac{1}{2} \left( x - \frac{x'}{2} \right)^2 - \frac{1}{2} \left( y - \frac{y'}{2} \right)^2 - x + a_1 x M + \frac{a_1 x y}{1 + b_1 M} + y + a_2 z (M + c_1) + a_2 c M \frac{a_2 y' + d_1 y' + d_2 z'}{1 + b_2 M} + x' - \frac{d_1 y + d_2 z'}{1 + b_2 M}
\]

It is easy to verify that \(\frac{dW}{dt} < 0\) under condition [4]. Therefore, the function \(W\) in the interior of the positive octant is a Lyapunov function. Hence, the equilibrium point \(E(x', y', z')\) is asymptotically stable in the positive octant.

4. Numerical studies

We perform the numerical simulation of system (3) with the following set of parameters, most of which are taken from HP [1] model. Parameter values are taken as \(a_1 = 5.0, a_2 = 0.1, b_1 = 2.0, c = 0.85, d_1 = 0.4, d_2 = 0.01\), which remain unchanged for all numerical simulations.

4.1. Verification of theoretical results

Using the above set of parameter values, the stable and saddle regions of axial equilibrium state \(E_0\) and boundary equilibrium state \(E_3\) are plotted in Fig. 1. The eigenvalues
of the Jacobian matrix \( J(E_A) \) are \( \frac{a_1}{b_1} - d_1 \) and \( \frac{a_2}{b_2} - d_2 \). The axial equilibrium point \( E_A \) is stable if \( \frac{a_1}{b_1} < d_1 \) and \( \frac{a_2}{b_2} < d_2 \). From Fig. 1(a) and (b), it is clear that the axial equilibrium \( E_A \) is stable in the region \( E_A \cap E_B \) for \( x = 1 \); in the region \( (E_A \cup E_R) \cap E_B \) for \( x = 5 \); in the region \( (E_A \cup E_R) \cap E_B \) for \( x = 10 \) and in the region \( (E_A \cup E_R) \cap E_B \) for \( x = 15 \), otherwise it is a saddle point. The boundary equilibrium point \( E_B(k, y, 0) \equiv (0.1053, 0.2355, 0) \) is stable if \( \frac{a_1}{b_1} < d_2, \sigma_1 > 0 \) and \( \sigma_2 > 0 \). But for above set of parameter values, two eigenvalues are 0.0547 \( \pm \) 0.5187i and other eigenvalue lies always on the region \( E_B \). So the boundary equilibrium state \( E_B \) is a saddle point for the above set of parameter values. The interior equilibrium state \( E^* \) exists if \( d_2(1 + x) > a_1 c_1 \) i.e., \( \xi < \frac{d_2}{a_1 c_1} \). Further, \( 0.1 > 0.02 \) and \( Q = 1 - b_1 < 0 \) i.e., \( Q < -2 \). Therefore, \( E^* \) exists for the above set of parameter values.

Periodic solution of system (3) for prey and intermediate predator in \( xy \) plane is shown in Fig. 2 for top-predator free system. From Fig. 2, it is confirmed that system (3) has a boundary periodic solution near boundary equilibrium state \( E_B(0.1053, 0.2355, 0) \). Calculating \( \zeta_i = -d_2 + \int_0^T \left( \frac{a_1}{b_2} \right) dx \) for above set of fixed parameter values and for \( x = 2, \xi = 0.1 \) with \( r = 1 \). T = 20, we find that \( \xi_1 > 0.0354 > 0 \) with initial condition \( (0.2, 0.8316, 0) \), which confirms the persistence of system (3).

We now present an example to verify the analytical results of global stability. We choose the above set of fixed parameter values and \( x = 2, \xi = 0.15 \). For the above set of parameter values we have \( \lambda_2 = -0.1859 < 0 \) and \( \Omega_1 = 1.1791 > 0, \Omega_2 = 0.1861 > 0, \Omega_3 = 0.001976 > 0, \Omega_4 = 0.21927 > 0 \), which means that the system (3) is locally asymptotically stable around positive interior equilibrium \( E = (0.9961, 0.0031, 13.5804) \). It is also observed that condition of Theorem 4 is satisfied and the trajectories with different initial conditions \( I_1 = (1.2, 0.5), I_2 = (1.1, 1), I_3 = (1.2, 1.1, 1.2, 3) \) and \( I_4 = (3.2, 1.5, 2) \) converge to the interior equilibrium \( E = (0.9961, 0.0031, 13.5804) \) where all three populations have stable coexistence (see Fig. 3). This indicates that the interior equilibrium \( E^* \) is globally asymptotically stable.

4.2. Phase portraits

Fig. 4 represents the phase portraits of system (3) for different values of \( \zeta \). From Fig. 4(a) presents chaotic dynamics of system (3) without additional food to top-predator (i.e., for \( \zeta = 0 \)), which is the well-known result of HP [1] model. After supplying small quantity of additional food \( \zeta = 0.02 \), we observe limit cycle oscillation of system (3) which is shown in Fig. 4(b). Period-2 oscillation and limit cycle oscillation of system (3) are depicted respectively for \( \zeta = 0.06 \) in Fig. 4(c) and for \( \zeta = 0.1 \) in Fig. 4(d). Therefore, we observe that the chaotic oscillation of system (3) can be controlled by supplying suitable quantity of additional food to top-predator.

4.3. One parameter bifurcation

In this subsection, we have done one parameter bifurcation analysis of system (3) with respect to quality and quantity of additional food, keeping one of them fixed and other parameter values are chosen as \( a_1 = 5, a_2 = 0.1, b_1 = 3, b_2 = 2.0, c = 0.85, d_1 = 0.4, d_2 = 0.01 \).

We have done bifurcation analysis of system (3) with respect to quality of additional food \( x \) within the range \( 0 \leq x \leq 12 \) taking \( \zeta = 0.1 \). Fig. 5 represents the bifurcation diagram of prey, intermediate predator and top-predator with respect to \( x \) for fixed \( \zeta \). From Fig. 5, we observe that all species have oscillatory coexistence within \( 0 \leq x < 9 \).
and stable coexistence after $a > 9$. From these bifurcation diagrams, it is evident that chaos totally disappears after $a > 9$. From these bifurcation diagrams, it is evident that chaos totally disappears from the system. We also observe that the systems behaviour gradually changes from period-4 to period-2; period-2 to limit cycle, and then limit cycle to steady state with the increase of quality of additional food. Now, we investigate the dynamical behaviour of the system under variation of quantity of supplied additional food $\xi$ keeping quality of additional food fixed. If condition $\frac{\alpha_1}{\alpha_2} < d_1$ i.e., $\xi < \frac{\alpha_1}{\alpha_2}d_1$ is satisfied then the top-predator cannot survive in the absence of intermediate predator, the top-predator also decreases after the decrease of the intermediate predator population and only the prey species dominates the system. So we have investigated effects of variation of quantity of additional food $\xi$ on the dynamics of system (3) with respect to quantity of additional food $\xi$ within the range $0 \leq \xi < 0.1538$ taking $a = 2$. Fig. 6, represents the bifurcation diagram of prey, intermediate predator and top-predator with respect to $\xi$ for fixed $a = 2$. Chaotic dynamics of system (3) is obvious from Fig. 6 in absence of additional food $\xi$ i.e., at $\xi = 0$. With the increase of $\xi$, periodic oscillations and chaotic bands are observed in the system. We observe oscillatory coexistence of species in the model in $0.042 \leq \xi < 0.1538$. Therefore, chaos to order transition is possible supplying suitable amount of additional food.

4.4. Two parameter bifurcation

The main goal of this section is to study the pattern of bifurcation that takes place as we vary the parameters.
quality ($\xi$) and quantity ($\alpha$) of additional food. This is actually done by studying the change in the eigenvalue of the Jacobian matrix and also following the continuation algorithm. To start with we consider a set of fixed point initial solution, $x_0 = 0.99253425$, $y_0 = 0.0066003743$, $z_0 = 8.5730761$, corresponding to a parameter set of values $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.0$, $b_2 = 2.0$, $c = 0.85$, $d_1 = 0.4$, $d_2 = 0.01$, most of which are taken from Hastings and Powell model [1]. The characteristics of Hopf point, the limit cycle may be explored using the software package MATCONT. This package is a collection of numerical algorithms implemented as a MATLAB toolbox for the detection, continuation and identification of limit cycles.

The existence of Hopf point (H), branch point (BP) and limit point (LP) are shown in the system. In Fig. 7 we obtain three Hopf points (H), one limit point (LP) and one branch point (BP) of system with respect to $\alpha$ for fixed $\xi = 0.1$. The first Hopf point is located at $(x,y,z,\alpha) = (0.8121083, 0.129132, 16.737646, 8.830530)$. For this Hopf point the first Lyapunov coefficient is $-0.08164002$, indicating the existence of supercritical Hopf bifurcation, which implies that the equilibrium point looses stability and bifurcates to a stable limit cycle. As the parameter increases, second Hopf point occurs at $(x,y,z,\alpha) = \ldots$
At this Hopf point the first Lyapunov coefficient is positive and is equal to 6.181509, indicating the existence of a subcritical Hopf bifurcation. At this point there is a conjugate pair of eigenvalues $k_1, k_2$ with real $k_1 = k_2 = 0$. This means there exists an unstable limit cycle, bifurcating from the equilibrium.

The third Hopf point is located at $(x,y,z) = (0.405149, 0.263572, 17.929612, 19.585774)$. At this Hopf point the first Lyapunov coefficient is positive and is equal to 6.181509, indicating the existence of a subcritical Hopf bifurcation. At this point there is a conjugate pair of eigenvalues $k_1, k_2$ with real $k_1 = k_2 = 0$. This means there exists an unstable limit cycle, bifurcating from the equilibrium.

The limit point is located at $(x,y,z) = (0.333333, 0.266667, 15.238888, 19.833333)$ with the eigenvalues of Jacobian matrix as $(0, -0.0328594, -0.528514)$. Negative real parts of the eigenvalues, indicate that the LP is stable. The branch point (BP) occurs at $\alpha = 17.336565$ (see, Appendix B). The continuation curve of equilibrium point for $\xi = 0.12, 0.15$ of the system (3) is also shown in the same Fig. 7. The curves of real and imaginary parts of each eigenvalue corresponding to different bifurcation points are presented in Fig. 8 taking $\xi = 0.1, 0.12, 0.15$.

Two parameter bifurcation of top-predator population with respect to quality ($\alpha$) and quantity ($\xi$) of additional food is plotted in Fig. 9. From Fig. 9 it is evident that the system has Hopf (H) points, limit point (LP) and branch point (BP) as quality and quantity of additional food vary. This two parameter bifurcation diagram predict that the dynamical behaviours of the system strongly depends on supplied additional food. The different dynamical behaviours of the system in $(\xi, \alpha)$ parameter plane is presented in Fig. 10. From Fig. 10, we observe that chaotic or oscillatory behaviour of the top-predator can be controlled and we obtain steady state behaviour by supplying large
quantity of low quality \((\alpha \geq 1)\) of additional food. Notice that the boundaries between different dynamical regions in Fig. 10 are not perfectly distinct, because there is some degree of sensitivity to small changes in parameters \(\alpha, \zeta\) resulting sharp transitions between different dynamical outcomes. Our above analysis indicates that a rich bifurcation structure exists in the predator–prey system in presence of additional food to top-predator.

5. Conclusion

We have demonstrated a systematic analysis of the dynamics of a chaotic tri-trophic predator–prey system supplying additional food to top-predator. This additional food is assumed to be a non-reproducing prey or some food source. It is assumed that the supply of additional food is not dynamic, but maintained at a specific constant level. We have determined theoretical as well as numerical results of the modified model. From our investigation, we observe that the modified tri-trophic predator–prey system with additional food to top-predator exhibits very interesting as well as complex dynamics. Stability and persistence conditions of the modified predator–prey system by supplying additional food to top-predator are determined. We have verified the theoretical results numerically (Figs. 1–3). The phase portraits of the system indicate the systematic change of dynamical behaviour of the model from chaos to order with the variation of quantity of additional food (Fig. 4). We have explored the bifurcation scenarios of the system when two biological parameters quality and quantity of additional food are varied. The interesting outcomes of the continuation process are the

![Bifurcation diagram of the top-predator](image)

**Fig. 8.** Real and imaginary parts of eigenvalues corresponding to continuation curves of equilibrium points.

![Continuation curves of equilibrium points](image)

**Fig. 9.** Bifurcation diagram of the top-predator \(z\) with respect to quality of additional food \(\alpha\) as well as quantity of additional food \(\zeta\) for \(a_1 = 5.0, a_2 = 0.1, b_1 = 3.0, b_2 = 2.0, c = 0.85, d_1 = 0.4, d_2 = 0.01\).
occurrence of various kinds of bifurcation points of the system. We predict Hopf point (H) and limit point (LP) of the system for suitable supply of additional food. By supplying suitable quality and quantity of additional food properly, one can control chaos and obtain oscillatory or stable coexisting dynamics in the model. The system gradually bifurcates from period-4 to period-2; period-2 to limit cycle; and then limit cycle to steady state for higher values of quality of additional food. The persistence of species in the system depends on supply of additional food to top-predator. Because, an arbitrary quality and quantity of additional food may have opposite effects (extinction risk) in the ecosystem (Fig. 6). But, the top-predator grows fast with increase of additional food ($\xi < 0.05$). Actually, when the top-predator encounters the additional food, they spend more time in consuming the additional food and a balancing number of the intermediate predator becomes free from predation pressure.

Therefore, our study reveals that for low to moderate levels of suitable additional food supply to the top-predator, the system changes from chaos to periodic or steady state behaviour. This study also shows that the chaos can be controlled with the increase of quality of additional food. Supply of high quality of additional food to predator can also control the chaotic dynamics of the system. Control of chaos refers to a process where a small perturbation is applied to a chaotic system, in order to obtain a desirable (periodic, or stable) behaviour. There has been many techniques to design effective control of chaotic systems but very few of these methods are applicable to control chaos in food chain model. Actually non-chemical method has the popularity as it has no side effects or infected disease factors. Our analysis confirm that the availability of suitable additional food (non-prey food) in a predator prey system can have significant impact on the dynamics of the system. Since insect outbreaks often cause serious ecological and economic problems, control of insect population plays a vital role in agriculture. Among many ways for insect control, biological and chemical control are considered as two most effective methods to beat agricultural insects. However, chemical control causes environmental pollution and also destroy biodiversity of the ecosystem. On the other hand non chemical control is eco-friendly. This new non-chemical method of chaos control will be very useful for biological conservation of species in real world biological systems and for designing optimal harvesting strategy in a fishery and pest control in agricultural systems.

**Appendix A. Hopf point bifurcation**

We consider the system (3) of the form

$$\dot{X} = f(X, \mu),$$

where $X \in \mathbb{R}^3$ and $\mu \in \mathbb{R}^n$ is a vector of control parameters where equilibria, limit cycles, etc. can be computed [41,42].

Suppose that the system (5) has an equilibrium point $X = X_0$ at $\mu = \mu_0$ and therefore, it represents

$$F(X) = f(X, \mu_0),$$

as $F(X) = AX + \frac{1}{Z} B(X, X) + \frac{1}{B} C(X, X, X),$$

where $A = f_x(0, \mu_0),$ and $B$, $C$ are expressions obtained from (5).

And

$$\dot{Y} = \sum_{i=1}^{4} \frac{\partial^2 F_i(\xi)}{\partial r_i^2} \bigg|_{r_i=0} x_i y_i.$$
We assume that $(X_0, \mu_0)$ is an equilibrium point of the system (5) where Jacobian matrix $A$ has a pair of purely imaginary eigenvalues on the imaginary axis: $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, i.e., $\Re \lambda_{1,2} = 0$.

A Hopf point $(X_0, \mu_0)$ is an equilibrium point of the system (5) where the Jacobian matrix $A$ has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$. At a Hopf point, a two-dimensional center manifold is well-defined, which is invariant under the flow generated by (5) and can be smoothly continued to nearby parameter values.

A Hopf point is called transversal if the curves of complex eigenvalues cross the imaginary axis with non-zero derivative.

In a neighbourhood of a transversal Hopf point with $l_1 = 0$ the dynamic behaviour of the system (5), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to the complex normal form

$$\dot{\omega} = (\gamma + i\omega)\omega + l_1\omega|\omega|^2,$$

(11)

$\omega \in \mathbb{C}$, $\gamma$, $\omega$ and $l_1$ are smooth continuations of $0$, $\omega_0$ and the first Lyapunov coefficient at the Hopf point [41], respectively. When $l_1 < 0$ ($l_1 > 0$) a family of (unstable) periodic orbits can be found on this family of center manifolds which shrink to the equilibrium point at the Hopf point.

Appendix B. Limit cycle bifurcation

While varying one parameter ($\mu \in \mathbb{R}$), one may encounter condensation 1 of fixed points. The eigenvalues of the Jacobian matrix $A = f'_x(0, \mu_0)$ of $f(X, \mu)$ are called multipliers. The fixed point is asymptotically stable if $|\mu| < 1$ for every multiplier $\mu$. If there exists a multiplier $\mu$ with $|\mu| > 1$, then fixed point is unstable. While following a curve of fixed points, three condensation 1 singularities can generally occur, namely a limitpoint (fold, LP) with multiplier $-1$, a period – doubling (flip, PD) point with a multiplier $-1$ and a Neimark–Sacker (NS) point with conjugate pair of complex multipliers $e^{\pm i\theta}$, $0 < \theta < \pi$. Generally, the curve of fixed points turns at an LP. In a PD point, a cycle of period two bifurcates from the fixed point of $f$ that changes stability. This bifurcation can be supercritical or subcritical, denoting the appearance of stable or unstable cycles for parameter values larger or smaller than critical one, respectively.

A branch point (BP) is a point where the Jacobian matrix $A = [f_x, f_y]$ of (5) is rank deficient. This is a nongeneric situation in one parameter problems where the implicit function theorem cannot be applied to ensure the existence of a unique smooth branch of solutions.

References

Chaos to Order: Role of Additional Food to Predator in a Food Chain Model

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Abstract  The chaos and its controllability is one of the main objectives of mathematical ecology today. In this paper, we propose a new chaos control technique for a three species predator prey system by supplying additional food to predators. A thorough mathematical analysis in this direction revealed the vital role of quality and quantity of the additional food in the controllability of the predator–prey system. The local stability analysis of boundary and interior equilibrium points of the system are done. We have determined the persistence conditions of the system. Bifurcation analysis of the proposed model is done with respect to quality and quantity of additional food. A Hopf point bifurcation phenomena is introduced to detect periodic behaviour of the system. By varying quality and quantity of additional food one can control the chaotic dynamics of a food chain. We establish that chaotic population dynamics of predator–prey system can be controlled by supplying additional food to predators. This study introduce a new chaos control mechanism in a predator–prey system which has applications in fishery management and biological conservation of prey predator species. This model has a special importance in the systems, such as caves, some small marine islands etc.

Keywords  Additional food · Chaos · Periodic · Stability · Persistence · Hopf point · Bifurcation

Introduction

Applications of additional food has been widely recognized as one of the important tools for biological control such as species conservation and pest management. There are studies which address the controllability and eradication of the pest through the predator by providing the
latter with alternative or additional food. In recent years, many biologist, experimentalists, and theoreticians have concentrated on investigating the consequences of providing additional food to predators in a predator–prey system [1–12]. van Baalen et al. [1] used the alternative food in a predator–prey system and they suggested that for stable prey density, one has to select suitable alternative food. They showed that for persistence of species sufficient amount of additional food is required. The concept of providing alternative food to predators for herbivores model is given by van Rijn et al. [2]. Harwood and Obrycki [3] examined the role of key species of alternative prey to generalist predators and discuss their impact in the context of biological control. The idea of alternative food for pest control and management is given by Sabelis and van Rijn [4]. They noticed that the short-term effect of plant provided food (PPF) will work most effectively when large areas of natural habitat surround agricultural fields and have low numbers of potential pest organisms. Huxel and McCann [5] investigated the impact of constant allochthonous inputs (additional food) to intermediate consumer on the stability of a simple food web model. They observed that low to moderate levels of allochthonous inputs stabilize food web dynamics when species preferentially feed on the autochthonous sources. On the other hand either increasing the input level or changing the feeding preference to favour allochthonous inputs, or both, led to a decoupling of the food chain that could result in the loss of one or all species. They also argued that allochthonous inputs are important sources of productivity in a food webs, in which more energy enters into the food web from allochthonous inputs than from autochthonous inputs. Huxel et al. [6] proposed a food web model with variable allochthonous inputs which are either one type available to both consumers and predators or two distinct types, one for consumers and other for predators. They observed that systems in which a single allochthonous resource available to both consumers and predators is more unstable than two allochthonous resources. Srinivasu et al. [7] reported qualitative behaviour of a predator–prey system in presence of additional food to the predators. They showed that handling times for the available foods to the predator play a key role in determining the eventual state of the system. It is interesting to observe that by varying quality and quantity of the additional food one can, not only control and limit the prey, but also limit and eradicate the predators. In the controllability studies by Srinivasu et al. [7], it is observed that, for a chosen quality and quantity of the additional food the asymptotic state of a solution of the system can either be an equilibrium or a limit cycle. Sahoo [8] investigated a predator–prey model with different growth rates and different functional responses in presence of additional food to predator. Again, Sahoo [9] reported that for biological conservation of species, additional food plays an important role in an ecosystem. Recently, Sahoo and Poria [12] introduced a non-chemical treatment for controlling disease of prey supplying alternative food to predator.

Srinivasu et al. [7] investigated the effects of additional food to predators in a di-trophic food chain model. The di-trophic food chain models are unable to capture the complexity of a real food chain and therefore the study of tri-trophic food chain is necessary. Srinivasu et al. [7] showed that for di-trophic food chain a target state can be reached providing suitable additional food to predators. In Science, so many fields are there [13–17] where chaos is appeared and various new techniques are developed for controlling chaos. But the controllability of chaos or high periodic oscillations of a food chain by supplying additional food is still unexplored. Our motivation is to investigate the chaos controllability aspects here by supplying suitable additional food to predators.

A non-chaotic predator–prey model is very useful for future prediction. Prediction of future population size is urgently required for biological conservation of species (to prevent extinction) and fishery management, and designing optimal harvesting strategies. Hastings and Powell (HP) [18] proposed a chaotic tri-trophic food chain model with Holling type-II
functional response. After the work of HP [18], many researchers explored this model by including various ecological factors to control chaos. Recently, Gakkhar and Singh [21] introduced an additional predator in HP [18] model to control chaos. In this paper, we introduce additional food to the predators in famous HP [18] model. The main goal of this study is to control the chaotic dynamics of the HP model by supplying additional food to predators. The paper is organized as follows: The model is formulated in “Model formulation” section. We have derived the different equilibrium points, their existence and persistence conditions of the system in “Theoretical study” section. The bifurcation scenarios of the system depending on some identified set of parameters is given in “Numerical results” section. Lastly, we draw a conclusion in “Conclusions” section.

Model Formulation

To examine the impact of additional food on a predator–prey system, we consider famous Hastings-Powell (HP) [18] tri-trophic food chain model. In HP model, the prey population grows as per logistic equation in the absence of predators and the interactions between preys, intermediate predators and top predators are of Holling type-II. The HP [18] model with pairwise interactions between three species, namely, $X, Y, Z$, which incorporates a Holling type-II functional interactions in both consumer species is the following

$$
\frac{dX}{dT} = R_0X \left(1 - \frac{X}{K_0}\right) - C_1A_1 \frac{XY}{B_1 + X},
$$

$$
\frac{dY}{dT} = A_1 \frac{XY}{B_1 + X} - A_2 \frac{YZ}{B_2 + Y} - D_1Y,
$$

$$
\frac{dZ}{dT} = C_2A_2 \frac{YZ}{B_2 + Y} - D_2Z.
$$

Here $X$ be the numbers of species at lowest level of the food chain, $Y$ the numbers of the species that preys upon $X$ and $Z$ the numbers of the species that preys upon $Y$. Here $T$ is time. The constant $R_0$ is the “intrinsic growth rate” and the constant $K_0$ is the “carrying capacity” of the species $X$. The constant $C_1 \leq 1$ and $C_2 \leq 1$ are conversion rates of prey to predators for species $Y$ and $Z$ respectively; $D_1$ and $D_2$ are constant death rates for species $Y$ and $Z$ respectively. The constants $A_1$ and $B_i$ for $i = 1, 2$ are maximal predation rate and half saturation constants for $Y$ and $Z$ respectively.

Here $h_1$ and $e_1, e_2$ are constants representing handling time (time spent on processing a food item) of the predators $Y, Z$ per prey item and ability (the rate at which the consumer encounters food items per unit of food density) of the predators to detect the prey respectively. The functional responses of $Y$ and $Z$ are respectively of the form

$$
e_1X \quad \text{and} \quad \frac{e_2Y}{1 + e_1h_1Y},
$$

i.e.,

$$
\frac{1}{1 + e_1h_1X} \quad \text{and} \quad \frac{1}{1 + e_2h_1Y}.
$$

Therefore, we have $A_i$ and $B_i$, representing the maximum predation rate and half saturation values of the predators $Y, Z$, to be $1/h_1$ and $1/e_1h_1, 1/e_2h_1$, respectively.

We now modify the model (1) supplying “additional food” to predators. We make the following assumptions:
(a) Predators are provided with additional food of constant biomass $A$ which is distributed uniformly. The constant supply level is maintained either by nature or by external agency.

(b) The number of encounters per predator with the additional food is proportional to the density of the additional food.

(c) The proportionality constant characterizes the ability of the predator to identify the additional food.

(d) For simplicity, we assume the handling time of both predators per unit quantity of additional food are same.

With the above assumptions, the model (1) takes the following form:

\[
\begin{align*}
\frac{dX}{dT} &= R_0X \left(1 - \frac{X}{K_0}\right) - C_1A_1 \frac{XY}{B_1 + \alpha \mu A + X}, \\
\frac{dY}{dT} &= A_1 \frac{(X + \mu A)Y}{B_1 + \alpha \mu A + X} - A_2 \frac{YZ}{B_2 + \alpha v A + Y} - D_1Y, \\
\frac{dZ}{dT} &= C_2A_2 \frac{(Y + \alpha A)Z}{B_2 + \alpha v A + Y} - D_2Z.
\end{align*}
\]

Here $h_2$ represents the handling time of both the predators $Y$, $Z$ per unit quantity of additional food and $e_3$, $e_4$ respectively represent the ability of the predators $Y$, $Z$ to detect the additional food. Now, the functional responses with additional food for $Y$ and $Z$ are respectively of the form

\[
\frac{e_1X}{1 + e_3h_2A + e_1h_1X} \quad \text{and} \quad \frac{e_2Y}{1 + e_4h_2A + e_2h_1Y},
\]

i.e.,

\[
\frac{1}{h_1X} \quad \text{and} \quad \frac{1}{h_1Y}
\]

Therefore, in model (2), we have $\mu = e_3/e_1$, $\nu = e_4/e_2$ and $\alpha = h_2/h_1$. The terms $\mu A$ and $\nu A$ represent effectual additional food for the predators $Y$ and $Z$ respectively.

Now, we define dimensionless variables $x = X/K_0$, $y = Y/K_0$, $z = Z/K_0$, $t = R_0T$ and we obtain the following system

\[
\begin{align*}
\frac{dx}{dt} &= x(1 - x) - \frac{a_1x}{1 + a_1^2 + b_1^2}y = F_1(x, y, z), \\
\frac{dy}{dt} &= \frac{\beta(x + c_\xi)}{1 + \alpha \eta + b_1^2}y - \frac{a_2y}{1 + \alpha \eta + b_2^2}z - d_1y = F_2(x, y, z), \\
\frac{dz}{dt} &= \frac{\gamma(y + e_\eta)}{1 + \alpha \eta + b_2^2}z - d_2z = F_3(x, y, z),
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= \frac{C_1A_1K_0}{R_0B_1}, \quad a_2 = \frac{A_2K_0}{B_2R_0}, \quad b_1 = \frac{K_0}{B_1}, \quad b_2 = \frac{K_0}{B_2}, \quad \beta = \frac{A_1K_0}{B_1R_0}, \quad \gamma = \frac{C_2A_2K_0}{R_0B_2}, \\
c &= \frac{B_1}{K_0}, \quad e = \frac{B_2}{K_0}, \quad \xi = \frac{\mu A}{B_1}, \quad \eta = \frac{v A}{B_2}, \quad d_1 = \frac{D_1}{R_0}, \quad d_2 = \frac{D_2}{R_0}.
\end{align*}
\]

Clearly $a_1 \geq \beta$ and $a_2 \geq \gamma$. The system (3) has to be analysed with the following initial conditions: $x(0) \geq 0$, $y(0) \geq 0$, $z(0) \geq 0$.

From the relation $\alpha = h_2/h_1$, it can be inferred that $\alpha$ is directly proportional to the handling time $h_2$ of the additional food. Hence the parameter $\alpha$ is proportionally related to the “quality” of the additional food. If the relation $h_2 < h_1$ holds, then the predator can
easily capture additional food than prey species which implies that the additional food is of high quality. Therefore for high quality of alternative food $\alpha$ is less than 1. Again, from the relation $\xi = \mu A/B_1$ and $\eta = \nu A/B_2$, it can be inferred that $\xi$ and $\eta$ are directly proportional to the biomass of the additional food ($A$) and thus $\xi$ and $\eta$ are representative of the “quantity” of the additional food that is supplied to predators [19]. The additional food is assumed to be a non-reproducing prey or some food source. We do not make any distinction regarding the additional food like complementary, essential or alternative. It is assumed that the supply of additional food is not dynamic, but maintained at a specific constant level. This simplification is justified for many arthropod predators because they can feed on plant-provided alternative food sources such as pollen and nectar, the availability of which is not influenced by predator’s consumption.

**Theoretical Study**

The functions of the right hand sides of system (3) are continuous and have continuous partial derivatives on the state space $\mathbb{R}_+^3 = \{(x(t), y(t), z(t)) : x(t) \geq 0, y(t) \geq 0, z(t) \geq 0\}$. Thus, they are Lipschitzian on $\mathbb{R}_+^3$ and therefore the solution of system (3) with non-negative initial condition exists and unique. Moreover, from Cao and Chen [20], it is easy to show that the state space $\mathbb{R}_+^3$ is an invariant domain of system (3).

In the following section, positivity and boundedness of solution for system (3) are established. Since the state variables $x, y, z$ represent populations, positivity implies that they never become zero and population always survive. The boundedness may be interpreted as natural restrictions to unlimited growth as a consequence of limited resources [21].

**Positive Invariance**

System (3) can be written in the matrix form as $\dot{X} = F(\bar{X})$ with $\bar{X}(0) = \bar{X}_0 \in \mathbb{R}_+^3$, where $\bar{X} = (x, y, z)^T \in \mathbb{R}_+^3$ and $F(\bar{X})$ is given by

$$F = F(\bar{X}) = \begin{pmatrix} x(1 - x) - \frac{a_1}{1 + \alpha \xi + b_1 x} y \\ \beta(x + c\xi) - \frac{a_2}{1 + \alpha \eta + b_2 y} y - \frac{y(y + e\eta)}{1 + \alpha \eta + b_2 y} z - d_1 y \\ \gamma y + e\eta - d_2 z \end{pmatrix},$$

where $F : \mathbb{R}_+ \to \mathbb{R}^3$ and $F \in C^\infty(\mathbb{R}^3)$.

It can be shown that whenever $\bar{X}(0) \in \mathbb{R}_+^3$ such that $X_i = 0$ then $F_i(\bar{X})|_{X_i=0} \geq 0$ (for $i = 1, 2, 3$). Now any solution of $\dot{X} = F(\bar{X})$ with $\bar{X}_0 \in \mathbb{R}_+^3$, say $\bar{X}(t) = \bar{X}(t, \bar{X}_0)$, is such that $\bar{X}(t) \in \mathbb{R}_+^3$ for all $t > 0$ [22].

**Boundedness**

**Theorem 1** All the solutions of the system (3) which start in $\mathbb{R}_+^3$ are uniformly bounded.

**Proof** Let $(x(t), y(t), z(t))$ be any solution of the system (3) with positive initial conditions. Let us consider that,

$$w = x + y + z.$$

i.e.,

$$\frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}.$$

$\square$ Springer
Using equations of (3), we have
\[
\frac{dw}{dt} = x(1 - x) - \frac{a_1xy}{1 + \alpha \xi + b_1x} + \frac{\beta(x + c\xi)y}{1 + \alpha \xi + b_1x} - \frac{a_2yz}{1 + \alpha \eta + b_2y} - d_1y + \frac{\gamma(y + \eta)z}{1 + \alpha \eta + b_2y} - d_2z.
\]
Therefore,
\[
\frac{dw}{dt} = x(1 - x) - \frac{(a_1 - \beta)xy}{1 + \alpha \xi + b_1x} + \frac{\beta c\xi y}{1 + \alpha \xi + b_1x} - \frac{(a_2 - \gamma)yz}{1 + \alpha \eta + b_2y} - d_1y + \frac{\gamma \eta z}{1 + \alpha \eta + b_2y} - d_2z,
\]
since \(a_1 \geq \beta, a_2 \geq \gamma\) we get the following expression:
\[
\frac{dw}{dt} \leq x(1 - x) + \frac{\beta c\xi y}{1 + \alpha \xi + b_1x} + \frac{\gamma \eta z}{1 + \alpha \eta + b_2y} - d_1y - d_2z,
\]
i.e.,
\[
\frac{dw}{dt} \leq x(1 - x) + \beta c\xi y + \gamma \eta z - d_1y - d_2z,
\]
i.e.,
\[
\frac{dw}{dt} \leq x - x^2 - (d_1 - \beta c\xi) y - (d_2 - \gamma \eta) z,
\]
i.e.,
\[
\frac{dw}{dt} \leq -k(x + y + z) - (x - 1)^2 + 1,
\]
where \(k = \min (1, d_1 - \beta c\xi, d_2 - \gamma \eta)\), provided \(d_1 > \beta c\xi, d_2 > \gamma \eta\).

\[
\frac{dw}{dt} + kw \leq 1 - (1 - x)^2.
\]

Hence,
\[
\frac{dw}{dt} + kw \leq 1; \text{ since } (1 - x)^2 \geq 0.
\]

Applying the theory of differential inequality we obtain
\[
0 < w < \frac{1 - e^{-kt}}{k} + w(x(0), y(0), z(0))e^{-kt}.
\]
For \(t \to \infty\), we have \(0 < w < 1/k\).

Thus, all solutions of the system (3) that initiate in \(\mathbb{R}_+^3\) are confined in the region \(B = \{(x, y, z) \in \mathbb{R}_+^3 : w = 1/k \pm \epsilon, \text{ for any } \epsilon > 0\}\). This proves the theorem. \(\square\)

Existence and Local Stability of Boundary Equilibrium Points

The system (3) always have two boundary equilibrium points. \(E_0(0, 0, 0)\) is the trivial equilibrium point. The axial equilibrium point is \(E_1(1, 0, 0)\). The third boundary equilibrium point \(E_2(\hat{x}, \hat{y}, 0)\) is the top-predator free equilibrium point, where
\[
\hat{x} = \frac{d_1(1 + \alpha \xi) - \beta c\xi}{\beta - b_1d_1} \quad \text{and} \quad \hat{y} = \frac{(1 - \hat{x})(1 + \alpha \xi + b_1\hat{x})}{a_1}.
\]
The top-predator free equilibrium point $E_2$ exists if $d_1(1 + \alpha \xi) > \beta c_\xi$, $\beta > b_1d_1$ or $d_1(1 + \alpha \xi) < \beta c_\xi$, $\beta < b_1d_1$ and $\hat{x} < 1$ i.e., $d_1(1 + b_1 + \alpha \xi) < \beta(1 + c_\xi)$. The Jacobian matrix $J$ of the system (3) at any arbitrary point $(x, y, z)$ is given by

$$J = \begin{pmatrix} F_{1x} & F_{1y} & 0 \\ F_{2x} & F_{2y} & F_{2z} \\ 0 & F_{3y} & F_{3z} \end{pmatrix}. \tag{4}$$

**Theorem 2** The trivial equilibrium point $E_0$ is always saddle. The axial equilibrium point $E_1$ is unstable if

$$\beta(1 + c_\xi) > d_1 \quad \text{and} \quad \frac{\gamma \eta}{1 + \alpha \eta} > d_2.$$

The top-predator free equilibrium point $E_2$ is locally stable if

$$a_1b_1\hat{y} < (1 + \alpha \xi + b_1\hat{x})^2 \quad \text{and} \quad \frac{\gamma(\hat{y} + \eta)}{1 + \alpha \eta + b_2\hat{y}} < d_2.$$

**Proof** The Jacobian matrix $J(E_0)$ at $E_0$ is given by

$$J(E_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix},$$

for which two of the eigenvalues are negative ($-d_1$ and $-d_2$) and it has one positive eigenvalue 1, giving a point at the origin with non-empty stable and an unstable manifold. So, $E_0$ is always saddle.

The Jacobian matrix $J(E_1)$ at $E_1$ is given by

$$J(E_1) = \begin{pmatrix} -1 & -a_1 & 0 \\ 0 & \frac{\beta(1 + c_\xi)}{1 + \alpha \xi + b_1} - d_1 & 0 \\ 0 & 0 & \frac{\gamma \eta}{1 + \alpha \eta - d_2} \end{pmatrix}.$$ 

From the Jacobian matrix $J(E_1)$, it is observed that it has one negative eigenvalue and two positive eigenvalues whenever

$$\beta(1 + c_\xi) > d_1 \quad \text{and} \quad \frac{\gamma \eta}{1 + \alpha \eta} > d_2.$$

Thus the equilibrium point $E_1$ has non-empty stable and an unstable manifold. Hence, the axial equilibrium point $E_1$ is unstable if

$$\beta(1 + c_\xi) > d_1 \quad \text{and} \quad \frac{\gamma \eta}{1 + \alpha \eta} > d_2.$$

The Jacobian matrix $J(E_2)$ at $E_2$ is given by

$$J(E_2) = \begin{pmatrix} \frac{a_1b_1\hat{y}\hat{x}}{(1 + \alpha \xi + b_1\hat{x})^2} - \hat{x} & -a_1\hat{x} & 0 \\ \frac{\beta(1 + \alpha \xi - c_\xi b_1)}{(1 + \alpha \xi + b_1\hat{x})^2} & 0 & -a_2\hat{y} \\ 0 & 0 & \frac{\gamma(\hat{y} + \eta)}{1 + \alpha \eta + b_2\hat{y}} - d_2 \end{pmatrix}.$$
The characteristic roots of the Jacobian matrix $J(E_2)$ are \( \frac{\gamma(y + e\eta)}{1 + \alpha\eta + b_2y} - d_2 \) and roots of the equation
\[
\lambda^2 - \left( \frac{a_1b_1\hat{x}\hat{y}}{(1 + a\xi + b_1\hat{x})^2} - \hat{x} \right) \lambda + \frac{\beta a_1\hat{x}\hat{y}(1 + a\xi - c\xi b_1)}{(1 + a\xi + b_1\hat{x})^2} = 0.
\]

The top-predator free equilibrium point $E_2$ is stable if
\[
a_1b_1\hat{y} < (1 + a\xi + b_1\hat{x})^2 \quad \text{and} \quad \frac{\gamma(y + e\eta)}{1 + \alpha\eta + b_2y} < d_2.
\]

Existence and Local Stability of Interior Equilibrium Point

The interior equilibrium point of the system (3) is given by $E^*(x^*, y^*, z^*)$, where
\[
y^* = \frac{d_2(1 + \alpha\eta) - \gamma e\eta}{\gamma - b_2d_2}, \quad z^* = \frac{1 + \alpha\eta + b_2y^*}{a_2}\left[ \frac{\beta(x^* + c\xi)}{1 + a\xi + b_1x^*} - d_1 \right]
\]
and $x^*$ is the positive root of the equation
\[
P x^{a_2} + Q x^* + R = 0,
\]
where, $P = b_1(\gamma - b_2d_2)$, $Q = (\gamma - b_2d_2)(1 + a\xi - b_1)$ and $R = a_1[d_2(1 + \alpha\eta) - \gamma e\eta]$.

The interior equilibrium point $E^*$ exists if either $\gamma > b_2d_2$, $d_2(1 + \alpha\eta) > \gamma e\eta$ or
\[
\gamma < b_2d_2, d_2(1 + \alpha\eta) < \gamma e\eta, \quad \frac{\beta(x^* + c\xi)}{1 + a\xi + b_1x^*} > d_1
\]
and $Q^2 = 4PR$, $Q < 0$.

**Theorem 3** The interior equilibrium point $E^*(x^*, y^*, z^*)$ for the system (3) is locally asymptotically stable if the following conditions are hold $\Omega_1 > 0$, $\Omega_3 > 0$ and $\Omega_1\Omega_2 - \Omega_3 > 0$.

where,
\[
\begin{align*}
\Omega_1 &= - \left[ \frac{a_1b_1x^*y^*}{(1 + a\xi + b_1x^*)^2} - x^* + \frac{a_2b_2y^*z^*}{(1 + \alpha\eta + b_2y^*)^2} \right], \\
\Omega_2 &= \left( \frac{a_2y^*}{1 + \alpha\eta + b_2y^*} \right) \left( \frac{\gamma z^*(1 + \alpha\eta - e\eta b_2)}{(1 + \alpha\eta + b_2y^*)^2} \right) \\
&\quad + \left( \frac{a_1b_1x^*y^*}{(1 + a\xi + b_1x^*)^2} - x^* \right) \left( \frac{a_2b_2y^*z^*}{(1 + \alpha\eta + b_2y^*)^2} \right) \\
&\quad + \left( \frac{-a_1x^*}{1 + \alpha\xi + b_1x^*} \right) \left( \frac{\beta y^*(1 + a\xi - c\xi b_1)}{(1 + \alpha\xi + b_1x^*)^2} \right), \\
\Omega_3 &= - \left[ \frac{\gamma z^*(1 + \alpha\eta - e\eta b_2)}{(1 + \alpha\eta + b_2y^*)^2} \left( \frac{a_1b_1x^*y^*}{(1 + a\xi + b_1x^*)^2} - x^* \right) \right] \left( \frac{a_2y^*}{1 + \alpha\eta + b_2y^*} \right) \right],
\end{align*}
\]

**Proof** The Jacobian matrix of the system (3) at the interior equilibrium point $E^*$ is
\[
J(E^*) = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{pmatrix}, \quad (5)
\]
where,
\[ A_{11} = \frac{a_1 b_1 x^* y^*}{(1 + \alpha \xi + b_1 x^*)^2} - x^*, \quad A_{12} = -\frac{a_1 x^*}{1 + \alpha \xi + b_1 x^*}, \]
\[ A_{13} = 0, \quad A_{21} = \frac{\beta y^* (1 + \alpha \xi - c \xi b_1)}{(1 + \alpha \xi + b_1 x^*)^2}, \]
\[ A_{22} = \frac{a_2 b_2 y^* z^*}{(1 + \alpha \eta + b_2 y^*)^2}, \quad A_{23} = -\frac{a_2 y^*}{1 + \alpha \eta + b_2 y^*}, \quad A_{31} = 0, \]
\[ A_{32} = \frac{\gamma z^* (1 + \alpha \eta - e \eta b_2)}{(1 + \alpha \eta + b_2 y^*)^2}, \quad A_{33} = 0. \]

The characteristic equation of the Jacobian matrix \( E^* \) is given by
\[ \lambda^3 + \Omega_1 \lambda^2 + \Omega_2 \lambda + \Omega_3 = 0 \]

Using the Routh–Hurwitz criteria we observe that the system (3) is stable around the positive equilibrium point \( E^* \) if the conditions \( \Omega_1 > 0, \Omega_3 > 0 \) and \( \Omega_1 \Omega_2 - \Omega_3 > 0 \) hold.

Persistence of the System

In recent years the concept of persistence has played an important role in mathematical ecology. Various definitions of persistence have been developed in order to analyze mathematical models. Butler et al. [23], Freedman and Waltman [24] developed the following definition of persistence:

If a population \( N(t) \) is such that \( N(t) > 0 \), then

(i) \( N(t) \) is said to be weakly persistent if \( \lim_{t \to \infty} \sup N(t) > 0 \),

(ii) \( N(t) \) is said to be (strongly) persistent if \( \lim_{t \to \infty} \inf N(t) > 0 \).

Further, if \( N(t) \in C \), where \( C \) is a certain class of function, then

(iii) \( N(t) \) is said to be weakly uniformly persistent if there exists \( \delta > 0 \) such that \( \lim_{t \to \infty} \sup N(t) > \delta \) for all \( N(t) \in C \),

(iv) \( N(t) \) is said to be uniformly persistent if there exists \( \delta > 0 \) such that \( \lim_{t \to \infty} \inf N(t) > \delta \) for all \( N(t) \in C \).

In a theoretical point of view, persistence definitions can be interpreted as instability or as boundedness of the system. In biological sense, persistence (permanence) means the survival of all populations of the system in future time. Mathematically, persistence of a system means that strictly positive solutions do not have omega (\( \Omega \)) limit points on the boundary of the non-negative cone.

**Theorem 4** Let the following conditions are satisfied

(i) \( \frac{\beta (1 + \xi \xi)}{1 + \alpha \xi + b_1} > d_1 \),

(ii) \( \frac{\gamma \eta}{1 + \alpha \eta + b_2} > d_2 \).

Further, if there exists a finite number (say, \( n \)) of periodic solutions \( x = \phi_r(t), y = \psi_r(t), r = 1, 2, 3, \ldots, n \) in the \( xy \) plane, then system (3) is uniformly persistent provided for each periodic solutions of period \( T \),

\[ \xi_r = -d_2 + \frac{1}{T} \int_0^T \gamma (\psi_r(t) + e \eta) \frac{\eta \psi_r}{1 + \alpha \eta + b_2 \psi_r} dt > 0, \quad r = 1, 2, 3, \ldots, n. \]
Proof Let \( X = (x, y, z) \) be a point in the positive quadrant and \( O(X) \) be orbit through \( X \) and \( \Omega \) be the omega limit set of the orbit through \( X \). Note that \( \Omega \) is bounded.

We claim that \( E_0 \notin \Omega(X) \). If \( E_0 \in \Omega(X) \) then by the Butler–McGehee lemma \([23]\) there exist a point \( P \) in \( \Omega(X) \cap W^s(E_0) \) where \( W^s(E_0) \) denotes the stable manifold of \( E_0 \). Since \( O(P) \) lies in \( \Omega(X) \) and \( W^s(E_0) \) is the \( yz \) plane, we conclude that \( O(P) \) is unbounded, which is contrary to the boundedness of the system.

Next \( E_1 \notin \Omega(X) \), for otherwise, since \( E_1 \) is a saddle point which follows from the conditions

\[
\frac{\beta(1 + c\xi)}{1 + \alpha\xi + b_1} > d_1 \quad \text{and} \quad \frac{\gamma\eta}{1 + \alpha\eta} > d_2,
\]

by the Butler–McGehee lemma there exist a point \( P \) in \( \Omega(X) \cap W^s(E_1) \). Now \( W^s(E_1) \) is the \( x \)-axis implies that an unbounded orbit lies in \( \Omega(X) \), a contradiction.

Lastly, we show that no periodic orbits in the \( xy \) plane or \( E_2 \in \Omega(X) \). Let \( r_i, i = 1, 2, \ldots, n \), denote the closed orbit of the periodic solution \((\phi_r(t), \psi_r(t)) \) in \( xy \) plane such that \( r_i \) lies inside \( r_{i-1} \). Let the Jacobian matrix \( J_r(\phi_r(t), \psi_r(t), 0) \) corresponding to \( r_i \) is given by

\[
J_r = \begin{pmatrix}
F_{1x}(\phi_r(t), \psi_r(t), 0) & F_{1y}(\phi_r(t), \psi_r(t), 0) & 0 \\
F_{2x}(\phi_r(t), \psi_r(t), 0) & F_{2y}(\phi_r(t), \psi_r(t), 0) & 0 \\
0 & 0 & F_{3z}(\phi_r(t), \psi_r(t), 0)
\end{pmatrix}.
\]

Here

\[
F_{3z} = -d_2 + \frac{\gamma(\psi_r(t) + \eta)}{1 + \alpha\eta + b_2\psi_r(t)}.
\]

Computing the fundamental matrix of the linear periodic system,

\[
M' = J_r(t)M, \quad M(0) = M_0,
\]

We find that its Floquet multiplier in the \( z \)-direction is \( e^{\xi r_i(t)} \). Then proceeding in an analogous manner like Kumar and Freedman \([25]\), we conclude that no \( r_i \) lies on \( \Omega(X) \). Thus, \( \Omega(X) \) lines in the positive quadrant and system \((3)\) is persistent. Finally, since only the closed orbits and the equilibria from the omega limit set of the solutions on boundary of \( \mathbb{R}^3_+ \) and system \((3)\) is dissipative. Now using a theorem of Butler et al. \([23]\), we conclude that system \((3)\) is uniformly persistent. \(\square\)

**Theorem 5** Let the following conditions are satisfied

(i) \( \frac{\beta(1 + c\xi)}{1 + \alpha\xi + b_1} > d_1 \),

(ii) \( \frac{\gamma\eta}{1 + \alpha\eta} > d_2 \),

(iii) \( -d_2 + \frac{\gamma(\psi_1(t) + \eta)}{1 + \alpha\eta + b_2\psi_1(t)} > 0 \).

If exists no limit cycle in the \( xy \) plane, the system \((3)\) is uniformly persistent.

Proof Proof of the theorem (5) is obvious and so omitted. \(\square\)

**Numerical Results**

To observe the dynamics of the modified system \((3)\), we consider same set of parameter values of HP model \([18]\). The parameter values taken from HP model \([18]\) are \( a_1 = 5.0, \)
Fig. 1 Continuation curves of equilibrium with the variation of the parameter $\alpha$ of the variable: a $x$, b $y$ and c $z$ for $\xi = 0.1$, $\eta = 0$. The label $H$ denotes Hopf point

$a_2 = 0.1$, $b_1 = 2.8$, $b_2 = 2.0$, $\beta = 4.6 (< a_1 = 5)$, $\gamma = 0.08 (< a_2 = 0.1)$, $d_1 = 0.4$, $d_2 = 0.01$ and other parameters are taken as $c = 0.95$, $e = 0.85$ [12]. The parameter set is being kept fixed throughout the numerical simulations. The remaining three parameters $\alpha$ (quality of additional food), $\xi$ and $\eta$ (quantity of additional food) are varied to observe the dynamics of the system (3). We have done the bifurcation analysis of the system (3) with respect to the quality of additional food $\alpha$, quantity of additional food $\xi$ and $\eta$.

Continuation Curve of Equilibrium Point

The main aim of this section is to study the pattern of bifurcation that takes place as we vary the parameters $\alpha$, $\xi$ and $\eta$. This is actually done by studying the change in the eigenvalue of the Jacobian matrix and also following the continuation algorithm. To start with we consider a set of fixed point initial solution, $x_0 = 0.79948116$, $y_0 = 0.14325663$, $z_0 = 13.058952$, corresponding to a parameter set of values $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 2.8$, $b_2 = 2.0$, $c = 0.95$, $e = 0.85$, $\beta = 4.6$, $\gamma = 0.08$, $d_1 = 0.4$, $d_2 = 0.01$. The characteristics of Hopf point, the limit cycle and the general bifurcation may be explored using the software package MATCONT2.5.1. [27–32]. This package is a collection of numerical algorithms implemented as a MATLAB toolbox for the detection, continuation and identification of limit cycles. In this package we use prediction–correction continuation algorithm based on the Moore–Penrose matrix pseudo inverse for computing the curves of equilibria, limit point (LP), along with fold bifurcation points of limit point (LP) and continuation of Hopf point ($H$), etc.

To start with we show in Fig. 1, the continuation curve from the interior equilibrium point taking $\alpha$ as the free parameter with fixed $\xi = 0.1$ and $\eta = 0$. From Fig. 1, it is evident that the system has a Hopf point ($H$) at $(x, y, z, \alpha) \equiv (0.898945, 0.166667, 2.059248, 47.293260)$ with purely imaginary eigenvalues $\pm 0.0348173i$ and the third one being $-0.82949$. For this Hopf point the first Lyapunov coefficient is $-0.06491249$, indicating a supercritical Hopf.
bifurcation. It being negative implies that a stable limit cycle bifurcates from the equilibrium when this looses stability. In the Fig. 2, we observe four Hopf points ($H_1$, $H_2$, $H_3$, $H_4$), one limit point (LP), and one branch point (BP) of the system with respect to $\alpha$ for fixed $\xi = 0$ and $\eta = 0.1$. In Fig. 3, we get one Hopf point ($H$) and one branch point (BP) of the system with respect to $\alpha$ for fixed $\xi = 0.1$ and $\eta = 0.1$. We list Hopf, LP and BP points in Table 1 together with first Lyaponuv coefficients.
Table 1  Parameter values of $\alpha$, $\xi$ and $\eta$ at the bifurcation points in Fig. 1, 2, and 3, together with normal form coefficients and eigenvalues (scaled, see [26])

<table>
<thead>
<tr>
<th>Label</th>
<th>$\alpha$</th>
<th>($\xi, \eta$)</th>
<th>$l_1$</th>
<th>First Lyapunov coefficients/normal form coefficient</th>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>47.293260</td>
<td>(0, 0.1)</td>
<td>$l_1 = -0.06491249$</td>
<td>$-0.82949, \pm i0.0348173$</td>
<td></td>
</tr>
<tr>
<td>$H_1$</td>
<td>4.839490</td>
<td>(0, 0.1)</td>
<td>$l_1 = -0.2218216$</td>
<td>$-0.536219, \pm i0.0637301$</td>
<td></td>
</tr>
<tr>
<td>$H_2$</td>
<td>12.032278</td>
<td>(0, 0.1)</td>
<td>$l_1 = 5.829647$</td>
<td>$-0.00100491, \pm i0.487601$</td>
<td></td>
</tr>
<tr>
<td>LP</td>
<td>12.271429</td>
<td>(0, 0.1)</td>
<td>$a = -0.0001031928$</td>
<td>0, 0.0355543 $\pm i0.528011$</td>
<td></td>
</tr>
<tr>
<td>BP$_1$</td>
<td>10.838843</td>
<td>(0, 0.1)</td>
<td>$l_1 = 3.025044$</td>
<td>0, 0.0502749 $\pm i0.515071$</td>
<td></td>
</tr>
<tr>
<td>HS</td>
<td>9.446627</td>
<td>(0, 0.1)</td>
<td>$l_1 = -2.872501$</td>
<td>$-0.0005838, \pm i0.338794$</td>
<td></td>
</tr>
<tr>
<td>BP$_2$</td>
<td>8.800000</td>
<td>(0, 0.1)</td>
<td>$l_1 = 3.025044$</td>
<td>0, 0.0304523 $\pm i0.45927$</td>
<td></td>
</tr>
<tr>
<td>$H_4$</td>
<td>8.810260</td>
<td>(0, 0.1)</td>
<td>$l_1 = -2.872501$</td>
<td>$-0.0689562, \pm i0.00509064$</td>
<td></td>
</tr>
<tr>
<td>$H$</td>
<td>6.666995</td>
<td>(0.1, 0.1)</td>
<td>$l_1 = -0.1292656$</td>
<td>$-0.559808, \pm i0.0617988$</td>
<td></td>
</tr>
<tr>
<td>BP</td>
<td>38.076859</td>
<td>(0.1, 0.1)</td>
<td>$l_1 = -0.1292656$</td>
<td>$-0.156458, \pm i0.342568$</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 4  Bifurcation diagram of prey and top-predator population with respect to quality of additional food $\alpha \in [0, 50.1]$ taking fixed $\xi = 0.1$, $\eta = 0$ of the system (3)

Bifurcation Scenarios with Respect to Quality of Additional Food

We have done bifurcation analysis of the system (3) with respect to quality of additional food $\alpha$ keeping other parameters fixed. We consider four cases of fixed parameter values of $\xi$ and $\eta$ when quality of additional food $\alpha$ is varied. In the first case, for $\xi = 0$ and $\eta = 0$, the system (3) reduces to same as Hastings–Powell model [18], which is well studied [33–35]. So, we omit this case. For second case we consider $\xi = 0.1$ and $\eta = 0$, i.e., the system (3) has additional food to intermediate predators only. In third case, we consider $\xi = 0$ and $\eta = 0.1$. In this case additional food is applied to top-predators only. And in fourth case, we consider $\xi = 0.1$ and $\eta = 0.1$. That is additional food is added to both predators.

The bifurcation diagram of the system (3) with respect to $\alpha \in [1, 50.1]$ for $\xi = 0.1$ and $\eta = 0$ is presented in Fig. 4. Figure 4 depicts that the system has period-3, period-2 and limit cycle oscillations. The system settles down to stable state also after $\alpha > 47$. Therefore, chaos can be controlled supplying high or low quality of additional food to intermediate predators.
Figure 5 shows the bifurcation diagram of prey and top-predator with respect to quality of additional food $\alpha \in [0, 5.8]$ taking fixed $\xi = 0$, $\eta = 0.1$ of the system (3). The system's dynamics shows period-3, period-2 and limit cycle oscillations for different ranges of $\alpha$. The system also settles down to steady state after certain values of $\alpha$. Figure 6 represents the bifurcation diagram when $\alpha$ is varied within $0 \leq \alpha \leq 7.5$ taking $\xi = 0.1$ and $\eta = 0.1$. Figure 6 predicts that the system has chaotic attractors, period-4, period-3, period-2 and limit cycle oscillations. It also settles down to steady state for higher values of $\alpha$. Figure 7 is the magnified parts of Fig. 6 in the range of $0 < \alpha < 2.35$. The chaotic bands, periodic windows and high periodic oscillations are more visible in this figure. Therefore, we predict that chaos of HP model can be controlled through the variation of quality of additional food.
Fig. 7 Magnified bifurcation diagram of Fig. 6 of prey and top-predator with respect to quality of additional food $\alpha \in [0.5, 2.5]$ keeping $\xi = 0.1$, $\eta = 0.1$ of the system (3)

Fig. 8 Bifurcation diagram of prey and top-predator population with respect to quantity of additional food $\xi \in [0, 0.135]$ keeping $\alpha = 3$, $\eta = 0.1$ of the system (3)

Bifurcation Scenarios with Respect to Quantity of Additional Food

We have investigated the dynamics of the system (3) through bifurcation analysis with respect to quantity of additional food $\xi$ within the range $0 \leq \xi \leq 0.135$ keeping $\alpha = 3$ and $\eta = 0.1$ fixed. Figure 8 represents bifurcation diagram of prey population and top-predator populations with respect to $\xi$ for fixed $\alpha$ and $\eta$. From Fig. 8 it is evident that the system (3) has period-3, period-2, limit cycle behaviour and after certain level of $\xi$, it depicts high periodic oscillations.

Bifurcation analysis of the system (3) is done with respect to quantity of additional food $\eta$ within the range $0.0025 \leq \eta \leq 0.26$ taking $\alpha = 3$ and $\xi = 0.1$ fixed. Bifurcation diagrams of prey population and top-predator population are shown in Fig. 9. Figure 9 shows that the system has chaotic bands, periodic windows, period-4, period-3, period-2 oscillations when $\eta$ varies. Therefore we can conclude that if we increase the quantity of additional food after certain level of concentration, the system shows periodic oscillations.
Now, we summarize the results which are shown in Figs. 4, 5, 6, 7, 8, and 9. Figure 4 is the bifurcation diagram of the system (3) with respect to quality of additional food ($\alpha$) incorporating additional food (quantity) to intermediate predator only ($\xi = 0.1, \eta = 0$) and Fig. 5 shows the bifurcation diagram of the system (3) with variation of quality of additional food ($\alpha$) where additional food (quantity) is supplied to top-predator only ($\xi = 0, \eta = 0.1$). From Figs. 4 and 5, we observe that system (3) has periodic behaviour and therefore, chaos disappears from the system in presence of additional food to intermediate predator only or top-predator only. But, supplying additional food (quantity) to both predators, we observe that system has chaotic behaviour for lower values of $\alpha$ and it also has to periodic nature for higher values of quality of additional food $\alpha$ (Figs. 6, 7). Therefore, from Figs. 4, 5, 6, and 7, a conclusion can be drawn that periodic solution exists for higher values of quality of additional food. On the other hand, from Figs. 8 and 9, we can conclude that the system has periodic oscillations with the variation of $\xi$, but chaos to order transition is observed with the variation of $\eta$. Therefore, for higher values of $\xi$ and $\eta$, the system always admits periodic solution. Thus, chaos to order behaviour can be obtained by supplying higher values of quality and quantity of additional food to predators.

Conclusions

In this paper, a tri-trophic food chain model in presence of additional food to predators is proposed and analyzed. We have derived the existence and local stability conditions of boundary and interior equilibrium points of the system. The persistence conditions of the system are also calculated. We have done the bifurcation diagrams with respect to quality of additional food $\alpha$ as well as quantity of additional food $\xi$ and $\eta$ respectively. From the bifurcation diagrams, we observe that the system dynamics have chaotic bands, periodic windows, limit cycle oscillations etc. Also, we have observed that increase of additional food up to a certain level remove the chaotic behaviour of the system and the system dynamics enters into periodic region. Furthermore, our bifurcation results, show that chaos can be controlled with proper choice of additional food for predators.
Our results suggest that additional food can stabilize a chaotic food chain for high quality of additional food. Similarly, for large quantity of additional food supply, one can control chaos of the system. A state of a di-trophic food chain converges to either a source or sink or saddle or a limit cycle as time goes to infinity. Srinivasu et al. [7] have shown that limit cycle oscillations or a divergence of a state can be controlled with proper choice of additional food. After that it was an important question whether chaos can be controlled or not with supply of additional food in tri-trophic or higher trophic food chain. None have reported the aspect of chaos control in a three species food chain model through the supply of additional food to both predators before this work. We have shown that chaos can be controlled with suitable supply of additional food to predators. Therefore, it is possible to control the dynamics of a tri-trophic food chain model by appropriate choice of additional food and its supply level. An arbitrary choice of additional food may have opposite effects in the model. So, we suggest that through proper choice of additional food to predators chaotic dynamics of a food chain can be controlled. Actually, a non-chaotic predator–prey model is very useful for future prediction. Prediction of future population size is urgently required for biological conservation of species and fishery management, and designing optimal harvesting strategies. There has been many techniques to design effective control of chaotic systems but very few of these methods are applicable to control chaos in real food chain model. However, our method is a new technique which can be easily applicable to control chaos in a real food chain model. Our analysis confirms that the availability of suitable additional food in a predator prey system can have significant impact on the dynamics of the system. This new non-chemical method of chaos control will be very useful for biological conservation of species in real world biological systems and for designing optimal harvesting strategy in a fishery and pest control in agricultural systems.

The results from our investigation imply that food chain with additional resources can exhibit increased stability. One can obtain stable steady state from chaos only supplying suitable additional food to predators. Effects of supplying additional food in a diffusive coupled food chain model is an introducing area for future investigations. In future, we must need to re-examine the structure, complexity, and dynamics of real food chains in presence of additional food.

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References

Effects of supplying alternative food in a predator–prey model with harvesting

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ABSTRACT

In this paper, we propose a tritrophic predator–prey model with harvesting where the top-predator population is partially supported with alternative food. We report the consequences of providing alternative food to the top-predator in a top-predator harvested model. The extinction criterion for top-predator population, local stability of equilibrium points and persistence conditions are discussed theoretically. Pontryagins maximum principle is used to characterize the optimal control of harvesting. We have derived the condition of Hopf bifurcation by varying harvesting effort. The bifurcation diagrams of the system with respect to harvesting effort in presence of alternative food are given. Our analysis show that alternative food can prevent top-predator extinction risk at higher harvesting effort and plays a vital role for biological conservation of species.

1. Introduction

It is now well known that most of the predators do not feed only on a single prey, but also depend on diverse prey species and are therefore involved in complex food web interactions [1,2]. Many species are also migratory and their spatial scale much longer than the habitat occupied by some of their prey. Therefore, for these types of species, alternative prey needs to be considered to develop a realistic prey–predator system. The impact of a generalist predator not only depend on the abundance and susceptibility of the focal prey, but also on the presence and abundance of alternative prey species [3]. These additional foods are an important component of most predators (e.g., coccinellid) diet, and although they receive less attention than basal prey. In scientific literature, these foods fundamentally shape the life histories of many predator species. The availability of suitable additional food (non-prey food) in a predator prey system can have significant role on the dynamics of the system. The role of alternative prey in sustaining predator populations has been widely reported in laboratory studies and field trials examining the fecundity, feeding behaviour, extinction risk and growth rates of species subjected to diets of varying quality. In the Channel Islands, golden eagle (Aquila chrysaetos) predation is thought to have reduced three resident fox populations by over 95%. The review report suggested that these predators are primarily sustained by hyper abundant alternative prey species [4]. For Spiny dogfish in the North West Atlantic and pacific hake, as spatial scale are much longer than the habitat occupied by some of their prey, alternative prey must be required as alternative source of food. Recently, the application of monoclonal antibody and molecular technology to study predation rates in the field has revealed the extent to which many predator communities rely on alternative prey before, during and after the immigration of pests into crops. Thus the dynamics of complex food web critically depend on an accurate understanding of the relative importance of direct and
indirect effects in multi-species prey-predator system [5]. Therefore, the consequences of providing additional food to predator and corresponding effects on the prey-predator dynamics and its importance in biological control have been existing as an interesting topic to the scientists.

Recently, many biologist, researchers, and theoreticians investigated the consequences of providing additional food to predators in predator–prey systems [6–12]. Huxel et al. [6,7] reported the impact of allochthonous inputs (additional food) on the stability of a simple food web model. They assumed constant and also variable allochthonous inputs (additional food) to consumers as well as to top-predators populations. They observed that low to moderate inputs of allochthonous resources can increase species persistences. Srinivasu et al. [8] studied qualitative behaviour of a di-trophic predator–prey system in presence of additional food to the predators and discussed the effect of both high to low quality food. Sahoo and Poria investigated a disease control aspects through the provision of alternative food to predators [11]. Recently, Sahoo and Poria reported that a chaotic dynamics can be controlled by supplying additional food to predator [12].

From the point of view of human needs, the exploitation of biological resources and harvesting of populations are commonly practiced in fishery, forestry. It is important to study bio-economic models to gain insight into the scientific management of renewable resources like fisheries and forestries [13]. This is related to the optimal management of renewable resources [14,15]. To determine Governments policy to avoid overexploitation to maximize profits of commercial harvesting. It is necessary to investigate the sustainability of harvesting of populations. The above application of the field experiment motivates us to focus on the predator–prey model with harvesting. One relevant work regarding alternative food source is done by Spencer and Collie [16] establishing a model of prey-predator fish with alternative prey in presence of harvesting. Recently, many researchers [5,17] investigated harvested predator–prey system with alternative prey.

In this paper, we first consider famous Hastings–Powell (HP) [18] food chain model incorporating alternative food to top-predator population. Next, we incorporate the constant harvesting effort to top-predator population in the model. We investigate the effects of harvesting as well as the effects of supplying alternative food into the model. Our main aim is to save top-predator population from extinction in presence of alternative food. Actually, we want to control the predator population by supplying alternative food with harvesting. The section-wise split of this paper is as follows: In Section 2, the model representing the dynamics of a three species food chain in the presence of additional food top-predator with harvesting effort is constructed. In Section 3, extinction criterion of top-predator population, the local stability analysis of equilibrium points, persistence criterion of the proposed model are determined. Section 4 is devoted to find the bionomic equilibrium of the system. In Section 5, an optimal control policy of the model is determined. Hopf-bifurcation analysis is developed in Section 6. The Section 7 illustrates some of the key findings through numerical simulations. In Section 8, the different routes of continuation of the associated bifurcations are investigated with the help of the software package MATCONT [19–22]. Finally, we draw a conclusion in Section 9.

2. Model formulation

We consider the base model as Hastings and Powell food chain model [18] composed of a logistic prey ($X$), a Holling type II intermediate predator ($Y$), and a Holling type II top-predator ($Z$). It is, given by the following system of ordinary differential equations:

\[
\begin{align*}
\frac{dX}{dT} &= R_0 X \left(1 - \frac{X}{K_0}\right) - C_1 A_1 \frac{XY}{B_1 + X}, \\
\frac{dY}{dT} &= A_1 \frac{XY}{B_1 + X} - A_2 \frac{YZ}{B_2 + Y} - D_1 Y, \\
\frac{dZ}{dT} &= C_2 A_1 \frac{YZ}{B_2 + Y} - D_2 Z.
\end{align*}
\]

(1)

where $T$ is time. The constant $R_0$ is the "intrinsic growth rate" and the constant $K_0$ is the "carrying capacity" of the species $X$. The constant $C_1$ and $C_2$ are conversion rates of prey to predators for species $Y$ and $Z$ respectively; $D_1$ and $D_2$ are constant death rates for species $Y$ and $Z$ respectively. The constants $A_i$ and $B_i$ for $i = 1, 2$ are maximal predation rate and half saturation constants for $Y$ and $Z$ respectively. In order to preserve the biological meaning of the model, the parameters are assumed to be strictly positive. Furthermore, to avoid the case where intermediate predator and top-predator cannot survive, even when their food is infinitely abundant, we assume that $A_1 > D_1$ and $A_1 C_2 > D_2$ [23] (see Appendix).

We now incorporate the constant harvesting effort $E$ on top-predator in the model (1) and we obtain the following model:

\[
\begin{align*}
\frac{dX}{dT} &= R_0 X \left(1 - \frac{X}{K_0}\right) - C_1 A_1 \frac{XY}{B_1 + X}, \\
\frac{dY}{dT} &= A_1 \frac{XY}{B_1 + X} - A_2 \frac{YZ}{B_2 + Y} - D_1 Y, \\
\frac{dZ}{dT} &= C_2 A_1 \frac{YZ}{B_2 + Y} - D_2 Z - EZ.
\end{align*}
\]

(2)
In the model (2), we have considered a predator–prey model where top-predator depends only on intermediate predator. But recently, some authors emphasized that the presence of an alternative prey (alternative food source) [6–11] to the system may change the dynamical behaviour of the system in several ways. As an example, alternative prey may reduce the predation on intermediate predator due to predator preference for the alternative prey resource. Feeding of such multiple prey seems appropriate for fish which are predators of the larvae and juveniles of numerous fish species. Thus, the growth rate of the top-predator $Z$ should be maximal when $Y$ is abundant but non-zero when $Y \to 0$. Hence the model (2) can be modified to describe a partially coupled predator–prey interaction:

$$
\begin{align*}
\frac{dX}{dt} &= R_x X \left(1 - \frac{X}{K_x}\right) - C_{1} A_{1} \frac{XY}{B_1 + X}, \\
\frac{dY}{dt} &= A_1 \frac{XY}{B_1 + X} - A_2 \frac{AYZ}{B_2 + Y} - D_1 Y, \\
\frac{dZ}{dt} &= C_2 A_2 Z \left(\frac{AY}{B_2 + Y} + (1 - A)\right) - D_2 Z - EZ,
\end{align*}
$$

(3)

where $A$ is a time independent constant and its origin is the alternative resource. If $A = 1$, the top-predator depends only on the intermediate predator species and thus it is clear that the system (2) is a special case of system (3). If $A = 0$, then both the intermediate predator and top-predator population grow without any interaction and the growth rate of the top-predator is determined by alternative prey. In such case, the predation pressure on intermediate predator is completely removed and top-predator population evolves in presence of alternative food only. But such decoupled system is out of our interest. A top-predator which alternates between two sources of prey can be represented within $0 < A < 1$.

We now nondimensionalize the system (2) with $x = \frac{X}{K_x}$, $y = \frac{Y}{K_y}$, $z = \frac{Z}{K_z}$, $t = \frac{K_t T}{T}$ and obtain the following system

$$
\begin{align*}
\frac{dx}{dt} &= x(1 - x) - \frac{a_{1}x}{1 + b_1x}, \\
\frac{dy}{dt} &= \frac{a_1xy}{1 + b_1x} - \frac{a_2AyZ}{1 + b_2Y} - d_1y, \\
\frac{dz}{dt} &= \frac{a_2Z}{1 + b_2Y} + c(1 - A) - d_2z - ez,
\end{align*}
$$

(4)

where $a_1 = \frac{A_1 K_x}{B_1 T}$, $a_2 = \frac{A_2 K_y K_z}{B_2 K_2 T}$, $b_1 = \frac{K_y}{K_x}$, $b_2 = \frac{K_z}{K_y}$, $c = \frac{C_2 K_z}{K_y}$, $d_1 = \frac{D_1 K_z}{K_y}$, $d_2 = \frac{D_2 K_z}{K_y}$, $e = \frac{E}{K_z}$. Here $a_1 = \frac{A_1 K_x}{B_1 T} > b_1 d_1 = \frac{D_1 K_z}{K_y}$ as $A_1 > D_1$ and $a_2 = \frac{A_2 K_y K_z}{B_2 K_2 T} > b_2 d_2 = \frac{D_2 K_z}{K_y}$ as $A_2 C_2 > D_2$.

Therefore the conditions for intermediate predator and top-predator persistence become $a_i > b_i d_i$, $i = 1, 2$. The system (4) has to be analysed with the following initial conditions: $x(0) \geq 0$, $y(0) \geq 0$, $z(0) \geq 0$. In the following section we shall analyse the dynamics of the system (3) theoretically as well as numerically.

3. Theoretical study

The functions of the right hand sides of the system (4) are continuous and have continuous partial derivatives on the state space $\mathbb{R}^3 = \{(x(t), y(t), z(t)) : x(t) \geq 0, y(t) \geq 0, z(t) \geq 0\}$. Therefore, they are Lipschitzian on $\mathbb{R}^3$ and hence the solution of the system (4) with non-negative initial condition exists and unique. Moreover, following Cao et al. [24], it is easy to show that the state space $\mathbb{R}^3$ is an invariant domain of the system (4).

In the following section, positivity and boundedness of solution for the system (4) are established. Since the state variables $x, y, z$ represent populations, positivity implies that they never become zero and population always survive. The boundedness may be interpreted as natural restrictions to unlimited growth as a consequence of limited resources [25].

3.1. Positive invariance

The system (4) can be written in the matrix form as $\dot{X} = F(X)$ with $X(0) = X_0 \in \mathbb{R}^3$, where $X = (x, y, z)^T \in \mathbb{R}^3$ and $F(X)$ is given by

$$
F = F(X) = 
\begin{pmatrix}
\frac{x(1 - x) - \frac{a_{1}x}{1 + b_1x}}{1 + b_1x} & \frac{a_1xy}{1 + b_1x} & \frac{-a_2AyZ}{1 + b_2Y} \\
\frac{a_1}{1 + b_1x} & \frac{-a_2AyZ}{1 + b_2Y} & -d_1y \\
\frac{-d_2}{1 + b_2Y} & c(1 - A) & -d_2z - ez
\end{pmatrix}.
$$

where $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $F \in C^0(\mathbb{R}^3)$.

It can be shown that whenever $X(0) \in \mathbb{R}^3$ such that $X_i = 0$ then $F_i(X)_{|X_i=0} \geq 0$ (for $i = 1, 2, 3$). Now any solution of $F = F(X)$ with $X_0 \in \mathbb{R}^3$, say $X(t) = X(t, X_0)$, is such that $X(t) \in \mathbb{R}^3$ for all $t > 0$ [26].
3.2. Boundedness

**Theorem 1.** All solutions of the system (4) which start in \( \mathbb{R}^3_+ \) are uniformly bounded if \( d_2 + e > a_2 c (1 - A) \) holds.

**Proof.** Let \((x(t), y(t), z(t))\) be any solution of the system (4) with positive initial conditions.

Let us consider that, \( w = x + y + z \).

\[
\frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}.
\]

Using equations of (4), we have

\[
\frac{dw}{dt} = x(1-x) - \frac{a_1 xy}{1 + b_1 x} + \frac{a_1 xy}{1 + b_2 x} - \frac{a_2 Ay}{1 + b_2 y} - d_1 y + a_2 z \frac{Ay}{1 + b_2 y} + c(1 - A) - d_2 z - ez.
\]

Therefore,

\[
\frac{dw}{dt} < -\theta(x + y + z) - (x - 1)^2 + 1,
\]

where \( \theta = \min(1, d_1, d_2 + e - a_2 c (1 - A)) \), provided \( d_2 + e > a_2 c (1 - A) \).

Hence,

\[
\frac{dw}{dt} + \theta w \leq 1.
\]

Applying the theory of differential inequalities we obtain

\[0 < w < \frac{1}{\theta} \exp{w(x(0), y(0), z(0))} e^{-rt}.
\]

For \( t \to \infty \), we have \( 0 < w < \frac{1}{\theta} \).

Hence all the solutions of (4) that initiate in \( \mathbb{R}^3_+ \) are confined in the region \( S = \{(x, y, z) \in \mathbb{R}^3_+: w = \frac{1}{\theta} + \eta, \text{ for any } \eta > 0\} \).

This proves the theorem.

**Note:** The condition \( d_2 + e > a_2 c (1 - A) \) implies that \( 1 - \frac{d_2 + e}{a_2 c} < A < 1 \). Therefore, for uniformly bounded solutions of the system (4), supply of alternative food depends on harvesting effort \( (e) \).

3.3. Extinction criterion for top-predator

**Lemma 1.** If \( e > \frac{a_1 a_2 (1 - A) - b_2}{b_2} \), then \( \lim_{t \to \infty} z(t) = 0 \).

**Proof.**

\[
\frac{dz}{dt} = a_2 z \left( \frac{Ay}{1 + b_2 y} + c(1 - A) \right) - d_2 z - ez.
\]

Therefore,

\[
z(t) \leq z(t_0) \exp \left( \int_{t_0}^{t} \left( \frac{a_2 A + b_2 a_2 c (1 - A) - (d_2 + e) b_2 y(r)}{1 + b_2 y(r)} \right) dr \right).
\]

Thus if \( e > \frac{a_1 a_2 (1 - A) - b_2}{b_2} \), then \( \lim_{t \to \infty} z(t) = 0 \).

3.4. Existence and local stability criteria of equilibrium points

The system (4) possesses following four equilibrium states:

(a) The trivial equilibrium state is \( E_T = (0, 0, 0) \). An eigenvalue associated with the Jacobian matrix at \( E_T \) is 1, positive, for which \( E_T \) is an unstable equilibrium point.

(b) The axial equilibrium state is \( E_A = (1, 0, 0) \). The Jacobian matrix at the equilibrium point \( E_A \) is

\[
J(E_A) = \begin{pmatrix}
-1 & -\frac{a_1}{1 + b_2} & 0 \\
0 & \frac{a_1}{1 + b_2} - d_1 & 0 \\
0 & 0 & a_2 c (1 - A) - d_2 - e
\end{pmatrix}.
\]

The axial equilibrium point \( E_A \) is stable if \( \frac{a_1}{1 + b_2} < d_1 \) and \( A > 1 - \frac{d_2 + e}{a_2 c} \). If at least one of these conditions is violated, then \( E_A(1, 0, 0) \) is unstable.
(c) The top-predator free equilibrium state is $E_0 \equiv (x, y, 0)$, where $x = \frac{d_1}{a_1 A}$ and $y = \frac{a_2 b_2 c}{(a_2 b_2 c + 1)}$. The top-predator free equilibrium point $E_0$ exists if $a_1 > (b_1 + 1) d_1$.

We now study the stability criteria of the top-predator free equilibrium point $E_0$. The Jacobian matrix at $E_0$ is given by

$$J(E_0) = \begin{pmatrix}
1 - 2x - \frac{a_1^2}{(1 + b_1 x)^2} & -\frac{a_1 x}{1 + b_1 x} & 0 \\
\frac{a_1 x}{(1 + b_1 x)^2} & \frac{a_1}{1 + b_1 x} - d_1 & -\frac{a_2 b_2 c}{(a_2 b_2 c + 1)} \\
0 & 0 & \frac{a_2 b_2 c}{(a_2 b_2 c + 1)} + a_2 c(1 - A) - d_2 - e
\end{pmatrix}.$$

The characteristic roots of the Jacobian matrix $E_0$ are $\frac{a_1 x}{(1 + b_1 x)^2} + a_2 c(1 - A) - d_2 - e$ and the roots of the equation $x^2 + \sigma_1 x + \sigma_2 = 0$, where

$$\sigma_1 = -\frac{a_1 x}{(1 + b_1 x)^2} - \frac{a_1 x}{1 + b_1 x} - d_1$$

and

$$\sigma_2 = \left(1 - 2x\right) \left(\frac{a_1 x}{1 + b_1 x} - d_1\right) + \frac{a_1 y d_1}{(1 + b_1 x)^2}.$$

The top-predator free equilibrium point $E_0(x, y, 0)$ is asymptotically stable if $\frac{a_1 x}{(1 + b_1 x)^2} + a_2 c(1 - A) < d_2 + e$, $\sigma_1 > 0$ and $\sigma_2 > 0$

(d) The interior equilibrium state is $E^* \equiv (x^*, y^*, z^*)$, where $y^* = \frac{a_1 x - a_2 b_2 c(1 - A)}{(a_2 b_2 c + 1)}$, $z^* = \frac{a_1 x}{1 + b_1 x} - d_1$ and $x^*$ is the positive root of the equation $P x^2 + Q x + R = 0$ where, $P = b_1, Q = 1 - b_1$ and $R = \frac{a_1 x}{(1 + b_1 x)^2} + a_2 c(1 - A) - d_2 - e$. The positive interior equilibrium point $E^*$ exists if $d_1 + e > a_2 c(1 - A); a_2 A > b_1 (d_1 + e - a_2 c(1 - A))$. The characteristic equation of the Jacobian matrix $E^*$ is given by

$$x^3 + \Omega_1 x^2 + \Omega_2 x + \Omega_3 = 0,$$

where,

$$\Omega_1 = -A_{11} - A_{22},$$

$$\Omega_2 = A_{11} A_{22} - A_{12} A_{21} - A_{12} A_{23},$$

$$\Omega_3 = A_{11} A_{22} A_{33}.$$

It is obvious that $A_{11} < 0; A_{12} < 0; A_{21} < 0; A_{23} < 0 and A_{32} > 0$. Here $\Omega_1 = -A_{11} - A_{22} > 0$ if $A_{22} < 0$ and obviously $\Omega_3 = A_{11} A_{22} A_{33} > 0$.

Now $\Omega_2 = \Omega_3 = A_{21} A_{22} A_{32} - (A_{11} + A_{22}) (A_{12} A_{22} - A_{12} A_{23})$.

If $A_{23} < 0$, then $A_{21} A_{22} A_{32} > 0; (A_{11} + A_{22}) < 0 and (A_{12} A_{22} - A_{12} A_{23}) > 0$. Thus, $\Omega_2 - \Omega_3 > 0$. Therefore the system (3) is locally stable at the interior equilibrium state $E^*$ if $\frac{a_1 x}{(1 + b_1 x)^2} < \frac{a_1 x}{1 + b_1 x} - d_1$.

Corollary 1. The system (4) is locally stable at the interior equilibrium state $E^* = (x^*, y^*, z^*)$ if $\frac{a_1 x}{(1 + b_1 x)^2} < \frac{a_1 x}{1 + b_1 x} - d_1$.

Proposition 1. Equilibrium level of intermediate predator biomass decreases monotonically with the increase of alternative resource $A$ if $a_2 c A < \left[(d_2 + e) - a_2 c(1 - A)\right](a_2 b_2 c + 1)$ and increases with the increase of harvesting effort $e$.

Proof. Differentiating $y^*$ with respect to $A$ we get

$$\frac{dy^*}{dA} = \frac{a_2 c A - \left[(d_2 + e) - a_2 c(1 - A)\right](a_2 b_2 c + 1)}{\left[a_2 A - b_2 \left((d_2 + e) - a_2 c(1 - A)\right)\right]^2} < 0,$$

if $a_2 c A < \left[(d_2 + e) - a_2 c(1 - A)\right](a_2 b_2 c + 1)$. 
Now, differentiating $y'$ with respect to $e$ we have
\[
\frac{dy'}{de} = \frac{a_2 A}{[a_2 A - b_2 ((d_2 + e) - a_2 c(1 - A))]^2} > 0.
\]
Hence the proof is completed. □

**Proposition 2.** Equilibrium level of top-predator biomass is an increasing function of additional resource $A$ if $b_2 c < 1$, but is an decreasing function of $e$.

**Proof.** Differentiating $z'$ with respect to $A$ we get
\[
\frac{dz'}{dA} = \frac{a_2 (1 - b_2 c)}{[a_2 A - b_2 ((d_2 + e) - a_2 c(1 - A))]^2} \left[ \frac{a_1 x'}{1 + b_1 x'} - d_1 \right].
\]
If $b_2 c < 1$, then $\frac{dz'}{dA} > 0$.
Again, differentiating $z'$ with respect to $e$ we have
\[
\frac{dz'}{de} = \frac{-d_2}{[a_2 A - b_2 ((d_2 + e) - a_2 c(1 - A))]^2} \left[ \frac{a_1 x'}{1 + b_1 x'} - d_1 \right] < 0.
\]
Hence the proof is completed. □

Therefore, from the above Propositions, it is clear that systems biomass depends on supply of alternative resource $A$ and harvesting effort $e$.

### 3.5. Persistence of the system

Butler et al. [27], Freedman and Waltman [28,29] developed the following definitions of persistence:
If a population $N(t)$ is such that $N(t) > 0$, then

(i) $N(t)$ is said to be weakly persistent if $\lim_{t \to \infty} \sup N(t) > 0$.
(ii) $N(t)$ is said to be (strongly) persistent if $\lim_{t \to \infty} \inf N(t) > 0$. Further, if $N(t) \in C$, where $C$ is a certain class of function,
then
(iii) $N(t)$ is said to be weakly uniformly persistent if there exists $\delta > 0$ such that $\lim_{t \to \infty} \sup N(t) > \delta$ for all $N(t) \in C$.
(iv) $N(t)$ is said to be uniformly persistent if there exists $\delta > 0$ such that $\lim_{t \to \infty} \inf N(t) > \delta$ for all $N(t) \in C$.

From biological point of view, persistence of a system means the long term survival of all populations of the system, no matter what the initial populations are. Mathematically, persistence of a system means that strictly positive solutions do not have omega ($\Omega$) limit points on the boundary of the non-negative cone.

**Theorem 2.** Let the following conditions are satisfied

(i) $\frac{a_1 x}{1 + b_1 x} > d_1$,
(ii) $A < 1 - \frac{a_1 x}{b_1 x}$
(iii) $1 + \frac{d_2}{1 + b_2 (x)} < d_1 + 2x + \frac{a_1 x}{1 + b_1 x}$
(iv) $\left( \frac{1 - 2x}{1 + b_2 (x)} - d_1 \right) > 0$,

and if there exists a finite number (say, $n$) of periodic solutions $x = \phi_i(t), y = \psi_i(t), r = 1, 2, 3, \ldots, n$, in the $xy$ plane, then system (4) is uniformly persistent provided for each periodic solutions of period $T$,
\[
\zeta_r = -d_2 - e + a_2 (1 - A) + \frac{1}{T} \int_0^T \left( \frac{AA_2 \psi_i(t)}{1 + b_2 \psi_i(t)} \right) dt > 0, \quad r = 1, 2, \ldots, n.
\]

**Proof.** Let $X = (x, y, z)$ be a point in the positive quadrant and $\Omega(X)$ be orbit through $X$ and $\Omega$ be the omega limit set of the orbit through $X$. Note that $\Omega(X)$ is bounded.

We claim that $E_T \notin \Omega(X)$. If $E_T \in \Omega(X)$ then by the Butler-McGehee lemma [28] there exist a point $P$ in $\Omega(X) \bigcap W^s(E_T)$ where $W^s(E_T)$ denotes the stable manifold of $E_T$. Since $O(P)$ lies in $\Omega(X)$ and $W^s(E_T)$ is the $yz$ plane, we conclude that $O(P)$ is unbounded, which is contrary to the boundedness of the system.

Next $E_3 \notin \Omega(X)$, for otherwise, since $E_3$ is a saddle point which follows from the conditions $\frac{a_1 x}{1 + b_1 x} > d_1$ and $A < 1 - \frac{a_1 x}{b_1 x}$, by the Butler-McGehee lemma there exist a point $P$ in $\Omega(X) \bigcap W^u(E_3)$. Now $W^u(E_3)$ is the $x$-axis implies that an unbounded orbit lies in $\Omega(X)$, a contradiction.
Lastly, we show that no periodic orbits in the xy plane or \( E_2 \in \Omega(X) \). The conditions (iii) and (iv) implies that the eigenvalue values of \( J(E_2) \) have non-negative real parts. Let \( \gamma_i, r = 1, 2, \ldots, n \) denote the closed orbit of the periodic solution \( (\phi_i(t), \psi_i(t)) \) in xy plane such that \( \gamma_i \) lies inside \( \gamma_{e-1} \). Let the Jacobian matrix \( J_i(\phi_i(t), \psi_i(t), 0) \) corresponding to \( \gamma_i \) is given by

\[
J_i = \begin{pmatrix}
F_{i1}(\phi_i(t), \psi_i(t), 0) & F_{i2}(\phi_i(t), \psi_i(t), 0) & 0 \\
F_{21}(\phi_i(t), \psi_i(t), 0) & F_{22}(\phi_i(t), \psi_i(t), 0) & 0 \\
0 & 0 & F_{23}(\phi_i(t), \psi_i(t), 0)
\end{pmatrix}
\]

Here \( F_{i2} = -d_2 - \bar{e} + \alpha_2(1 - A) + \frac{\alpha_1\gamma_i}{(1 + \theta_1)} \). Computing the fundamental matrix of the linear periodic system, we obtain \( \lambda_j = \frac{1}{\gamma_j} \). Then proceeding in an analogous manner like Kumar and Freedman [30], we conclude that no \( \gamma_i \) lies on \( \Omega(X) \). Thus, \( \Omega(X) \) lies in the positive quadrant and system (4) is persistent. Finally, since only the closed orbits and the equilibria from the omega limit set of the solutions on boundary of \( R^3 \), the system (4) is dissipative. Now using a theorem of Butler et al. [28], we conclude that system (4) is uniformly persistent. ∎

**Theorem 3.** Let the following conditions are satisfied

(i) \( \frac{d_2}{d_1} > 1 \),

(ii) \( A < 1 - \frac{d_2}{d_1} \),

(iii) \( 1 + \frac{d_1}{d_2} > d_1 + 2x + \frac{4d_1}{1 + \theta_1} \),

(iv) \( \frac{1}{2d_2} \left( \frac{a_0}{1 + \theta_1} - d_1 \right) + \frac{a_0}{1 + \theta_1} > 0 \),

(v) \( \frac{d_2 - \bar{e} + a_2(1 - A) + \frac{\alpha_1\gamma_i}{(1 + \theta_1)}}{d_1} > 0 \).

Then if there exists no limit cycle in the xy plane, the system (4) is uniformly persistent.

**Proof.** Proof is obvious and so omitted.

4. Bionomic equilibrium

The term bionomic equilibrium is an amalgamation of biological equilibrium and economic equilibrium. The biological equilibrium of the system (4) satisfy the equations

\[
\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0.
\]

The economic equilibrium is said to be achieved when the TR (total revenue obtained by selling the harvested predator) equals TC (the total cost for the effort devoted to harvesting). At first we consider the term \( e \) to be the non-dimensional measure of the harvesting effort, \( p \) is the constant price per unit biomass, \( h \) is the constant cost of harvesting effort and \( \alpha \) is the economic constant. Then the economic rent (net revenue) at any time is given by

\[
\pi(z, e) = TR - TC = (p - \omega e)z - he.
\]

For convenience, we take \( p \) and \( h \) to be constant. So, the economic equilibrium can be obtained from the Eq. (5) and using the equation of zero profit line

\[
\pi(z, e) = 0 \text{ i.e., } (p - \omega z)e - he = 0,
\]

Therefore, \( (p - \omega)z - he = 0 \).

(7)

Therefore, using the Eqs. (5) and (7) one can obtain the feasible economic equilibrium \((x, \bar{y}, \bar{z})\). The optimal economic rent is calculated in the next section.

5. Optimal control policy

In commercial exploitation of renewable resources the fundamental problem is to determine the optimal trade-off between present and future harvests. The emphasis of this section is on the profit-making aspect of the model. It is a thorough study of the optimal harvesting policy to optimize the profit earned by harvesting considering quadratic costs and conservation of top-predator. The prime reason for using quadratic costs is that it allows to derive an analytical expression for the optimal harvesting. It is assumed that price is a function which decreases with the increase of harvested biomass. Thus, to maximize the total discounted net revenues from the model, the optimal control problem can be formulated [31] as
\[ J(e) = \int_{t_0}^{T} e^{-\delta t}(p - c_e)z - he dt, \]  

(8)

where \( \delta \) is the instantaneous annual discount rate. The problem (8) can be solved by applying Pontryagins maximum principle subject to model (4) and control constraints \( 0 < e < e_{\text{max}} \). The convexity of the objective function with respect to \( e \), the linearity of the differential equations in the control and the compactness of the range values of the state variables can be combined to give the existence of the optimal control. Suppose \( e_i \) is an optimal control with corresponding states \( x_i, y_i, z_i \). We take \( A_i(x_i, y_i, z_i) \) as optimal equilibrium point. Here we intend to derive optimal control \( e_i \) such that

\[ J(e_i) = \max \{ J(e) : e \in U \}, \]

where \( U \) is the control set defined by \( U = \{ e : [x_0, t_f] \rightarrow [0, e_{\text{max}}] \} \) is Lebesgue measurable.

Now, the Hamiltonian of this optimal control problem is \( H = (p - c_e)z - he + \lambda_1 [x(1 - x) - \frac{a_i y_i}{1 + b_i x_i} - \frac{a_i y_i}{1 + b_i x_i}] + \lambda_2 (a_i y_i (1 - y_i) + c(1 - A) - d_i z - ez), \) where \( \lambda_1, \lambda_2, \lambda_3 \) are adjoint or costate variables.

Here the transversality conditions give \( \lambda_i(t_f) = 0, i = 1, 2, 3. \)

Now, it is possible to find the characterization of the optimal control \( e_i \).

On the set \( t : 0 < e_i(t) < e_{\text{max}} \), we have

\[ \frac{\partial H}{\partial e} = pz - 2a_i z e - h - \lambda_3 z. \]

Thus at \( A_i(x_i, y_i, z_i), e = e_i(t) \) and

\[ \frac{\partial H}{\partial e} = pz - 2a_i z e - h - \lambda_3 z = 0. \]

This implies that, \( e_i = \frac{p z_i - h - \lambda_3 z_i}{2a_i z_i}. \)

(10)

Now, the adjoint equations at the point \( A_i(x_i, y_i, z_i) \) are

\[ \frac{d\lambda_1}{dt} = \delta \lambda_1 - \frac{\partial H}{\partial x_i} \big|_{A_i} = \delta \lambda_1 - \left[ \lambda_1 \left( 1 - 2x_i - \frac{a_i y_i}{1 + b_i x_i} \right) + \lambda_2 \left( \frac{a_i y_i}{1 + b_i x_i} \right) \right], \]

(11)

\[ \frac{d\lambda_2}{dt} = \delta \lambda_2 - \frac{\partial H}{\partial y_i} \big|_{A_i} \text{ Therefore, } \frac{d\lambda_2}{dt} = \delta \lambda_2 - \left[ \lambda_1 \left( - \frac{a_i y_i}{1 + b_i x_i} \right) + \lambda_3 \left( \frac{a_i y_i}{1 + b_i x_i} \right) \right], \]

(12)

\[ \frac{d\lambda_3}{dt} = \delta \lambda_3 - \frac{\partial H}{\partial z_i} \big|_{A_i} \text{ Thus, } \frac{d\lambda_3}{dt} = \delta \lambda_3 - \left[ \lambda_2 \left( - \frac{a_i y_i}{1 + b_i x_i} \right) + \lambda_3 \left( \frac{a_i y_i}{1 + b_i x_i} \right) \right]. \]

(13)

The three Eqs. (11)–(13) are first order system of simultaneous differential equations and it is easy to get the analytical solution of the equations with the help of initial conditions \( \lambda_i(t_f) = 0, i = 1, 2, 3. \) Using the value of \( \lambda_i \) and Eqs. (5) and (9) one can get the feasible optimum harvesting equilibrium \( (x_i, y_i, z_i) \). Therefore the optimum economic rent or net revenue at any time is obtained using the value of \( z_i \) from the Eq. (6). In this regard, it is important to note that we have formulated the optimal control problem considering fishing effort as control parameter. The optimal control problem is solved using a forward–backward sweep technique of 4th order Runge–Kutta method. We summarize the above analysis by the following theorem.

**Theorem 4.** There exists an optimal control \( e_i \) and corresponding solutions of the system (4) \( (x_i, y_i, z_i) \) maximizes \( J(e) \) over \( U \). Furthermore, there exists adjoint functions \( \lambda_1, \lambda_2, \lambda_3 \) satisfying Eqs. (11)–(13) with transversality conditions \( \lambda_i(t_f) = 0, i = 1, 2, 3. \) Moreover, the optimal control is given by \( e_i = \frac{p z_i - h - \lambda_3 z_i}{2a_i z_i}. \)

6. Hopf bifurcation

With variation of model parameters, qualitatively different dynamical behaviour may occur and the critical parameter values at which such transitions happen are called bifurcation points. The purpose of this study is to determine the stability of the system with the variation of different parameters of the system. We have considered \( e \) as bifurcation parameter and \( e_{\text{cr}} \) represent the critical value or the bifurcating value of the concerned parameters [32].
Theorem 5. The necessary and sufficient conditions for occurrence of Hopf bifurcation at \( e = e_{cr} \) are the following

(i) \( \Omega(e_{cr}) > 0, \quad i = 1, 2, 3. \)
(ii) \( \Omega_i(e_{cr}) = 0, \quad i = 1, 2, 3. \)
(iii) \( \text{Re}\left[\frac{d\lambda_i}{de}\right]_{e=e_{cr}} \neq 0, \quad i = 1, 2, 3. \)

Proof. The condition \( \Omega_1, \Omega_2, \Omega_3 = 0 \) gives \( A_{12}A_{21}A_{23} - (A_{11} + A_{22})(A_{13}A_{22} - A_{12}A_{23}) = 0 \), where all \( A_{ij} \) for \( i, j = 1, 2, 3 \) and \( \Omega \)'s are given in Subsection 3.4.

Since \( \Omega_2 > 0 \) at \( e = e_{cr} \), there exists an interval containing \( e \) in \( (e_{cr} - \epsilon, e_{cr} + \epsilon) \), for every \( \epsilon > 0 \). Therefore, for \( e \in (e_{cr} - \epsilon, e_{cr} + \epsilon) \) the characteristic equation cannot have roots containing negative real parts. For \( e = e_{cr} \), we have
\[
(\lambda^2 + \Omega_2)(\lambda + \Omega_1) = 0.
\]

It has three roots namely, \( \lambda_1 = -i\sqrt{\Omega_2}, \quad \lambda_2 = -i\sqrt{\Omega_2}, \quad \lambda_3 = -\Omega_1. \)

For \( e \in (e_{cr} - \epsilon, e_{cr} + \epsilon) \), the roots of the characteristic equation are in following general form
\[
\lambda_1(e) = \eta_1(e) - i\eta_2(e), \\
\lambda_2(e) = \eta_1(e) + i\eta_2(e), \\
\lambda_3(e) = -\Omega_1(e).
\]

We now verify the transversality condition \( \text{Re}\left[\frac{d\lambda_i}{de}\right]_{e=e_{cr}} \neq 0, i = 1, 2, 3. \)

Substituting \( \lambda_1(e) = \eta_1(e) - i\eta_2(e) \) in (14) and calculating the derivative, we get
\[
M_1(e)\eta_1(e) - M_2(e)\eta_2(e) + N_1(e) = 0, \\
M_2(e)\eta_1(e) + M_1(e)\eta_2(e) + N_2(e) = 0,
\]
where,
\[
M_1(e) = 3\eta_2^2(e) + 2\Omega_1(e)\eta_1(e) + \Omega_2(e) - \Omega_1(e)^2, \\
M_2(e) = 6\eta_1(e)\eta_2(e) + 2\Omega_1(e)\eta_2(e), \\
N_1 = \eta_1^2(e)\Omega_2(e) + \eta_2^2(e)\Omega_1(e) + \Omega_1(e)\eta_2^2(e), \\
N_2 = 2\eta_1(e)\eta_2(e)\Omega_2(e) + \Omega_1(e)\eta_2^2(e).
\]

Since \( M_1(e_{cr})N_1(e_{cr}) + M_2(e_{cr})N_2(e_{cr}) \neq 0, \) we have \( \text{Re}\left[\frac{d\lambda_i}{de}\right]_{e=e_{cr}} = \frac{M_1(e_{cr})N_1(e_{cr}) - M_2(e_{cr})N_2(e_{cr})}{M_1(e_{cr})N_2(e_{cr}) - M_2(e_{cr})N_1(e_{cr})} \neq 0, i = 1, 2, 3, \) and \( \lambda_3(e_{cr}) = -\Omega_1(e_{cr}). \)

Hence the theorem is proved. \( \square \)

A Hopf point occurs at \( e = e_{cr} \), if the Jacobian matrix \( J(E) \) has a pair of purely imaginary eigenvalues namely, \( \lambda_{1,2} = \pm i\omega_0, \quad \omega_0 > 0, \) and no other critical eigenvalue. A Hopf point is called transversal if the curves of complex eigenvalues cross the imaginary axis with non-zero derivative. At the Hopf point [33], when the first Lyapunov coefficient \( l_1 < 0 (l_1 > 0) \), a family of stable (unstable) periodic orbits can be found and the equilibrium point looses its stability. Therefore, the Hopf bifurcation with first lyapunov coefficient \( l_1 < 0 (l_1 > 0) \) indicates supercritical(subcritical) Hopf bifurcation.

7. Numerical study

We perform the numerical simulation of the system (4) with the following set of parameters, most of which are taken from HP [18] model. A set of parameter values are taken as \( a_1 = 5.0, a_2 = 0.1, b_1 = 3, b_2 = 2.0, d_1 = 0.4, d_2 = 0.01, e = 0.45, \) which remain unchanged throughout the simulations. We examine the effects of alternative resource parameter \( A \) within \( 0 < A < 1 \) and harvesting effort \( e \) in the system.

We first simulate the dynamics of top-predator population \( z \) for different harvesting effort \( e \) and alternative resource \( A \) in Figs. 1 and 2. From Fig. 1(a), we observe that top-predator population \( z \) extinct for harvesting effort \( e = 0.01 \), when no alternative resource is available. But in presence of alternative resource, we observe that the top-predator population survives for the same harvesting effort \( e = 0.01 \) (Fig. 1(b)-(d)). Fig. 2 depicts the dynamics of the top-predator population with harvesting effort \( e = 0.015 \) for different values of alternative resource parameter. From Fig. 2, it is evident that the top-predator population extinct in absence of alternative resource, but in presence of alternative resource, the top-predator population may survives in the system. Therefore, from Figs. 1 and 2, it is clear that alternative resource supply prevent the extinction risk of top-predator population in the system.

One parameter bifurcation analysis is done with respect to harvesting effort of the system 4 in absence of alternative resource (i.e., for \( A = 1 \)) in Fig. 3. We observe from the Fig. 3 that the system has chaotic or high periodic oscillations within \( 0 < e < 0.0026 \), period-2 oscillation for \( 0.0026 < e < 0.0029 \), limit cycle for \( 0.0029 < e < 0.0034 \), stable steady state for \( 0.0034 < e < 0.0067 \) and extinction risk for \( e > 0.0067 \). Therefore, the top-predator population has extinction risk for \( e > 0.0067 \) without alternative resource. We now demonstrate the bifurcation diagram of top-predator population in pres-
ence of alternative resource in Fig. 4. From Fig. 4, it is evident that top-predator population survive if $e \leq 0.0095$ for $A = 0.9$, if $e \leq 0.0124$ for $A = 0.8$, if $e \leq 0.0182$ for $A = 0.6$ and if $e \leq 0.0209$ for $A = 0.5$. Therefore, for every cases there is a critical harvesting effort below that top-predator survives and above that top-predator extinct. Notice that the critical value of harvesting effort increases with the increase of alternative resource. It is important to note that dynamics of top-predator population strongly depends on harvesting effort as well as supply of alternative resource. So, we plot the dynamics of top-predator population in $(A,e)$-plane in Fig. 5. Fig. 5 depicts regions of chaotic or periodic oscillation, steady state and extinction of top-predator's population. Notice that in Fig. 5 the boundaries between the different dynamical regions are not perfectly distinct. Because, in Fig. 5 that the high periodic oscillations and chaotic region there are small areas. This occurs because there is some degree of sensitivity to small changes in parameter values resulting in sharp transitions between different dynamical outcomes.
It is obvious that the adjoint variables are directly related to the change of the value of the Hamiltonian as the time derivatives of the adjoint variables are negative of the corresponding partial derivatives of the Hamiltonian, $H$ with respect to the state variables. We now study the nature of adjoint variables in Fig. 6. We observe that control variables $k_1, k_2, k_3$ change significantly with time. This means that to maximize the objective functional given in $8$ the rate of change of the Hamiltonian, $H$, should be changed according to the behaviour of adjoint variables. The effects of variation constant price per unit biomass of catch and alternative resource supply on optimal control of harvesting effort $e$ on the model is described in the Figs. 7 and 8. It is evident from the Fig. 7 that for low price rate, large amount of alternative resource is required for maximum harvesting. For high price rate, optimal harvesting effort increases for small amount of alternative resource supply. Therefore, it is natural that for maximum harvesting effort, sufficient alternative resource supply is very important. However, the controllability of the fishing effort with the price per unit biomass of catch is also necessary for constant alternative resource. It is noted from Fig. 8 that when price per unit biomass of catch decreases, the consequent harvesting effort will decrease. But with the increasing time, the optimal efforts correspond to the higher price per unit biomass of catch increases more rapidly compared to the optimal efforts correspond to lower price per unit biomass of catch.
8. Bifurcation of equilibrium and Hopf continuation

The main goal of this section is to study the bifurcation pattern of the system (4) with respect to harvesting effort \( e \) for different alternative resource parameter \( A \). This is actually done by studying the change in the eigenvalue of the Jacobian matrix and also following the continuation algorithm. To start with we consider a set of fixed point initial solution \( x_0 = 0.90582312, y_0 = 0.06745432 \) and \( z_0 = 5.059978 \), corresponding to a parameter set of values \( a_1 = 5.0, a_2 = 0.1, b_1 = 3, b_2 = 2.0, d_1 = 0.4, d_2 = 0.01, c = 0.45, p = 2.5, c_r = 7.5, h = 0.1 \) and \( A = 0.3, 0.5 \) and \( 0.9 \) respectively in Fig. 9. For \( A = 0.3 \), the first Hopf point \((H_1)\) is located at \((x, y, z, e) \) \((1.000000, 0.000000, 14.166668, 0.053000)\) with purely imaginary eigenvalues \( \pm 0.000621646i \) and the third one being \(-1\). For this Hopf point the first Lyapunov coefficient turns out to be \(-0.06482337\) indicating a super-critical Hopf bifurcation. It being negative implies that a stable limit cycle appears and the equilibrium point looses its stability. The second Hopf point \((H_2)\) occurs at \((x, y, z, e) \) \((0.713680, 0.179869, 16.680729, 0.060937)\) with purely imaginary eigenvalues \( \pm 0.0829487i \) and the third one being \(-0.323779\). For this second Hopf point the first Lyapunov coefficient turns
Fig. 7. Variation of optimal control of harvesting effort with the increasing time for $a_1 = 5.0, a_2 = 0.1, b_1 = 3, b_2 = 2.0, d_1 = 0.4, d_2 = 0.01, c = 0.45, h = 0.1$ and with the variation of alternative resource $A = 0.3, 0.5, 0.9$. (a) $p = 2.5, \omega = 7.5$, (b) $p = 3.5, \omega = 10.5$.

Fig. 8. Variation of optimal control of harvesting effort with the increasing time with the variation of constant price per unit biomass of catch for $a_1 = 5.0, a_2 = 0.1, b_1 = 3, b_2 = 2.0, d_1 = 0.4, d_2 = 0.01, c = 0.45, h = 0.1$ and $A = 0.5$. (a) $\omega = 7.5$, (b) $\omega = 10.5$.

Fig. 9. Continuation curves of equilibrium with the variation of harvesting effort $e$ of top-predator $z$ for constant alternative resource $A = 0.3, 0.5, 0.9$. 
out to be $-1.050970$, indicating a supercritical Hopf bifurcation. The third Hopf point ($H_3$) is situated at $\langle x; y; z; e \rangle = (0.333333, 0.266667, 11.074074, 0.063435)$ with the eigenvalues as $(0, -0.07536 \pm 0.524449)$. The real part being negative, indicates that the LP is stable.

A branch point $BP$ occurs at $\langle x; y; z; e \rangle = (0.052391, 0.063435, 0.075362, 0.5).$ The equilibrium bifurcation results for different alternative resource supply together with eigenvalues, first Lyapunov coefficients and normal form coefficient are listed in Table 1.
To proceed further, for supply of alternative resource $A = 0.5$, we start from the Hopf point $(H_2)$ which is predicted in Fig. 9 as the initial point at $e = 0.048301$, and computing in backward continuation process, get a family of stable limit cycles bifurcating from this Hopf point. This phenomenon is shown in Fig. 10, where again the harvesting effort is the only free parameter. At $e = 0.048301$, we observe a LPC point with period $58.40594$. At this situation two cycles collide and disappear. The critical cycle has a double multiplier equal to 1. From this it follows that a stable branch occurs after the LPC point.

At $e = 0.04776136$, we get a period doubling (PD) with period $62.47194$ and two of the multiplier is equal to 1. We also get another LPC and PD points at different values of $e$, which are listed in Table 2. One can observe LPC and Neimark–Sacker (NS) points by varying harvesting effort $e$ starting from the Hopf point at $e = 0.051601$, presented in Fig. 9 and its results are shown in Fig. 11. We also list the bifurcation points and its periodicity in Table 3, when harvesting effort is varying for alternative resource $A = 0.5$.

### Table 3

<table>
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<th>Label</th>
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<th>Normal form coefficient</th>
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</thead>
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<td>LPC</td>
<td>15.65692</td>
<td>-0.1524734</td>
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<tr>
<td>0.05005751</td>
<td>LPC</td>
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<td>0.004383309</td>
</tr>
<tr>
<td>0.04635609</td>
<td>NS</td>
<td>19.82018</td>
<td>–</td>
</tr>
</tbody>
</table>

To proceed further, for supply of alternative resource $A = 0.5$, we start from the Hopf point $(H_2)$ which is predicted in Fig. 9 as the initial point at $e = 0.048301$, and computing in backward continuation process, get a family of stable limit cycles bifurcating from this Hopf point. This phenomenon is shown in Fig. 10, where again the harvesting effort is the only free parameter. At $e = 0.048301$, we observe a LPC point with period $58.40594$. At this situation two cycles collide and disappears. The critical cycle has a double multiplier equal to 1. From this it follows that a stable branch occurs after the LPC point. At $e = 0.04776136$, we get a period doubling (PD) with period $62.47194$ and two of the multiplier is equal to 1. We also get another LPC and PD points at different values of $e$, which are listed in Table 2. One can observe LPC and Neimark–Sacker (NS) points by varying harvesting effort $e$ starting from the Hopf point at $e = 0.051601$, presented in Fig. 9 and its results are shown in Fig. 11. We also list the bifurcation points and its periodicity in Table 3, when harvesting effort is varying for alternative resource $A = 0.5$.

### 9. Conclusion

A food chain model providing alternative resource (additional food) to top-predator species in a predator–prey system is proposed and analysed with constant harvesting. Choosing a suitable alternative resource $A$, one can reduce the predation pressure on prey as well as remove extinction possibility of top-predator population. We assume that the supply of alternative resource is not dynamic, but maintained at a specific constant level. This simplification is justified for many arthropod predators, because they can feed on plant-provided alternative food sources such as pollen or nectar, the availability of which is unlikely to be influenced by predator’s consumption [34]. This model is especially important in the systems, such as caves, headwater streams, and some small marine islands, in which more energy enters the food web from alternative resource.

We have presented a systematic analysis of the model with the variation of harvesting effort in presence of suitable alternative food. From our investigation, we conclude that suitable alternative resource $(A)$ can reduce the extinction risk of top-predator population (Figs. 1 and 2) up to some critical harvesting effort. It is also clear that one can achieve the permission to obtain higher harvesting effort for suitable supply of alternative resource (Figs. 3 and 4). Therefore, one can conclude that the extinction risk will decrease for suitable supply of alternative food even when harvesting effort is high. It seems to be one of the important result of this paper. For an example, we draw numerically a functional relationship between alternative resource and harvesting effort (Fig. 5). On the other hand, from the bifurcation analysis, we observe chaos or high periodic oscillation to steady state transition is possible by increasing harvesting effort. Therefore, chaos of a food chain model can be controlled with harvesting effort. Reasonable harvesting policies are indisputably one of the major and interesting
problems from ecological and economic point of view. Optimal harvesting policy is also derived using Pontryagin’s Maximum Principle and the problem is solved both analytically and numerically. A Hopf bifurcation analysis is carried out when harvesting effort is increasing in presence of alternative resource. The bifurcation scenarios of top-predator population in presence of alternative resource is plotted in Fig. 9 when harvesting effort is varied for a considerable range. The continuation curve (LPC, PD etc.) starting from Hopf points are plotted by varying harvesting effort to identify the nature of dynamical system.

Therefore, our analysis predicts that extinction possibility of harvested species can be eliminated providing a suitable amount of alternative food to the species. We have also shown that chaos control is possible by harvesting in presence of alternative food. We have shown that biological control is possible supplying alternative food to the system. This study enables management to develop optimal harvesting strategies in presence of alternative food. Therefore, alternative food plays a very important role for biological conservation of species.

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Appendix

From the second equation of the system (1), we have
\[
\frac{dY}{dt} = A_1 \frac{XY}{B_1 + X} - A_2 \frac{YZ}{B_2 + Y} - D_1 Y,
\]
\[
= A_1 \frac{Y}{B_1/X + 1} - A_2 \frac{YZ}{B_2 + Y} - D_1 Y,
\]
\[
\leq (A_1 - D_1) Y - A_2 \frac{YZ}{B_2 + Y}, \quad \text{since} \quad \frac{B_1}{X} + 1 > 1.
\]

Therefore, condition for persistence of intermediate predator (\(Y\)) is \(A_1 > D_1\).

Again, from the third equation of the system (1), we obtain that
\[
\frac{dZ}{dt} = C_A \frac{YZ}{B_2 + Y} - D_2 Z.
\]
\[
= C_A \frac{Z}{B_2/Y + 1} - D_2 Z.
\]
\[
\leq (C_A - D_2) Z, \quad \text{since} \quad \frac{B_2}{Y} + 1 > 1.
\]

Therefore, condition for persistence of top-predator (\(Z\)) is \(C_A > D_2\).

References

Effects of Allochthonous Resources in a Three Species Food Chain Model with Harvesting

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Abstract We formulate a three species prey–predator model supplying allochthonous inputs to the top predator in the presence of top predator harvesting. All the equilibria of the system are determined and their stability nature are investigated. With the help of Pontryagin’s maximum principle, fishing effort is used as an optimal controller to investigate the optimal solution. The system dynamics for different values of preference parameter ($c$), catchability coefficient ($q$), allochthonous inputs ($A$) and harvesting effort ($E$) are presented. Bifurcation analysis is done with respect to harvesting and existence of chaos, 2-cycle, 4-cycle, limit cycle and steady state behaviours are observed. The existence of Hopf bifurcation, limit point, limit point cycle, period doubling and branch point are observed in the system. The dynamics of the system in the A–E parameter plane is presented. We observe that predator extinction risk decreases with the increasing of available allochthonous resources in the presence of harvesting. Our results suggest that allochthonous inputs at high levels can stabilize food webs and remove the possibility of species extinction in the presence of harvesting.

Keywords Predator–prey · Allochthonous resources · Harvesting · Extinction · Economic equilibrium · Hopf-bifurcation · Chaos

Introduction

A quantitative and qualitative understanding of the interaction of different species is crucial for the conservation and management of species. The exploitation of biological resources and harvesting of species are commonly practiced in fishery, forestry, agriculture and wildlife management [1]. Harvesting has a strong impact on the dynamic evolution of a population, depending on the nature of the applied harvesting strategy. In the absence of harvesting, a
population can be free of extinction risk; however, harvesting can lead to extinction in finite time. Extensive and unregulated harvesting of species lead to the extinction of several species. The problems of predator–prey systems in the presence of harvesting have been investigated by many authors. Most of them have focused their attention on optimal exploitation guided entirely by profits from harvesting [2–5]. The harvesting strategy and associated implementation guidelines aim to ensure that key commercial fish species are managed for long term biological sustainability and economic profitability.

Allochthonous resources (alternative resources) are an important component of most predators (e.g., coccinellid) diet, and although they receive less attention than basal prey in the scientific literature, these foods fundamentally shape the life histories of many predator species. Energetic resources (allochthonous resources) frequently enter into a food chain to increase local productivity and influence community structure and their stabilities [6–11]. Actually, in nature, fluxes across habitats often bring both nutrient and energetic resources into areas of low productivity from areas of higher productivity. Allochthonous resources can alter consumption rates of predator species in the recipient food systems, thereby influencing food web stability [9]. Thus the availability of suitable additional food (non-prey food, energetic resource etc.) in a predator–prey system can have a significant impact on the dynamics of the system. In reality, most predators do not only feed on a single prey but rather move to alternative foods and are therefore involved in a complex food web of interactions [12]. Polis and Strong [13] investigated that detrital and other allochthonous inputs play a central role in consumer-resource interactions and food web dynamics. The consequences of providing additional food to predator and the corresponding effects on the predator prey dynamics and its utility in biological control (such as species conservation and pest management) have been the topics of great attention for many scientists [14]. In recent years, many biologists, experimentalists, and theoreticians investigated the consequences of providing additional food to predators in a predator–prey system [11,14–17].

Huxel and McCann [9] examined the impact of allochthonous inputs on food web stability, using the Yodzis and Innes [18] parametrization on the Hastings and Powell [19] tri-trophic food chain model. Huxel et al. [9] described a tri-trophic food chain composed of a logistic prey (x), a Holling type II intermediate predator (y), and a Holling type II top-predator (z), with allochthonous inputs (A) into the top-predator population in the following model

\[
\begin{align*}
\frac{dx}{dt} &= x \left(1 - \frac{x}{k}\right) - \frac{a_1 a_2 x y}{b_1 + x}, \\
\frac{dy}{dt} &= \frac{a_1 a_2 x y}{b_1 + x} - \frac{a_3 a_4 (1 - c) y z}{b_2 + c A + (1 - c) y} - a_1 y, \\
\frac{dz}{dt} &= \frac{a_3 a_4 ((1 - c) y + c A) z}{b_2 + c A + (1 - c) y} - a_3 y.
\end{align*}
\]

where \(b_1\) and \(b_2\) are the half saturation constants for the functional response between the resource and consumer levels and the functional response between the consumer and top predator levels, respectively; \(a_1\) and \(a_3\) are the mass-specific metabolic rate of consumer and predator trophic level, measured relative to the production-to-biomass ratio of the resource density; \(a_2\) is a measure of the ingestion rate per unit metabolic rate of the basal trophic level \(x\) by consumer \(y\); and \(a_4\) is a measure of the ingestion rate per unit metabolic rate of \(y\). \(c\) (\(0 < c < 1\)) is the parameter describing the feeding preference for the allochthonous input by the top-predator. The analyses is performed over a range of feeding preference from 0 (feeding only on autochthonous/classical food sources) to 1 (feeding only on the allochthonous sources). The allochthonous input is a constant, and feeding on that resource
only depends on the amount of input and the feeding preference parameter. The parameters can be deemed biologically plausible as they represent realistic predator–prey ratios in body size found in surveys (Peters [20]; Cohen et al. [21]).

The impact of harvesting renewable resources considered and studied in detail in many works [22–25]. The functional form of the harvest is generally considered using the phrase catch-per-unit-effort (CPUE) hypothesis [22] to describe an assumption that catch per unit effort is proportional to the stock level. Since, allochthonous inputs are important sources of productivity in many food chains, then their influence need to be studied in a harvested model. Recently Kar et al. [26] focused on long-run sustainability of a harvested prey predator system in the presence of alternative prey. Kar et al. [26] also reported the consequences of providing additional food to the predator with both the species harvesting. In both the cases Kar et al. [26] considered a two species prey predator model and they had not considered the Huxel’s type of interaction function in the presence of alternative food. Since, the dynamics of three species prey predator model is complex [9] and Holling type-II interaction function in the presence of alternative food is more realistic, therefore, it is worth investigating the consequences of harvesting in Huxel’s model [9]. In this paper, we modify the Huxel et al. model [9] in the presence of top predator harvesting and obtain the following model:

\[
\begin{align*}
\frac{dx}{dt} &= x \left( 1 - \frac{x}{k} \right) - \frac{a_1 a_2 xy}{b_1 + x}, \\
\frac{dy}{dt} &= \frac{a_1 a_2 xy}{b_1 + x} - \frac{a_3 a_4 (1 - c) yz}{b_2 + cA + (1 - c)y} - a_1 y, \\
\frac{dz}{dt} &= \frac{a_2 a_4 [(1 - c) y + cA] z}{b_2 + cA + (1 - c)y} - a_3 y - qEz,
\end{align*}
\]

where \( q \) is the catchability coefficient of the top predator and \( E \) is the harvesting effort used to harvest the predator. The system is analysed with initial conditions \( x(0) \geq 0, y(0) \geq 0, z(0) \geq 0 \).

The main aim of this paper is to study consequences of supplying allochthonous inputs to the top-predator population in a three species population model incorporating a harvesting policy. The section-wise split of this paper is as follows. In “Positivity and Boundedness of Solutions” section, the conditions of positivity and boundedness of solutions are derived. Stability analysis of all equilibria are done in “Stability Analysis” section. “Biological and Economic Equilibrium” section illustrates the conditions for biological and economic equilibrium. The optimal harvesting policy is determined in “Optimal Control Policy” section. Conditions for existence of Hopf bifurcation is calculated theoretically in “Hopf Bifurcation” section. “Simulation Results” section demonstrates numerical results of the system. We also study the stability zones in parameter plane. Different routes of continuation of the associated bifurcation is investigated with the help of the software package MATCONT2.5.1 [27–29]. Finally conclusions are given in “Conclusions” section.

**Positivity and Boundedness of Solutions**

**Theorem 2.1** If the initial conditions are positive, then the solutions of system (2) remain positive for all times.

**Proof** The first equation of system (2) can be written as

\[
\frac{dx}{x} = \left( 1 - \frac{x}{k} \right) \frac{a_1 a_2 y}{b_1 + x} dt,
\]
which is of the form
\[
\frac{dx}{x} = \phi(x, y)dt,
\]
where
\[
\phi(x, y) = \left[ x \left( 1 - \frac{x}{k} \right) - \frac{a_1 a_2 x y}{b_1 + x} \right].
\]
Now integrating the above differential equation over the interval \([0, t]\) we get
\[
x(t) = x(0)e^{\int \phi(x,y)dt} > 0, \quad \text{for all } t.
\]
The second equation of system (2) can be written as
\[
\frac{dy}{y} = \left\{ \frac{a_1 a_2 x}{b_1 + x} - \frac{a_3 a_4 (1 - c) z}{b_2 + cA + (1 - c)y} - a_1 \right\} dt,
\]
i.e.,
\[
\frac{dy}{y} = \psi(x, y, z)dt,
\]
where
\[
\psi(x, y, z) = \left[ \frac{a_1 a_2 x y}{b_1 + x} - \frac{a_3 a_4 (1 - c) y z}{b_2 + cA + (1 - c)y} - a_1 y \right].
\]
Now integrating the above differential equation over the interval \([0, t]\) we have
\[
y(t) = y(0)e^{\int \psi(x,y,z)dt} > 0 \quad \text{for all } t.
\]
Similarly, the third equation of system (2) can be written as
\[
\frac{dz}{z} = \left\{ \frac{a_3 a_4 [(1 - c)y + cA]}{b_2 + cA + (1 - c)y} - a_3 - qEz \right\} dt,
\]
which can be written as
\[
\frac{dz}{z} = \theta(y, z)dt,
\]
where
\[
\theta(y, z) = \left[ \frac{a_3 a_4 [(1 - c)y + cA]z}{b_2 + cA + (1 - c)y} - a_3 z - qEz \right].
\]
Now integrating the above differential equation over the interval \([0, t]\) we have,
\[
z(t) = z(0)e^{\int \theta(y,z)dt} > 0 \quad \text{for all } t.
\]
Hence we can conclude that all solutions of the system (2) are always positive. □

**Theorem 2.2** All solutions of system (2) are bounded if \(a_3 + qE > a_3 a_4 cA\) holds.

**Proof** The first equation of system (2), it follows that
\[
\frac{dx}{dt} \leq x(k - x),
\]
Therefore, \(x(t) \leq k\) as \(t \to +\infty\)
Now we define
\[ w = x + y + z, \]
i.e.,
\[ \frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}. \]
Using equations of (2), we can write
\[ \frac{dw}{dt} \leq x \left( 1 - \frac{x}{k} + 1 \right) - x - a_1 y - (a_3 + qE - a_3a_4cA)z, \]
i.e.,
\[ \frac{dw}{dt} \leq k - \theta(x + y + z), \]
where \( \theta = \min(1, a_1, a_3 + qE - a_3a_4cA) > 0 \), provided \( a_3 + qE > a_3a_4cA \).
\[ \frac{dw}{dt} + \theta w \leq k \]
Applying the theory of differential inequalities we obtain
\[ 0 < w < \frac{k - e^{-\theta t}}{\theta} + w(x(0), y(0), z(0))e^{-\theta t}. \]
Now, as \( t \to \infty \), we have \( 0 < w < \frac{k}{\theta} \).

Hence all the solutions of (2) that initiate in \( R^3_+ \) are confined in the region \( S = \{ (x, y, z) \in R^3_+ : w = \frac{k}{\theta} + \epsilon, \text{ for any } \epsilon > 0 \} \). This proves the theorem. \( \square \)

**Stability Analysis**

The biologically possible equilibrium states of system (2) are identified and their local stability criteria are determined in this section.

(a) The trivial equilibrium state \( E_0 \equiv (0, 0, 0) \). An eigenvalue associated with the Jacobian matrix at \( E_0 \) is 1, therefore \( E_0 \) is an unstable equilibrium.

(b) The axial equilibrium state \( E_1 \equiv (k, 0, 0) \). The Jacobian matrix at \( E_1 \) is
\[ J(E_1) = \begin{pmatrix} -1 & -\frac{a_1a_2k}{k+b_1} & 0 \\ 0 & \frac{a_2k}{k+b_1} - a_1 & 0 \\ 0 & 0 & \frac{a_3a_4cA}{b_2+cA} - a_3 - qE \end{pmatrix}. \]
The axial equilibrium \( E_1 \) is stable if
\[ \frac{a_2k}{k+b_1} < 1 \quad \text{and} \quad \frac{a_3a_4cA}{b_2+cA} < a_3 + qE. \]
The equilibrium \( E_1 \) is a saddle point if either
\[ \frac{a_2k}{k+b_1} > 1 \quad \text{or} \quad \frac{a_3a_4cA}{b_2+cA} > a_3 + qE \]
or both holds.
(c) The boundary equilibrium state $E_2 \equiv (\bar{x}, \bar{y}, 0)$, where

$$\bar{x} = \frac{b_1}{a_2 - 1} \quad \text{and} \quad \bar{y} = \frac{b_1[ka_2 - (k + b_1)]}{ka_1(a_2 - 1)^2}.$$  

The boundary equilibrium $E_2$ exists if $ka_2 > (k + b_1)$. 

The Jacobian matrix at $E_2$ is given by

$$J(E_2) = \begin{pmatrix}
1 - \frac{2\bar{x} - a_1a_2b_1\bar{y}}{ka_1(b_1 + \bar{x})^2} & \frac{a_1a_2\bar{x}}{b_1 + \bar{x}} & 0 \\
\frac{a_1a_2\bar{x}}{b_1 + \bar{x}} & \frac{a_1a_2\bar{x}}{b_1 + \bar{x}} & -a_1 - \frac{a_1a_4(1-c)\bar{y}}{b_2 + cA + (1-c)\bar{y}} \\
0 & 0 & \frac{a_1a_4(1-c)\bar{y} + cA}{b_2 + cA + (1-c)\bar{y}} - a_3 - qE
\end{pmatrix}.$$  

The characteristic roots of the Jacobian matrix $J(E_2)$ are

$$\sigma_1 = \frac{a_3a_4[(1-c)\bar{y} + cA]}{b_2 + cA + (1-c)\bar{y}} - a_3 - qE$$

and the roots of the equation $\lambda^2 + \sigma_1 \lambda + \sigma_2 = 0$, where

$$\sigma_2 = \left(1 - \frac{2\bar{x}}{k} \right) \left\{ \frac{a_1a_2\bar{x}}{b_1 + \bar{x}} - a_1 \right\} + \frac{a_1^2a_2^2b_1\bar{x}\bar{y}}{(b_1 + \bar{x})^3}.$$  

The boundary equilibrium $E_2(\bar{x}, \bar{y}, 0)$ is stable if

$$\frac{a_3a_4[(1-c)\bar{y} + cA]}{b_2 + cA + (1-c)\bar{y}} < a_3 + qE, \quad \sigma_1 > 0 \quad \text{and} \quad \sigma_2 > 0.$$  

(d) The interior equilibrium state $E^* \equiv (x^*, y^*, z^*)$, where

$$y^* = \frac{(b_2 + cA)(a_3 + qE) - a_3acA}{a_3a_4 - (a_3 + qE)(1-c)}, \quad z^* = \frac{b_2 + cA + (1-c)y^*}{a_3a_4(1-c)} \left[ \frac{a_1a_2x^*}{b_1 + x^*} - a_1 \right]$$

and $x^*$ is the positive root of the equation $Px^2 + Qx^* + R = 0$ where, $P = 1, Q = b_1 - k$ and $R = a_1a_3ky^* - b_1k$. The positive interior equilibrium $E^*$ exists if

$$(b_2 + cA)(a_3 + qE) > a_3acA, a_3a_4 > (a_3 + qE), \frac{a_2x^*}{b_1 + x^*} > 1$$

and $Q^2 \geq 4PR, Q < 0$.

The Jacobian matrix at $E^*$ is given by

$$J(E^*) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33} \end{pmatrix},$$

where

$$A_{11} = 1 - \frac{2x^*}{k} - \frac{a_1a_2b_1y^*}{(b_1 + x^*)^2}, \quad A_{12} = -\frac{a_1a_2x^*}{b_1 + x^*}.$$
\[
A_{13} = 0, \quad A_{21} = \frac{a_1 a_2 b_1 y^*}{(b_1 + x^*)^2}, \\
A_{22} = \frac{a_1 a_2 x^*}{b_1 + x^*} - \frac{a_3 a_4 (1 - c) (b_2 + c A) z^*}{(b_2 + c A + (1 - c) y^*)^2} - a_1, \\
A_{23} = -\frac{a_3 a_4 (1 - c) y^*}{b_2 + c A + (1 - c) y^*}, \\
A_{31} = 0, \quad A_{32} = \frac{a_3 a_4 (1 - c) b_2 z^*}{(b_2 + c A + (1 - c) y^*)^2}, \quad A_{33} = 0.
\]

The characteristic equation of the Jacobian matrix \( J(E^*) \) is given by

\[
\lambda^3 + \Omega_1 \lambda^2 + \Omega_2 \lambda + \Omega_3 = 0,
\]

where

\[
\Omega_1 = -[A_{11} + A_{22}], \\
\Omega_2 = [A_{11} A_{22} - A_{12} A_{21} - A_{32} A_{23}], \\
\Omega_3 = A_{11} A_{32} A_{23}.
\]

Obviously, we can determine that \( A_{11} < 0, A_{12} < 0, A_{21} > 0, A_{23} < 0 \) and \( A_{32} > 0 \).

Here \( \Omega_1 = -[A_{11} + A_{22}] > 0 \) if \( A_{22} < 0 \) and obviously \( \Omega_3 = [A_{11} A_{32} A_{23}] > 0 \).

Now \( \Omega_1 \Omega_2 - \Omega_3 = A_{22} A_{23} A_{32} - (A_{11} + A_{22}) (A_{12} A_{21} - A_{12} A_{22}). \) If \( A_{22} < 0 \), then \( A_{22} A_{23} A_{32} > 0, (A_{11} + A_{22}) < 0 \) and \( (A_{11} A_{22} - A_{12} A_{21}) > 0 \). Thus \( \Omega_1 \Omega_2 - \Omega_3 > 0 \).

Therefore, system (2) is locally stable at the interior equilibrium state \( E^* \) if

\[
\frac{a_1 a_2 x^*}{b_1 + x^*} < \frac{a_3 a_4 (1 - c) (b_2 + c A) z^*}{(b_2 + c A + (1 - c) y^*)^2} + a_1.
\]

**Corollary 1** System (2) is locally stable at the interior equilibrium if

\[
\frac{a_1 a_2 x^*}{b_1 + x^*} < \frac{a_3 a_4 (1 - c) (b_2 + c A) z^*}{(b_2 + c A + (1 - c) y^*)^2} + a_1.
\]

Now, we study the effects of allochthonous input \( A \) and harvesting effort \( E \) on the steady state population size in the following propositions.

**Proposition 1** The equilibrium level of prey biomass always monotonically increases with respect to the allochthonous input level \( A \) and decreases with respect to harvesting effort \( E \).

**Proof** Differentiating \( x^* \) with respect to \( A \) we obtain

\[
\frac{dx^*}{dA} = (Q^2 - 4 PR)^{-\frac{1}{2}} a_1 a_2 k - \frac{c}{(1 - c)} > 0,
\]

where \( P = 1, Q = b_1 - k \) and \( R = a_1 a_2 k y^* - b_1 k \).
Now, differentiating $x^*$ with respect to $E$ we have
\[
\frac{dx^*}{dE} = -(Q^2 - 4PR)^{-\frac{1}{2}}a_1a_2k \frac{bqa_3a_4}{(1-c)(a_3a_4 - (a_3 + qE))^2} < 0.
\]
Hence the proof is completed. \qed

**Proposition 2** The equilibrium level of intermediate predator biomass always monotonically decreases with respect to the allochthonous input level $A$ and increases with respect to harvesting effort $E$.

**Proof** Differentiating $y^*$ with respect to $A$ we get
\[
\frac{dy^*}{dA} = -\frac{c}{1-c} < 0.
\]
Now, differentiating $y^*$ with respect to $E$ we have
\[
\frac{dy^*}{dE} = \frac{bqa_3a_4}{(1-c)(a_3a_4 - (a_3 + qE))^2} > 0,
\]
Hence the proof is completed. \qed

**Proposition 3** The equilibrium level of top-predator biomass always increases with respect to the allochthonous inputs $A$. But the equilibrium biomass of the top-predator decreases with respect to $E$ if
\[
a_2x^*(1-c)(b_1 + x^*) < a_1(b_1 + x^*)^2(1-c) + (b_2 + cA + (1-c)y^*)a_1a_2^2b_1k.
\]

**Proof** Differentiating $z^*$ with respect to $A$ we get
\[
\frac{dz^*}{dA} = \frac{b_2 + cA + (1-c)y^*}{a_3a_4(1-c)} \frac{a_1a_2b_1}{(b_1 + x^*)^2} \frac{dx^*}{dA} > 0.
\]
Again, differentiating $z^*$ with respect to $E$ we have
\[
\frac{dz^*}{dE} = \frac{bq}{(1-c)(a_3a_4 - (a_3 + qE))^2}
\]
\[
\cdot \left[ \frac{a_1a_2x^* - a_1}{b_1 + x^*} - \frac{b_2 + cA + (1-c)y^*}{1-c} \frac{a_1a_2b_1}{(b_1 + x^*)^2} (Q^2 - 4PR)^{-\frac{1}{2}}a_1a_2k \right] < 0
\]
if
\[
a_2x^*(1-c)(b_1 + x^*) < a_1(b_1 + x^*)^2(1-c) + (b_2 + cA + (1-c)y^*)a_1a_2^2b_1k.
\]
Hence the proof is completed. \qed

From the above propositions, it is clear that equilibrium of the prey population increases with an increased rate of allochthonous inputs $A$ and decreases with an increased rate of harvesting effort. The equilibrium level of the intermediate predator decreases with an increase of $A$ and the population size increases when harvesting effort is applied to the top-predator. The equilibrium of the top-predator population increases with an increased rate of $A$ and decreases for an increased rate of harvesting effort provided
\[
a_2x^*(1-c)(b_1 + x^*) < a_1(b_1 + x^*)^2(1-c) + (b_2 + cA + (1-c)y^*)a_1a_2^2b_1k.
\]
From a biological point of view, it is obvious that the equilibrium of the top-predator will decrease for harvesting effects and therefore, the equilibrium of the intermediate predator
will increase and as a result, the prey population will decrease. For allochthonous inputs, the equilibrium of the top-predator increases and therefore the equilibrium of the intermediate predator will decrease and hence equilibrium of the prey will increase.

**Biological and Economic Equilibrium**

The study of the economic equilibrium of system (2) is of practical significance to test the applicability of the model in a real situation.

The biological equilibrium can be obtained from the isoclines given by system (2). Using the third equation of system (2) we have

\[ \frac{a_3a_4[(1 - c)y + cA]}{b_2 + cA + (1 - c)y} - a_3 - qE = 0. \]  

(3)

Now combining the first two isoclines of (2) with (3), we have

\[ y = \frac{(a_3 + qE)(b_1 + x)z}{a_1x(a_2 - 1) - a_1b_1} - \frac{cA}{1 - c}, \]  

(4)

and

\[ B_3x^3 + B_2x^2 + B_1x + B_0 = 0, \]  

(5)

where

\[
B_0 = (1 - c)ka_1b_1^2 + a_1a_2k(a_3 + qE)(1 - c)b_1z + cAa_1^2a_2b_1k,
\]

\[
B_1 = -[(1 - c)kb_1a_1(a_2 - 1) - (k - b_1)a_1b_1(1 - c) - a_1a_2k(1 - c)(a_3 + qE)z + cAa_1^2a_2k(a_2 - 1)],
\]

\[
B_2 = -[(1 - c)((k - b_1)a_1(a_2 - 1) + a_1b_1)]B_3 = (1 - c)a_1(a_2 - 1).
\]

The points satisfy both Eqs. (4) and (5) are referred to as the biological equilibrium curve.

The economic equilibrium is said to be achieved when the \( TR \) (total revenue obtained by selling the harvested predator \( z \)) equals \( TC \) (the total cost for the effort devoted to harvesting). At first we define some symbols for this purpose. The term \( E \) is the non-dimensional measure of the harvesting effort. The term \( q \) denotes the catchability coefficient of the top-predator. \( p \) is the constant price per unit biomass. The symbol \( h \) denotes the constant cost of harvesting effort and \( \omega \) is the economic constant. Then the economic rent (net revenue) at any time is given by

\[ \pi(z, E) = TR - TC = (p - \omega q Ez)qEz - hE. \]  

(6)

For convenience, we take \( p, q, \omega, h \) to be constant. So, the economic equilibrium can be obtained from the Eqs. (4), (5) and using the equation of zero profit line

\[ \pi(z, E) = 0 \quad \text{i.e.,} \quad (p - \omega q E z)qEz - hE = 0, \]

\[ \omega q E z^2 - pqz + h = 0. \]  

(7)

Therefore, from Eqs. (4), (5) and (7) one can obtain the feasible economic equilibrium \((\hat{x}, \hat{y}, \hat{z})\). The optimal economic rent is calculated in the next section.
Optimal Control Policy

In commercial exploitation of renewable resources the fundamental problem, is to determine the optimal trade-off between present and future harvests. The emphasis of this section is on the profit-making aspect of the model. It is a thorough study of the optimal harvesting policy and the profit earned by harvesting, focusing on quadratic costs and conservation of the top-predator population by constraining the latter to always stay above a critical threshold. The prime reason for using quadratic costs is that it allows to derive an analytical expression for the optimal harvest; the resulting solution is different from the bang-bang solution which is usually obtained in the case of a linear cost function. It is assumed that price rate decreases with increasing biomass. Thus, to maximize the total discounted net revenues from the model, the optimal control problem can be formulated \(^{[30]}\) as

\[
J(E) = \int_{t_0}^{t_f} e^{-\delta t} [\(p - \omega q E z\) q E z - h E] dt,
\]

where \(\delta\) is the instantaneous annual discount rate. Problem \((8)\) can be solved by applying Pontryagin’s maximum principle subject to model \((2)\) and control constraints \(0 \leq E \leq E_{max}\). The convexity of the objective function with respect to \(E\), the linearity of the differential equations in the control and the compactness of the range values of the state variables can be combined to give the existence of the optimal control.

Suppose \(E_{\delta}\) is an optimal control with corresponding states \(x_{\delta}, y_{\delta}\) and \(z_{\delta}\). We take \(A_{\delta}(x_{\delta}, y_{\delta}, z_{\delta})\) as an optimal equilibrium. Here we intend to derive optimal control \(E_{\delta}\) such that

\[
J(E_{\delta}) = \max\{J(E) : E \in U\},
\]

where \(U\) is the control set defined by \(U = \{E : [t_0, t_f] \rightarrow [0, E_{max}] \ | \ E\) is Lebesgue measurable\}. 

Now the Hamiltonian of this optimal control problem is

\[
H = (p - \omega q E z) q E z - h E + \lambda_1 \left[ x \left( 1 - \frac{x}{k} \right) \right] \frac{a_1 a_2 x y}{b_1 + x} + \lambda_2 \left[ \frac{a_1 a_2 x y}{b_1 + x} - \frac{a_3 a_4 (1 - c) y z}{b_2 + c A + (1 - c) y} - a_1 y \right] + \lambda_3 \left[ \frac{a_3 a_4 (1 - c) y + c A}{b_2 + c A + (1 - c) y} - a_3 - q E z \right],
\]

where \(\lambda_1, \lambda_2, \lambda_3\) are adjoint or costate variables.

Here the transversality conditions give \(\lambda_i(t_f) = 0, i = 1, 2, 3\). Now, it is possible to find the optimal control \(E_{\delta}\). On the set \(t : 0 < E_{\delta}(t) < E_{max}\), we have

\[
\frac{\partial H}{\partial E} = pq z - 2 \omega q^2 z^2 E - h - \lambda_3 q z.
\]

Thus at \(A_{\delta}(x_{\delta}, y_{\delta}, z_{\delta}), E = E_{\delta}(t)\) and

\[
\frac{\partial H}{\partial E} = pq z_{\delta} - 2 \omega q^2 z_{\delta}^2 E_{\delta} - h - \lambda_3 q z_{\delta} = 0.
\]

This implies that

\[
E_{\delta} = \frac{pq z_{\delta} - h - \lambda_3 q z_{\delta}}{2 \omega q^2 z_{\delta}^2}.
\]
Now, the adjoint equations at the point \( A_δ(x_δ, y_δ, z_δ) \) are
\[
\frac{dλ_1}{dt} = \delta λ_1 - \frac{∂H}{∂x} |_{A_δ} = \delta λ_1 - \left[ λ_1 \left( 1 - \frac{2x_δ}{k} - \frac{a_1a_2b_1y_δ}{(b_1 + x_δ)^2} \right) + λ_2 \left( \frac{a_1a_2b_1y_δ}{(b_1 + x_δ)^2} \right) \right],
\]
\[
\frac{dλ_2}{dt} = \delta λ_2 - \frac{∂H}{∂y} |_{A_δ}
\]
Therefore,
\[
\frac{dλ_2}{dt} = \delta λ_2 - \left[ λ_1 \left( \frac{a_1a_2x_δ}{b_1 + x_δ} \right) + λ_2 \left( \frac{a_1a_2x_δ}{b_1 + x_δ} - \frac{a_3a_4(1-c)(b_2 + cA + (1-c)y_δ)^2}{(b_2 + cA + (1-c)y_δ)^2} - a_1 \right) \right. \\
\left. \quad + λ_3 \left( \frac{a_3a_4(1-c)b_2z_δ}{(b_2 + cA + (1-c)y_δ)^2} \right) \right],
\]
\[
\frac{dλ_3}{dt} = \delta λ_3 - \frac{∂H}{∂z} |_{A_δ}
\]
Thus,
\[
\frac{dλ_3}{dt} = \delta λ_3 - \left[ λ_2 \left( \frac{a_3a_4(1-c)y_δ}{b_2 + cA + (1-c)y_δ} \right) \right. \\
\left. \quad + λ_3 \left( \frac{a_3a_4[(1-c)y_δ + cA]}{b_2 + cA + (1-c)y_δ} - a_3 - qE_δ \right) \right] - (pqE_δ - 2ωq^2E_δ^2z_δ).
\]
The three Eqs. (11), (12), (13) is a first order system of simultaneous differential equations and it is easy to get the analytical solution of the equations with the help of initial conditions 

\[ λ_i(t_f) = 0, \quad i = 1, 2, 3. \]
Using the value of \( λ_3 \) and Eqs. (4), (5), (9) one can get the feasible optimum harvesting equilibrium \((x_δ, y_δ, z_δ)\). Therefore, the optimum economic rent or net revenue at any time is obtained using the value of \( z_δ \) from Eq. (6). Notice that we have formulated the optimal control problem by considering the fishing effort as a control parameter and the optimal control problem will be numerically solved using a forward-backward sweep technique of 4th order Runge-Kutta method to pursue numerical simulations in the next section. We summarize the above analysis by the following theorem.

**Theorem 5.1** There exists an optimal control \( E_δ \) and corresponding solutions to the system (2) \( x_δ, y_δ \) and \( z_δ \) that maximize \( J(E) \) over \( U \). Furthermore, there exists adjoint functions \( λ_1, λ_2 \) and \( λ_3 \) satisfying Eqs. (11), (12), (13) with transversality conditions \( λ_i(t_f) = 0, \quad i = 1, 2, 3. \) Moreover, the optimal control is given by
\[
E_δ = \frac{pqz_δ - c - λ_3qz_δ}{2ωq^2z_δ^2}.
\]

**Hopf Bifurcation**

Predator–prey models with constant parameters are often found to approach a steady state in which the species coexist in equilibrium. But if parameters used in the model are changed, other types of dynamical behaviour may occur and the critical parameter values at which such transitions happen are called bifurcation values. The purpose of this study is to determine the stability behaviours of the system in presence of different density-dependent factors.
of the predator–prey interactions. To study the transition of the system with respect to the small changes in the density dependent factors, we consider $E$ as a bifurcation parameter and $E_{cr}$ represents the critical value or the bifurcating value of the concerned bifurcation parameter [31].

**Theorem 6.1** Necessary and sufficient conditions for occurrence of Hopf bifurcation at $E = E_{cr}$ are the following

(i) $\Omega_i(E_{cr}) > 0$, $i = 1, 2, 3$,

(ii) $\Omega_1(E_{cr})\Omega_2(E_{cr}) - \Omega_3(E_{cr}) = 0$,

(iii) $\text{Re}\left[\frac{d\lambda_1}{dE}\right]_{E=E_{cr}} \neq 0$, $i = 1, 2, 3$.

**Proof** The condition $\Omega_1\Omega_2 - \Omega_3 = 0$ gives $A_{22}A_{33} - (A_{11} + A_{22})(A_{11}A_{22} - A_{12}A_{21}) = 0$, where all $A_{ij}$, for $i, j = 1, 2, 3$ and $\Omega_i$’s are given in “Stability Analysis” (d) section.

Since $\Omega_2 > 0$ at $E = E_{cr}$, there exists an interval containing $E$ in $(E_{cr} - \epsilon, E_{cr} + \epsilon)$, for every $\epsilon > 0$. Therefore, for $E \in (E_{cr} - \epsilon, E_{cr} + \epsilon)$ the characteristic equation cannot have roots containing negative real parts. For $E = E_{cr}$, we have

$$\lambda^2 + \Omega_2 \lambda + \Omega_1 = 0. \quad (14)$$

It has three roots namely, $\lambda_1 = +i\sqrt{\Omega_2}$, $\lambda_2 = -i\sqrt{\Omega_2}$, and $\lambda_3 = -\Omega_1$.

For $E \in (E_{cr} - \epsilon, E_{cr} + \epsilon)$, the roots of the characteristic equation are in following general form $\lambda_i(E) = \eta_i(E) + i\zeta_i(E), \lambda_2(E) = \eta_1(E) - i\eta_2(E), \lambda_3(E) = -\Omega_3(E)$.

We now verify the transversality condition $\text{Re}\left[\frac{d\lambda_i}{dE}\right]_{E=E_{cr}} \neq 0$, $i = 1, 2, 3$.

Substituting $\lambda_i(E) = \eta_i(E) + i\zeta_i(E)$ in (14) and calculating the derivative, we get

$$M_1(E)\eta_1(E) - M_2(E)\eta_2(E) + N_1(E) = 0, M_2(E)\eta_1(E) + M_1(E)\eta_2(E) + N_2(E) = 0,$$

where, $M_1(E) = 3\eta_1(E) - 2\Omega_1(E)\eta_1(E) + \Omega_2(E) - 3\eta_2(E)$, $M_2(E) = 6\eta_1(E)\eta_2(E) + 2\Omega_1(E)\eta_2(E) + \Omega_2'(E)\eta_1(E) + \Omega_2''(E) - \Omega_1(E)\eta_2(E)$, $N_1 = 2\eta_1(E)\Omega_1(E) + \Omega_1'(E)\eta_2(E)$, $N_2 = 2\eta_1(E)\Omega_1(E) + \Omega_2'(E)\eta_2(E)$. Since $M_1(E_{cr})N_1(E_{cr}) + N_2(E_{cr})N_2(E_{cr}) \neq 0$, we have

$$\text{Re}\left[\frac{d\lambda_i}{dE}\right]_{E=E_{cr}} = \frac{M_1N_1 + M_2N_2}{M_1^2 + M_2^2} |_{E=E_{cr}} \neq 0, \quad i = 1, 2, 3.$$

and $\lambda_3(E_{cr}) = -\Omega_1(E_{cr})$. Hence the theorem is proved. □

It is to be noted that a Hopf bifurcation occurs at $E = E_{cr}$ if the Jacobian matrix $J(E^*)$ has a pair of purely imaginary eigenvalues namely, $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, and no other critical eigenvalue. A Hopf bifurcation is called transversal if the curves of complex eigenvalues cross the imaginary axis with non-zero derivative. At the Hopf bifurcation, when the first Lyapunov coefficient $l_1 < 0$ ($l_1 > 0$), a family of stable (unstable) periodic orbits can be found at the equilibrium. Therefore, the Hopf bifurcation with first Lyapunov coefficient $l_1 < 0$ ($l_1 > 0$) indicates supercritical(subcritical) Hopf bifurcation [32,33].

**Simulation Results**

Numerical simulations are carried out with a set of parameter values shown in Table 1. We shall discuss the dynamics of the proposed system by varying allochthonous inputs $A$ ($0 \leq A < 1$) [9] as well as harvesting effort $E$.

Table 2 shows the numerical values of biological equilibria and economic equilibria for different allochthonous inputs and harvesting efforts for the above set of parameter values. The variation of the equilibrium level of the prey, intermediate predator and top-predator
Table 1  Parameters values

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
<th>Parameter value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>Carrying capacity</td>
<td>1 [9]</td>
</tr>
<tr>
<td>$a_1$</td>
<td>Mass-specific metabolic rate of consumer</td>
<td>0.4 [18]</td>
</tr>
<tr>
<td>$a_2$</td>
<td>Measure of the ingestion rate per unit metabolic rate of $y$</td>
<td>2.009 [18]</td>
</tr>
<tr>
<td>$a_3$</td>
<td>Mass-specific metabolic rate of top predator</td>
<td>0.08 [18]</td>
</tr>
<tr>
<td>$a_4$</td>
<td>Measure of the ingestion rate per unit metabolic rate of $z$</td>
<td>5 [18]</td>
</tr>
<tr>
<td>$b_1$</td>
<td>Half saturation constant of consumer</td>
<td>0.16129 [18]</td>
</tr>
<tr>
<td>$b_2$</td>
<td>Half saturation constant of top predator</td>
<td>0.5 [18]</td>
</tr>
<tr>
<td>$c$</td>
<td>Preference parameter</td>
<td>0.1 $(0 &lt; c &lt; 1)$ [9]</td>
</tr>
<tr>
<td>$q$</td>
<td>Catchability coefficient of top predator</td>
<td>0.1 [30]</td>
</tr>
</tbody>
</table>

Table 2  Biological and economic equilibria for different harvesting efforts with various allochthonous inputs

<table>
<thead>
<tr>
<th>Allochthonous input</th>
<th>Harvesting effort</th>
<th>Biological equilibrium</th>
<th>Economic equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = 0.1$</td>
<td>$E = 0.1$</td>
<td>(0.8941, 0.1391, 0.5032)</td>
<td>(0.2237, 0.3719, 0.2931)</td>
</tr>
<tr>
<td></td>
<td>$E = 0.3$</td>
<td>(0.8502, 0.1885, 0.5277)</td>
<td>(0.2450, 0.3816, 0.3105)</td>
</tr>
<tr>
<td></td>
<td>$E = 0.5$</td>
<td>(0.7936, 0.2453, 0.5512)</td>
<td>(0.2720, 0.3926, 0.3333)</td>
</tr>
<tr>
<td>$A = 0.5$</td>
<td>$E = 0.1$</td>
<td>(0.8278, 0.2419, 0.5608)</td>
<td>(0.2622, 0.3887, 0.3333)</td>
</tr>
<tr>
<td></td>
<td>$E = 0.3$</td>
<td>(0.8030, 0.1552, 0.5348)</td>
<td>(0.2376, 0.3786, 0.3105)</td>
</tr>
<tr>
<td>$A = 0.8$</td>
<td>$E = 0.1$</td>
<td>(0.9475, 0.0724, 0.5138)</td>
<td>(0.2136, 0.3670, 0.2931)</td>
</tr>
<tr>
<td></td>
<td>$E = 0.3$</td>
<td>(0.9085, 0.1218, 0.5411)</td>
<td>(0.2314, 0.3756, 0.3105)</td>
</tr>
<tr>
<td>$A = 1$</td>
<td>$E = 0.1$</td>
<td>(0.7524, 0.2815, 0.6059)</td>
<td>(0.2067, 0.9038, 0.3900)</td>
</tr>
<tr>
<td></td>
<td>$E = 0.3$</td>
<td>(0.8594, 0.1786, 0.5692)</td>
<td>(0.2538, 0.3854, 0.3333)</td>
</tr>
<tr>
<td></td>
<td>$E = 0.8$</td>
<td>(0.9462, 0.0502, 0.5169)</td>
<td>(0.2190, 0.3654, 0.2931)</td>
</tr>
<tr>
<td></td>
<td>$E = 0.1$</td>
<td>(0.9264, 0.0996, 0.5449)</td>
<td>(0.2277, 0.3741, 0.3105)</td>
</tr>
<tr>
<td></td>
<td>$E = 0.3$</td>
<td>(0.8792, 0.1564, 0.5741)</td>
<td>(0.2489, 0.3834, 0.3333)</td>
</tr>
<tr>
<td></td>
<td>$E = 0.8$</td>
<td>(0.7783, 0.2593, 0.6149)</td>
<td>(0.2671, 0.3907, 0.3900)</td>
</tr>
</tbody>
</table>

Biomass with respect to allochthonous input $A$ and harvesting effort $E$ are shown in Table 2. The results of all postulates are verified and listed in Table 2. Now, we demonstrate the importance of supplying allochthonous inputs to the top-predator population in the model. We start our discussion with the parameter values taken from Table 1. We collect the results of maximum density of top-predator population supplying allochthonous inputs to the system. Fixing the harvesting rate and taking $(x_0, y_0, z_0) = (0.1, 0.2, 0.2)$ as the initial condition, we calculate the maximum density of the top-predator population within 365 days for various allochthonous inputs and the results are presented in Figs. 1 and 2. From Figs. 1 and 2, it is evident that for constant harvesting effort, the density of the top-predator population $(z)$ increases with the increase of allochthonous inputs. Therefore, allochthonous input plays an important role for the growth of the top-predator in the system.

Now we discuss the dynamic behaviours of system (2) for various values of preference parameter $(c)$, catchability coefficient $(q)$, allochthonous inputs $(A)$ and harvesting effort $(E)$
Fig. 1  Plot of maximum density of top-predator population ($z$) within 365 days for various allochthonous inputs and others as in Table 1 fixing harvesting effort $a E = 0.2$ and $b E = 0.4$. It depicts that density of the predator $z$ increases for supplying increase rate of allochthonous inputs with constant harvesting.

Fig. 2  Plot of maximum density of top-predator population ($z$) within 365 days for various allochthonous inputs and others as in Table 1 fixing harvesting effort $a E = 0.5$ and $b E = 0.8$. It depicts that density of the predator $z$ increases for supplying increase rate of allochthonous inputs with constant harvesting.

keeping other parameters fixed as in Table 1. Dynamics of system (2) for $E = 0.5$, $c = 0.2$, $A = 0.7$, $q = 0.05$ are presented in Fig. 3 and chaotic oscillation of the population size is observed. Figure 4 depicts period-3 behaviour for $E = 0.5$, $c = 0.2$, $A = 0.7$, $q = 0.1$. For $E = 0.85$, $c = 0.1$, $A = 0.8$, $q = 0.1$, we observe limit cycle oscillation of the system which is presented in Fig. 5. From Figs. 3, 4, and 5, we conclude that from chaos to order transition is possible in the system (2) for suitable allochthonous inputs and catchability coefficients and preference parameters.

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Numerically, we have analysed the optimal control problem considering fishing effort as control parameter using a 4th order Runge-Kutta method for a time span of 10 years. We take here some hypothetical data (Table 1) for the purpose of illustrating the established theoretical results of previous sections. Variation of optimal harvesting effort with time is plotted in Fig. 6 using the parameter values taken from Table 1 and taking $p = 1.5$, $\omega = 1.5$, $\delta = 0.02$, $h = 0.05$ with initial conditions $( \lambda_1 = 0.2, \lambda_2 = 0.2, \lambda_3 = 0.405 )$. It is evident from Fig. 6a that fishing effort of the harvested system increases with the increase of allochthonous inputs to the top-predator. For low allochthonous inputs, the optimal rate of harvesting decreases due to a slow growth rate of the harvested population. The impact of constant price per unit biomass of catch on the harvested population is described in Fig. 6b. It is clear that harvesting effort decreases up to 5 years and then it increases for moderate to high price of unit biomass. For the increasing price per unit biomass of catch, optimal harvesting effort
Fig. 5  Dynamics of the system (2) for \( E = 0.85, \ c = 0.1, \ A = 0.8 \) and \( q = 0.1 \) and others are taken from Table 1. It depicts limit cycle oscillation.

Fig. 6  Variation of optimal harvesting effort with time is plotted. It depicts the variation of optimal harvesting effort for various allochthonous inputs to top-predator and for different price per unit biomass of catch on the harvested population.

increases with time. Notice that when the price per unit biomass of catch decreases, the consequent harvested population increases with decreasing fishing effort. However, the price function of the harvested population is fluctuating year to year based on market demand and harvesting effort. Therefore, the controllability of the fishing effort of a harvested population depends on the supply of allochthonous inputs and price per unit biomass of the catch.

We now study the pattern of bifurcation that takes place as we vary the parameters \( E \) and \( A \). The characteristics of the Hopf point and the limit cycle may be explored using the software package MATCONT2.5.1. This package is a collection of numerical algorithms implemented as a MATLAB toolbox for the detection, continuation and identification of limit cycles. In this package we use a prediction-correction continuation algorithm based on the Moore-Penrose matrix pseudo inverse for computing the curves of equilibria, limit point (LP), along with fold bifurcation points of the limit point (LP) and continuation of Hopf points (H), etc. This
is actually done by studying the change in the eigenvalue of the Jacobian matrix and also following the continuation algorithm [34]. We consider a set of fixed point initial solution, $x_0 = 0.93154905$, $y_0 = 0.10675029$, $z_0 = 0.27859482$ corresponding to a parameter set of values as in Table 1. We show in Fig. 7 the continuation curve from the equilibrium with $E$ as the free parameter and for fixed $A = 0.8$. Existence of Hopf point (H), one limit point (LP) and one branch point (BP) of the system in the positive quadrant is observed. The Hopf value is located at $(x, y, z, E) \equiv (0.676064, 0.337543, 0.610824, 0.937015)$. For this Hopf point the first Lyapunov coefficient turns out to be $-1.519071$, indicating a supercritical Hopf bifurcation. Indeed, there are two complex eigenvalues of the equilibrium with real $\lambda_{1,2} \approx 0$ at this parameter and the eigenvalues at this Hopf point are $(-0.328998, \pm 0.156158i)$. The first Lyapunov coefficient is negative implies that a stable limit cycle appears when the equilibrium loses stability. As the parameter is increased, the limit point (LP) is located at $(x, y, z, E) \equiv (0.419354, 0.419548, 0.479802, 1.111430)$ with the eigenvalues as $(0, -0.0355633 \pm 0.327414i)$. The branch point (BP) occurs at $(x, y, z, E) \equiv (0.159851, 0.335747, 0.000000, 0.932870)$ with eigenvalues as $(0, 0.129171 \pm 0.389997i)$. We list these dynamical results in Table 3.
Fig. 8 Family of limit cycle bifurcating from the Hopf point with the variation of harvesting effort $E$. a predicts limit point cycle (LPC) and period doubling (PD) dynamics and b indicates period of LPC and PD with respect to harvesting effort $E$ for allochthonous inputs $A = 0.8$.

<table>
<thead>
<tr>
<th>$E$</th>
<th>Label</th>
<th>Period</th>
<th>Normal form coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9370149</td>
<td>LPC</td>
<td>40.23598</td>
<td>0.5669778</td>
</tr>
<tr>
<td>0.7928537</td>
<td>PD</td>
<td>41.73442</td>
<td>-0.01389971</td>
</tr>
</tbody>
</table>

Table 4 Periodicity and normal form coefficient of equilibria when harvesting effort $E$ is varying starting from Hopf value at $E = 0.937015$

To proceed further we start from the Hopf bifurcation labelled as H obtained in Fig. 7, as the initial point and get a family of stable limit cycles bifurcating from this Hopf value. This phenomenon is shown in Fig. 8a, where harvesting effort $E$ is the only free parameter. We observe that at $E = 0.937015$, we have a LPC point with period 40.23598. At this situation two cycles collide and disappear. The critical cycle has a double multiplier equal to 1. From this it follows that a stable branch occurs after the LPC point. At $E = 0.7928537$ there is a period doubling with period 41.73442 and two of the multipliers are equal to 1. Table 4 contains periodicity and normal form coefficients for these results. If we choose $E$ and period of the cycles as free parameters and start from the Hopf point, then the corresponding variation of $E$ versus period is shown in Fig. 8b. We also plot the maximum and minimum of $z$ starting from the Hopf value in Fig. 9. The important observation is that there is an unstable dynamics before Hopf point and after Hopf point a stable behaviour is observed. It should be noted that if we continue the bifurcation process starting from Hopf value (II) and the period-doubling point (PD), the different dynamical natures of the top-predator population $z$ are observed for various allochthonous inputs. The dynamical nature of top-predator population is also studied with respect to harvesting effort for various allochthonous inputs in Figs. 10 and 11.

The bifurcation scenarios of the top-predator population with respect to harvesting effort $E$ are depicted in Fig. 10 and 11 at constant allochthonous inputs $A = 0.2$, $A = 0.5$, $A = 0.8$, and $A = 1$ for the parameter values given in Table 1. We observe different scenarios with
Fig. 9 Stable and unstable behaviour of Hopf point (H) with the variation of harvesting effort $E$ at a fixed allochthonous inputs $A = 0.8$.

Fig. 10 Bifurcation diagram of the top-predator population $z$ with respect to harvesting effort $E$ for different fixed allochthonous inputs $A$. It depicts chaotic attractors, 2–4 cycle, limit cycle and steady state behaviours with respect to harvesting effort $E$. Figure 10a predicts that the top-predator population has an extinction risk for low allochthonous inputs ($A = 0.2$) and for a high level of harvesting. But for low to moderate allochthonous inputs, the top-predator population has no extinction risk. For $A = 0.2$, we observe a Hopf value at $E = 0.79805$, and for a backward bifurcation process from Hopf value, period-2 cycle occurs at $E = 0.6265$, a period-4 cycle observed at $E = 0.5152$ and also chaos is observed. Accordingly, for $A = 0.5$, we observe a Hopf value at $E = 0.8715$, a period-2 cycle at $E = 0.7135$, a period-4 cycle at $E = 0.58989$ and also
Fig. 11  Bifurcation diagram of the top-predator population $z$ with respect to harvesting effort $E$ for different fixed allochthonous inputs $A$. It depicts chaotic attractors, 2–4 cycle, limit cycle and steady state behaviours.

Fig. 12  Dynamical results of the system (2), as a function of allochthonous input $A$ and harvesting effort $E$. It shows parametric region of different dynamical behaviours for $A \in [0, 0.5]$.

other complex dynamical scenarios (Fig. 10b). For $A = 0.8$, we observe a Hopf value at $E = 0.937015$, a period-2 cycle at $E = 0.7928538$, a period-4 cycle at $E = 0.63298$ and other complex dynamics (Fig. 11a). Also for $A = 1$, we observe a Hopf value at $E = 0.982512$, and in a backward technique from Hopf value we observe a period-2 cycle at $E = 0.84251$, a period-4 cycle at $E = 0.65188$ and other complex dynamics (Fig. 11b). Therefore, from these bifurcation diagrams, we observe that the Hopf value, periodic doubling behaviours occur at different harvesting efforts for different supply of allochthonous inputs. So the stability of a system strongly depends on the supply of allochthonous inputs. We also notice that the density of the top-predator species gradually increases for low to moderate allochthonous
inputs. From the bifurcation diagrams we can conclude that the chaos can be controlled by a large harvesting effort.

Different dynamical behaviour of the system in the \((A, E)\) parameter plane are plotted in Figs. 12 and 13. Notice in Figs. 12 and 13 that the boundaries between the different dynamical regions are not perfectly distinct. For example, in Fig. 13 in the \(n\)-cycle and chaotic region there are small areas of two to four cycles and chaotic bands. From Figs. 12 and 13 one can observe that the dynamics of system (2) is highly complex and can be controlled by a large harvesting effort. Increasing harvesting effort \(E\), leads to a bifurcation sequence from chaos \(\rightarrow\) 2–4 cycle \(\rightarrow\) limit cycle \(\rightarrow\) steady state. For very low allochthonous inputs, increase of harvesting effort can lead to extinction of the top-predator. However, at high levels of allochthonous inputs, the system has no extinction risk and reaches a steady state from chaos through different bifurcation sequences with increase of harvesting. Therefore, an increase of the allochthonous inputs leads to the persistence of the top predator species even for a high harvesting effort.

Conclusions

Supplying constant allochthonous inputs to a top predator population in a tri-trophic predator–prey model with top-predator harvesting is proposed and analysed in this paper. We discuss the existence criteria of various steady states and derive their local stability conditions. We compare the biological and economic equilibria of the system for practical purpose. We analyze the harvesting effects to investigate the optimal utilization of the resource, sustainability properties of the stock and the resource rent earned from the resource. Pontryagin’s maximum principle is used to characterize the optimal control. We examine the importance of allochthonous inputs to the top-predator with harvesting effects numerically. We observe that for absence of allochthonous inputs or the low allochthonous inputs may lead to the
top-predator extinction for some critical harvesting effort (Fig. 2). We also observe that allochthonous inputs play an important role to enrich the density of the top-predator species in the system (Fig. 1). The system dynamics for different values of preference parameter (c), catchability coefficient (q), allochthonous inputs (A) and harvesting effort (E) are presented (Figs. 3, 4, and 5). We calculate Hopf-value when harvesting effort is varied. The bifurcation analysis is also done systematically.

Occurrence of Hopf bifurcation in the neighbourhood of the interior equilibrium is observed with harvesting effort as the bifurcation parameter (Fig. 7) for some constant allochthonous inputs. The internal bifurcation scenarios (Hopf, LPC, PD etc.) of system (2) are computed with the help of MATCONT2.5.1 software. System (2) depicts some complex dynamical behaviours like chaos, 2–4 cycles, period-doubling cascade when harvesting effort is varying in the presence of different levels of allochthonous inputs (Figs. 10 and 11). Different dynamical phenomena are investigated in the E/A parameter plane (Figs. 12 and 13). We have shown that occurrence of the Hopf value strongly depends on allochthonous inputs. For higher allochthonous inputs, the Hopf bifurcation occurs at a higher harvesting effort. Therefore, we can conclude that the harvesting effort leads to a stable dynamic behaviour from an unstable motion through Hopf bifurcation when these parameters pass through the critical value. From the point of view of the bioeconomic context, optimal control policy is formulated and solved both analytically and numerically.

The prime purpose of this article is to draw attention to the direct and the indirect consequences of large-scale transfer of allochthonous resources from natural sources to natural communities. A fundamental prediction we make here is that the existence of high level of allochthonous resources can stabilize food chain in the presence of harvesting. Availability of suitable allochthonous resources may remove a species extinction possibility from the system in the presence of harvesting. Our model may be very useful for conservation of key commercial fish species and for long-term biological sustainability and economic profitability of a fishery.

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References