Chapter 5

THE JAYNES-CUMMINGS MODEL WITH CONTINUOUS EXTERNAL PUMPING

In the standard JCM [1] the atom and the field in the cavity get so entangled as to form a single entity. This is revealed in the vacuum field Rabi splitting [2] which has been observed by Thompson et al [3]. An external probe is necessary to get information about the system in the cavity. One of the ways would be to use a strong laser pump externally which would drive the atom in the cavity. Alsing et al [4] studied the driven JCM and obtained the steady state solution of it where they investigate the effect of field strengths on the dynamic Stark splitting induced. They solved the eigenvalue problem and found the dressed states of the driven system. They studied two situations. In the first one, the external field pumps the cavity mode which corresponds to the experiment of Thompson et al. In the second situation the field drives the atom directly. In this chapter we study the dynamics of the latter system numerically and analytically illustrate the model's correspondence with the standard JCM where the coherent field enters via the initial conditions. Our model is a simple example of the two-channel JCM [5] with the external laser making the second channel between the two states of the atom.

In Section 5.1 we present a brief mathematical review of the standard JCM demonstrating the sinusoidal Rabi oscillations and the periodic collapse and revivals for different initial field states. In Section 5.2 we present the state vector approach to the JCM with
continuous pumping. Numerical results for various coherent field amplitudes are presented. Large coherent amplitudes are shown to demonstrate collapse and revivals as in the standard JCM. In Section 5.3 the origin of this phenomenon is understood using a simple analytic technique making use of unitary transformations. In Section 5.4 the continuously pumped JCM is studied when the external field is detuned from the atom and the cavity.

5.1 The Standard Jaynes - Cummings Model

The fully quantized model of a single atom interacting with a single cavity mode constitutes the standard JCM. A lot of study has been done on this fundamental model in radiation-matter interaction. For a detailed description of the JCM and its numerous extensions see the reviews by Haroche and Raimond [6], Yoo and Eberly [7], Barnett et al [8], Meystre and Sargent [9], Meystre [10] and Shore and Knight [11].

The interaction of a single atom of transition frequency $\omega_o$ with a single mode of the radiation field of frequency $\omega$; in a cavity in the dipole and rotating wave approximations is described by the Jaynes-Cummings Model Hamiltonian [1]

$$H = \hbar \omega_o S^z + \hbar \omega b^\dagger b + \hbar g (S^+ b + S^- b^\dagger)$$

which is derived from the general Hamiltonian in Eq.(1.44) in the introductory Chapter 1. Here $g$ is the strength of the coupling between the radiation mode and the atom, which contains the transition dipole moment matrix element. Now $S^*$, $S^*$ are the atomic spin operators and $S^*$ are the bosonic ladder operators.

The interaction part of the Hamiltonian indicates the single photon processes that the atom undergoes. The $S^+ b$ term describes the excitation of the atom from the ground state to the excited state with the absorption of a photon from the cavity mode. The
The Jaynes-Cummings Model

In Eqs.(5.4), (5.5) and (5.6) C’s are the complex amplitudes.

Due to the conservation laws (5.2) and (5.3) the JCM which is infinite dimensional reduces to a two dimensional problem. The Hamiltonian matrix has a block diagonal form consisting of many (2 x 2) matrices along the main diagonal. Hence there are an

\[ \langle S^z \rangle^2 + \langle S^x \rangle^2 + \langle S^y \rangle^2 = \frac{3}{4} \]  

(5.2)

and another which conserves total excitation in the field-atom system

\[ \langle S^z \rangle + \langle b^\dagger b \rangle = \text{constant} \]  

(5.3)

The atomic state vector at time \( t \) is

\[ |\psi_A(t)\rangle = C^e|e\rangle + C^g|g\rangle \]  

(5.4)

The field state vector at time \( t \) is expressed in terms of the complete set of Fock states as

\[ |\psi_F(t)\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \]  

(5.5)

The state vector for the system of atom and the field mode is a direct product of the atom and field state vectors and is given by

\[ |\psi(t)\rangle = |\psi_A(t)\rangle \otimes |\psi_F(t)\rangle \]

\[ = \sum_{n=0}^{\infty} C_n^e |n, e\rangle + C_n^g |n, g\rangle \]  

(5.6)

In Eqs.(5.4), (5.5) and (5.6) C’s are the complex amplitudes.

S"b term describes the de-excitation of the atom from excited state to the ground state with the emission of a photon into the cavity mode.

Let \(|e\rangle\) and \(|g\rangle\) be the eigenstates of the \(S^z\) operator with eigenvalues \(\frac{1}{2}\) and \(-\frac{1}{2}\) where \(|e\rangle\) and \(|g\rangle\) represent the excited and ground states of the atom. From the structure of the Hamiltonian we observe two conservation laws. One is

\[ \langle S^z \rangle^2 + \langle S^x \rangle^2 + \langle S^y \rangle^2 = \frac{3}{4} \]  

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and another which conserves total excitation in the field-atom system

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Due to the conservation laws (5.2) and (5.3) the JCM which is infinite dimensional reduces to a two dimensional problem. The Hamiltonian matrix has a block diagonal form consisting of many (2 x 2) matrices along the main diagonal. Hence there are an
infinite set of uncoupled two state Schrödinger equations, each pair identified by the number of photons in the ground state. For a fixed $n$ the state vector is given by

$$|\psi(t)\rangle = C_n^n|n,e\rangle + C_{n+1}^n|n+1,g\rangle \quad (5.7)$$

The Hamiltonian matrix for the pair of basis vectors is

$$
\begin{pmatrix}
(n+1)\hbar\omega - \hbar\omega_e/2 & h\sqrt{n+1} \\
\hbar\sqrt{n+1} & nh\omega + \hbar\omega_e/2
\end{pmatrix}
$$

(5.8)

From the diagonalization of $H$ the whole dynamics of JCM is studied. The eigenvalues and eigenfunctions of $H$, given by $H|\psi_n^\pm\rangle = \hbar\omega_n^\pm|\psi_n^\pm\rangle$ are defined from

$$H|\psi_n^\pm\rangle = \left(\left(n + \frac{1}{2}\right)\hbar\omega \pm \frac{\hbar\Omega_{n\Delta}}{2}\right)|\psi_n^\pm\rangle \quad (5.9)$$

where

$$\Omega_{n\Delta}^2 = 4g^2(n+1) + \Delta^2 \quad {\text{and}}\quad \Delta = \omega_e - \omega \quad (5.10)$$

and

$$|\psi_n^\pm\rangle = \begin{pmatrix}
\cos \theta_n \\
-\sin \theta_n
\end{pmatrix}|n+1,g\rangle + \begin{pmatrix}
\sin \theta_n \\
\cos \theta_n
\end{pmatrix}|n,e\rangle \quad (5.11)$$

where $n = 0,1,2,\ldots$ and $\theta_n$ is defined from

$$\tan \theta_n = \frac{2g\sqrt{n+1}}{(\Omega_{n\Delta} - \Delta)} \quad (5.12)$$

At exact resonance, when the transition frequency is equal to the frequency of the radiation mode, i.e., $\Delta = 0$ and at $n = 0$, $H_n A = 2g$, the Rabi splitting frequency. Here $2g$ is called the single photon Rabi frequency and $Q_n k$ is called the generalized Rabi frequency. The evolution operator is found to be

$$\exp[-itH/\hbar] = \exp[-it(\omega - \omega_e)/2]|0,g\rangle\langle 0,g| + \sum_{n=0}^{\infty} \exp(-i\omega_n^+ t)|\psi_n^+\rangle\langle \psi_n^+|$$

$$\quad + \sum_{n=0}^{\infty} \exp(-i\omega_n^- t)|\psi_n^-\rangle\langle \psi_n^-| \quad (5.13)$$
If the atom is initially in the excited state and the radiation field is in the Fock state, with \( n \) photons then the state of the system at any time \( t \) is given by

\[
|\psi(t)\rangle = \exp\left[-\frac{iHt}{\hbar}\right]|n, e\rangle \\
= A_{n,e}|n, e\rangle + B_{n,e}|n + 1, g\rangle
\]  \hspace{1cm} (5.14)

where

\[
A_{n,e} = \sin^2 \theta_n \exp(-i\omega_n^+ t) + \cos^2 \theta_n \exp(-i\omega_n^- t) \\
B_{n,e} = \cos \theta_n \sin \theta_n [\exp(-i\omega_n^+ t) - \exp(-i\omega_n^- t)] .
\]  \hspace{1cm} (5.15)

The probability that the atom emits a photon into the cavity mode and goes to the ground state is

\[
P_g(t) = P_{n,e} \rightarrow P_{n+1,g} = \frac{4g^2(n + 1)}{(\Omega_n^+)^2} \sin^2 \frac{\Omega_{n,\Delta} t}{2}.
\]

Eq.(5.16) predicts sinusoidal Rabi oscillations much like the classical case. However, even when the field is initially in vacuum state, i.e., \( n = 0 \), the probability exists. This is a purely quantum feature of periodic reversible spontaneous emission. (Probability would be zero for zero classical field amplitude). The vacuum fluctuations stimulate spontaneous emission from the atom which is followed by reabsorption and so on.

If the radiation mode is initially in a coherent state the inversion of the atom is given by

\[
\langle S^z \rangle = \sum_{n=0}^{\infty} \frac{e^{-\bar{n} n}}{n!} \cos 2g\sqrt{n}t .
\]  \hspace{1cm} (5.17)

Numerical evaluation of the infinite sum leads to the collapse and revival phenomena. If the atom is in the ground state and the field is coherent initially, then the expectation value of the energy of the atom oscillates sinusoidally initially and then decays rapidly to a constant value even though the Hamiltonian describing the system is Hermitian (no damping). Following the decay are periodic revivals to large amplitudes on a much
larger time scale. On a still larger time scale neighbouring revivals begin to overlap and the regions of overlap gradually include more distant revivals causing the envelope of atomic inversion signal to appear irregular and noisy. Narozhny et al [12,13] gave analytic formulas for the collapse function, the revival period and the amplitude of the revival envelope. Recently with the success of the one-atom maser the collapse and revival phenomenon has been observed [14].

5.2 The Jaynes-Cummings Model with Continuous Pumping

In this model in addition to the atom interacting with a single cavity mode, it is driven continuously by an external field of frequency $U \gg L$ as shown in Figure 5.1. There are two processes connecting the two states of the atom. One is the single photon process occurring due to the interaction with the cavity mode. The other is the direct coherent excitation by the external field. The total Hamiltonian for this system can be written as

$$H = h u > o S^z - f h u > f b + h g (S^+ b + S^- f) + h a (S^+ \exp (-i u > t) - f S^- \exp (i u > t)) .$$  \hspace{1cm} (5.18)

where $a$ is the Rabi frequency of the external pumping field and $g$ is the single photon Rabi frequency.

Let $<t>$ be the state of the system at time $t$. The dynamics of the system is governed by the Schrodinger equation

$$i \hbar \frac{\partial |\phi\rangle}{\partial t} = H |\phi\rangle \hspace{1cm} (5.19)$$

To get rid of the optical frequency in the Hamiltonian in Eq.(5.18) we go to a frame rotating with the frequency $U \gg L$ by the unitary transformation.

$$|\psi\rangle = U^\dagger |\phi\rangle \hspace{1cm} (5.20)$$

where

$$U^\dagger = \exp (i \omega_L (S^z + b^\dagger b) t) \hspace{1cm} (5.21)$$
Fig. 5.1: A two-level atom in a cavity which supports a quantized mode 6, as well as being pumped by an external field a.
Differentiating Eq.(5.20) w.r.t. time we get

$$\frac{\partial |\psi\rangle}{\partial t} = \frac{\partial U^\dagger}{\partial t} |\phi\rangle + U^\dagger \frac{\partial |\phi\rangle}{\partial t}. \tag{5.22}$$

Using Eqs. (5.19) and (5.21) in above equation

$$\frac{\partial |\psi\rangle}{\partial t} = i\omega_L (S^z + b^\dagger b) U^\dagger |\phi\rangle - \frac{iU^\dagger H |\phi\rangle}{\hbar}. \tag{5.23}$$

Using the unitarity property $UW = 1$ in Eq. (5.23),

$$\frac{\partial |\psi\rangle}{\partial t} = i\omega_L (S^z + b^\dagger b) |\psi\rangle - \frac{iU^\dagger H U}{\hbar} |\psi\rangle. \tag{5.24}$$

The second term in Eq.(5.24) is evaluated as follows.

$$U^\dagger H U = h\omega_o S^z + h\omega b^\dagger b + \exp (i\omega_L (S^z + b^\dagger b)t) \times \left[ \hbar g (S^+ b + S^- b^\dagger) + \hbar \alpha (S^+ \exp (-i\omega_L t) + S^- \exp (i\omega_L t)) \right] \times \exp (-i\omega_L (S^z + b^\dagger b)t). \tag{5.25}$$

Using the bosonic and atomic commutation relations Eq.(5.25) reduces to

$$U^\dagger H U = h\omega_o S^z + h\omega b^\dagger b + \hbar g [S^+ b + S^- b^\dagger] + \hbar \alpha [S^+ + S^-]. \tag{5.26}$$

Substituting (5.26) in Eq.(5.24) Schrodinger equation in the rotating frame is obtained as

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \mathcal{H} |\psi\rangle \tag{5.27}$$

where the effective Hamiltonian in the rotating frame is given by

$$\mathcal{H} = \hbar (\omega_o - \omega_L) S^z + h(\omega - \omega_L) b^\dagger b + 2\hbar \alpha S^x + g\hbar (S^+ b + S^- b^\dagger). \tag{5.28}$$

In order to understand the basic features of the JCM with continuous pumping, we first consider the case of exact resonance $u_o = u_i = u$. The Hamiltonian (5.28) reduces to

$$\mathcal{H} = 2\hbar \alpha S^x + g\hbar (S^+ b + S^- b^\dagger). \tag{5.29}$$
Because of the structure of the term corresponding to the coherent pump it is useful to work in a representation in which \( S^x \) is diagonal. Let \(|±\rangle\) be the eigenstates of the \( S^x \) operator with eigenvalues \( ±\). The two sets of eigenstates are related by

\[
|+\rangle = \frac{|e\rangle + |g\rangle}{\sqrt{2}} , \quad |-\rangle = \frac{|e\rangle - |g\rangle}{\sqrt{2}} ; \quad S^x |±\rangle = \pm \frac{1}{2} |±\rangle .
\] (5.30)

The state vector at time \( t \) can be expressed in terms of the complete set of states as

\[
|\psi(t)\rangle = \sum_{n=0}^{\infty} C_n^- |-, n\rangle + \sum_{n=0}^{\infty} C_n^+ |+, n\rangle
\] (5.31)

where \( C_n^\pm \) are the complex amplitudes. Using the Schrodinger equation in (5.27),

\[
i \left[ \sum_{n=0}^{\infty} \hat{C}_n^- |-, n\rangle + \sum_{n=0}^{\infty} \hat{C}_n^+ |+, n\rangle \right] = -\alpha \left[ \sum_{n=0}^{\infty} C_n^- |-, n\rangle - \sum_{n=0}^{\infty} C_n^+ |+, n\rangle \right] \\
- \frac{g}{2} \left[ \sum_{n=0}^{\infty} C_n^- \sqrt{n} - \sum_{n=0}^{\infty} C_n^+ \sqrt{n} \right] \left[ |-, n-1\rangle + |+, n-1\rangle \right] \\
- \frac{g}{2} \left[ \sum_{n=0}^{\infty} C_n^- \sqrt{n+1} + \sum_{n=0}^{\infty} C_n^+ \sqrt{n+1} \right] \left[ |-, n+1\rangle - |+, n+1\rangle \right]
\] (5.32)

Taking the scalar product with \((-|p\rangle\) and \((+\langle p|\) on both sides Eq. (5.32), the equation of motion for the coefficients \( C_n^\pm \) are found to be

\[
i \dot{C}_n^- = -\alpha C_n^- - \frac{g}{2} (C_{p-1}^- + C_{p-1}^+) \sqrt{p} + \frac{g}{2} (C_{p+1}^- - C_{p+1}^+) \sqrt{p + 1}
\]

\[
i \dot{C}_n^+ = \alpha C_n^+ + \frac{g}{2} (C_{p-1}^- + C_{p-1}^+) \sqrt{p} + \frac{g}{2} (C_{p+1}^- - C_{p+1}^+) \sqrt{p + 1}
\] (5.33)

where \( p = 0, 1, 2, \cdots \infty \).

Unlike the standard JCM where the cavity mode is prepared in a coherent state initially, in our model we assume the cavity field to be in vacuum. The atom is assumed to be in the ground state \(|2\rangle\). The initial values of the \( C^\prime \)'s are deduced as follows:

Initially, at time \( t = 0 \) the state of the system is

\[
|\psi(0)\rangle = C_o^- |-, 0\rangle + C_o^+ |+, 0\rangle
\] (5.34)
The probability of the system to be in the ground state \(|2,0\rangle\) initially is equal to 1. Taking scalar product \((2,0|\) on both sides of Eq.(5.34),

\[
C_I - C_J = \sqrt{2} .
\]  (5.35)

Similarly from the stipulation that probability of the system to be in the excited state initially is zero, we get

\[
C_+ + C_J = 0 .
\]  (5.36)

Solving the simultaneous equations (5.35) and (5.36) the initial conditions on C's are found to be \(C^* = 4, C** = A^\dagger\) with all other C's zero. The coupled equations are solved using the Fehlberg's Runge Kutta method [15] for various values of the external field amplitudes, \(a\), relative to the atom-field mode coupling.

The infinite number of photons present in the sum in Eq.(5.31) indicates that the convergence of the equations in (5.33) has to be tested for each \(a\). Convergence is indicated by stable, unchanging values of the coefficients C's. The number of photons needed in the infinite sum in Eq.(5.31) for convergence is roughly estimated from the photon number distribution of the coherent external held. The coherent state has a Poissonian distribution with Poisson coefficients \(\lambda\). For small \(n\) values \(a^n\) dominates over \(\sqrt{n!}\), till the mean photon number \(n = |a|^2\), where the distribution increases to a maximum. After the mean photon number is reached, the factorial in the denominator starts influencing the coefficient \(\sqrt{n}\) and the distribution falls off till any increase in \(n\) does not influence it. The photon number \(n\) around this point is the convergency limit. For example for \(a = 3.0\), \(n\) is taken from 0 to \(~~ 12 — 15\) for convergence.

We study the dynamics of the photon statistics of the cavity mode and the atomic excitation as in the standard JCM. We specifically present results for the evolution of the photon number \((\delta\sigma)\) and the atomic population \(P_c(t)\), \(the\) probability that the atom is
in the excited state). In terms of the complex $C$'s defined in Eq. (5.31) these are

\begin{align}
(b^\dagger b) &= \sum_{n=0}^{\infty} n \left\{ |C_n^- (t)|^2 + |C_n^+ (t)|^2 \right\} \\
P_e(t) &= \frac{1}{2} \sum_{n=0}^{\infty} \left| C_n^+ (t) + C_n^- (t) \right|^2.
\end{align}

(5.37)

The behavior of $(bH)$ and $P_e(t)$ for the values $\alpha = 1.0$ and $\alpha = 3.0$ when $g = 1.0$ (this implies that time is scaled as $gt$) is shown in Figures. 5.2(a), (b) and 5.3(a), (b). For $\alpha = 1.0$, there are no pronounced collapse and revivals. There are modulated sinusoidal oscillations. For the larger value of $\alpha = 3.0$, $P_e(t)$ exhibits collapses and revivals like in the standard JCM where for increasing $\alpha$ the Rabi oscillations are affected by the Poisson distribution of photon numbers. Even though the cavity mode was initially in vacuum state, the atomic excitation probability exhibits the collapse and revival phenomena. The field mode initially with zero photons gains energy, becomes finite and oscillates with time even as the atom also exhibits modulated Rabi oscillations. The transfer of energy between the coherent laser field and the cavity mode is mediated by the atom due to simultaneous excitation pathways of the two fields. A pictorial comparison of the standard JCM with the JCM with external pumping is given in Figure 5.4. In the next section we explain the origin of such collapses and revivals in the continuously pumped model. Another interesting feature is that the number of photons in the cavity increases as the intensity of the pump increases.

Note that the usual JCM is characterized by the Hamiltonian $H$ obtained from (5.29) by setting $\alpha = 0$

\[ \hat{H} = \hbar g (S^+ b + S^- b^\dagger). \]

(5.38)

The initial conditions are now different. The atom may be in any state but the field is say in a coherent state $|\beta\rangle$. Thus the initial state for the atom-field system is

\[ |\psi(0)\rangle = |\psi_A(0)\rangle |\beta\rangle \]

(5.39)
Fig. 5.2: Dynamical results for the JCM with continuous pumping for $a = 1.0$

a) Mean number of photons in the cavity mode, $(\langle b b^\dagger \rangle)$ as a function of time

b) $P_e(t)$ (Probability that the atom is in the excited state) as a function of time.
Fig. 5.3: Dynamical results for the JCM with continuous pumping for a 3.0a) Same as in Fig. 5.2(a). b) Same as in Fig. 5.2(b).
Fig. 5.4: The eigenstates of the standard JCM and the JCM with external pumping are depicted in a) and b) respectively.
and the time dependent wave function is

$$|\psi(t)\rangle = \exp\left(-\frac{it\mathcal{H}}{\hbar}\right) |\psi_0(0)\rangle |\beta\rangle. \quad (5.40)$$

### 5.3 Analytical Results

In this section we use unitary transformations to obtain some analytical results. We rewrite the Hamiltonian (5.29) in the form

$$\mathcal{H} = \hbar g \left[ S^+ \left( b + \frac{\alpha}{g} \right) + S^- \left( b^\dagger + \frac{\alpha^*}{g} \right) \right]. \quad (5.41)$$

Introducing the displacement operator $D(0)$ defined by $D(0) = \exp (\beta b^\dagger - /3 b)$, $D\sqrt{3}bD(3) = 6 + r^2$, the Hamiltonian (5.41) reduces to

$$\mathcal{H} = \hbar g D^\dagger(\beta)(S^+ b + S^- b^\dagger) D(\beta) , \quad \beta = \frac{\alpha}{a}. \quad (5.42)$$

The time evolution of the wave function is then given by

$$|\psi(t)\rangle = \exp\left(-\frac{it\mathcal{H}}{\hbar}\right) |\psi(0)\rangle$$

$$= \exp\left(-\frac{it}{\hbar} D^\dagger(\beta)\mathcal{H}D(\beta)\right) |\psi(0)\rangle \quad (5.43)$$

where we have used (5.38).

Expanding the exponential in (5.43) gives

$$|\psi(t)\rangle = \left[ 1 - \frac{it}{\hbar} D^\dagger(\beta)\mathcal{H}D(\beta) + \left(\frac{it}{\hbar}\right)^2 (D^\dagger(\beta)\mathcal{H}D(\beta))^2 + \cdots \right] |\psi(0)\rangle. \quad (5.44)$$

Using the unitary property of displacement operator i.e., $D^\dagger(\beta)D(\beta) = 1$, Eq.(5.44) becomes

$$|\psi(t)\rangle = D^\dagger(\beta) \left[ 1 - \frac{it}{\hbar} \mathcal{H} + \left(\frac{it}{\hbar}\right)^2 \mathcal{H}^2 + \cdots \right] D(\beta) |\psi(0)\rangle. \quad (5.45)$$

Hence

$$|\psi(t)\rangle = D^\dagger(\beta) \exp\left(-\frac{it\mathcal{H}}{\hbar}\right) D(\beta) |\psi(0)\rangle \quad (5.46)$$
We write the initial condition in the form
\[
|\psi(0)\rangle = |\psi_A(0)\rangle |0\rangle .
\] (5.47)

Here \( |0\rangle \) is the vacuum state of the field and \( |I/M(0)\rangle \) is the initial state of the atom. Note that the displacement operator acting on the vacuum yields the coherent state \( |\boldsymbol{\gamma}\rangle \) of the field i.e.,
\[
D(\beta)|0\rangle = |\beta\rangle .
\] (5.48)

Thus (5.46) becomes
\[
|\psi(t)\rangle = D^\dagger(\beta) \exp \left( \frac{-i\hat{\mathcal{H}}}{\hbar} \right) |\psi_A(0)\rangle |\beta\rangle .
\] (5.49)

To understand the dynamics of the system we use the density operator formalism. The state vector in Eq.(5.49) completely describes the state of the system of the interacting atom and cavity mode. Such a state is called a pure state. The density operator, \( \rho \), for such a pure state is defined as
\[
|\beta\rangle = W |I\rangle .
\] (5-50)

Substituting the state vector from Eq.(5.49) the density operator of the coupled system of the atom and the radiation field is given by
\[
\rho = D^\dagger(\beta) \exp \left( \frac{-i\hat{\mathcal{H}}}{\hbar} \right) |\psi_A(0)\rangle \langle \beta| |\psi_A(0)\rangle \langle \psi_A(0)| \exp \left( \frac{i\hat{\mathcal{H}}}{\hbar} \right) D(\beta) .
\] (5.51)

The reduced density matrix \( \rho_A \) for the atom is obtained by taking a trace over the field variables as
\[
\rho_A(t) = \text{Tr}_F D^\dagger(\beta) \exp \left( \frac{-i\hat{\mathcal{H}}}{\hbar} \right) |\psi_A(0)\rangle \langle \beta| |\psi_A(0)\rangle \langle \psi_A(0)| \exp \left( \frac{i\hat{\mathcal{H}}}{\hbar} \right) D(\beta)
= \text{Tr}_F \exp \left( \frac{-i\hat{\mathcal{H}}}{\hbar} \right) |\psi_A(0)\rangle \langle \beta| |\psi_A(0)\rangle \langle \psi_A(0)| \exp \left( \frac{i\hat{\mathcal{H}}}{\hbar} \right) .
\] (5.52)

In obtaining Eq.(5.52) we used the cyclic property of the trace which states that \( \text{Tr}[ABC] = \text{Tr}[BCA] \) where \( A, B, C \) are represented as matrices in matrix notation and
the unitary property of displacement operator. Note that the right hand side of (5.52) is nothing but the solution (5.40) for the standard JCM. Thus we have shown that

$$p_A(t) = P_A(t)$$  \hspace{1cm} (5.53)

i.e., the dynamical properties of the atomic system for a continuously pumped JCM are identical to the dynamical properties of the atomic system for the standard JCM in which the coherent field appears in the initial conditions. This explicit connection explains the existence of the collapse and revival of the Rabi oscillations in Figure 5.3.

We next examine the reduced density matrix $\rho_r$ for the field mode which is obtained by tracing over the atomic variables in Eq.(5.51) as

$$\rho_F(t) = Tr_A D^\dagger(\beta) \exp \left( -\frac{it\mathcal{H}}{\hbar} \right) |\psi_A(0)\rangle \langle \beta | (\psi_A(0)) | \exp \left( \frac{it\mathcal{H}}{\hbar} \right) D(\beta)$$

$$= D^\dagger(\beta) Tr_A \left[ \exp \left( -\frac{it\mathcal{H}}{\hbar} \right) |\psi_A(0)\rangle \langle \beta | (\psi_A(0)) | \exp \left( \frac{it\mathcal{H}}{\hbar} \right) \right] D(\beta) .$$  \hspace{1cm} (5.54)

Note further that the term in the square bracket in (5.54) is just the field density matrix for the standard JCM and hence

$$\rho_F(t) = D^\dagger(\beta) \tilde{\rho}_F(t) D(\beta) .$$  \hspace{1cm} (5.55)

This relation indicates that the moments of the field variables are very simply related.

For example the mean photon number of the cavity mode is calculated. In the density matrix formalism the expectation value of an operator is defined as $\langle A \rangle = Tr p A$ . Using Eq.(5.55) the mean number of photons in cavity mode is found to be

$$\langle b^\dagger(t) b(t) \rangle = Tr D^\dagger(\beta) \tilde{\rho}_F(t) D(\beta) b^\dagger b$$

$$= Tr \tilde{\rho}_F(t) D(\beta) b^\dagger b D^\dagger(\beta)$$

$$= Tr \tilde{\rho}_F(b^\dagger - \beta^*)(b - \beta)$$

$$= \langle (b^\dagger - \beta^*)(b - \beta) \rangle .$$  \hspace{1cm} (5.56)
The results obtained by using the solution of the standard JCM model and the equations (5.53) and (5.56) agree with the ones calculated in Section 5.2 from the direct numerical integration of the Schrödinger equation.

Hence results in Eq. (5.53) and Eq. (5.56) indicate that in the cavity QED experiments which need the cavity mode to be in the coherent state initially, could use an external coherent laser field to prepare the cavity mode which initially is in a vacuum state, to be in a coherent state. The situation would be not very different from the case where the cavity mode was initially in a coherent state which is a difficult proposition. The atomic properties will be identical at all times whereas field properties will be related in a simple manner. Our model can also be viewed as an exactly soluble example of the two-channel JCM.

5.4 Non-zero Detuning

In this section we consider the effect of detuning the cavity and the external field from the atomic resonance i.e., \( U?L \neq W \nu \). We find that the results proved in the Section 5.3 also hold if the external field is on resonance with the cavity field i.e., when \( LJ = u^\wedge \).

We define \( Ai \) as the detuning between the atom and the external pump i.e., \( Ai = \nu \omega - LJ \) and \( A2 \) as the detuning between the cavity and the external pump i.e. \( A2 = w - UJI \). The general Hamiltonian (5.28) is thus rewritten as

\[
\mathcal{H} = \hbar \Delta_1 S^z + \hbar \Delta_2 b^\dagger b + 2\hbar \alpha S^x + \hbar g(S^+ b + S^- b\dagger) . \tag{5.57}
\]

Working in the \( S^x \) basis as before, the state vector at time \( t \) can be expressed in terms of the complete set of states as

\[
|\psi(t)\rangle = \sum_{n=0}^{\infty} D^-_{n}|-, n\rangle + \sum_{n=0}^{\infty} D^+_{n}|+, n\rangle \tag{5.58}
\]

where the complex coefficients \( D \)'s are determined from the Schrödinger equation (5.19).
The equations of motion for the coefficients D's are found to be

\[
i \dot{D}_p^- = \frac{1}{2} \Delta_1 D_p^+ + (p \Delta_2 - \alpha) D_p^- - \frac{g}{2} (D_{p-1}^- + D_{p+1}^+) \sqrt{p} + \frac{g}{2} (D_{p+1}^+ - D_{p-1}^-) \sqrt{p + 1}
\]

\[
i \dot{D}_p^+ = \frac{1}{2} \Delta_1 D_p^- + (p \Delta_2 + \alpha) D_p^+ + \frac{g}{2} (D_{p-1}^+ + D_{p+1}^-) \sqrt{p} + \frac{g}{2} (D_{p+1}^- - D_{p-1}^+) \sqrt{p + 1}
\]

(5.59)

where \( p = 0, 1, 2 \ldots \infty \).

As before assuming the atom to be in the ground state \( |2\rangle \) and the cavity field to be in vacuum, the initial conditions on \( V_s \) are \( D^+ = 4^+, D^- = -4^- \) with all other D's zero. The coupled equations are solved using Fehlberg's Runge Kutta method. The dynamics of the system is studied for different values of the external field and different detunings relative to the atom field coupling \( (g = 1.0) \). As before we present the results for the evolution of the photon number \( (b^* b) \) and \( P_e(t) \) (the probability that the atom is in the excited state).

It is observed that \( A_2 \), the detuning between the cavity and the external field alone affects the behavior of \( (6^6) \) and \( P_e(t) \). For instance, for \( A_2 = 0.2 \), \( (6^6) \) and \( P_e(t) \) behave identically for any value of \( A_1 \) like 0.1, 0.2, 0.3 etc. For the case where \( a = 3.0 \) the behavior of \( (6^6) \) and \( P_e(t) \) is shown in Figure 5.5 for the representative set of the detunings i.e., \( A_i = 0.0, A_2 = 0.2 \). The mean photon number \( (6^6) \) reaches a maximum 18.63 which is very much less than the maximum of the exact resonance case. Thus the amount by which the field is detuned from the cavity affects the dynamics of the system in a quantitative way. In Figure 5.6 we give the dynamical behavior of the field and the atom when both the external field and the cavity field are detuned from the atomic transition.

In conclusion we have studied the dynamical properties of an example of the two-channel JCM where the atom is driven continuously by an external laser field in addition
Fig. 5.5: Dynamical results for the JCM with continuous pumping for $a = 3.0$. Now the external field is detuned $A2 = u - u|i = 0.2$, $Ai = u_o - UJI = 0.0$ a) Mean number of photons in the cavity mode, $\langle \delta'^6 \rangle$ as a function of time $6) P_e(t)$ (Probability that the atom is in the excited state) as a function of time.
Fig. 5.6: Same as in Fig. 5.5 but now all transition frequencies are different i.e. $UL^u > o \; \ell \; w$. $A_1 = 1.0$, $A_2 = 1.0$
to being coupled to a cavity mode. We compare the standard JCM with our model in the
resonance case and establish a one to one correspondence between the results for
the JCM with continuous pumping and the standard JCM where the field enters via the
initial conditions. The situation however becomes more complex if the external field is
detuned from the cavity field.

In a recent work P. L. Knight and coworkers [16] discussed the dynamics of the JCM
with external pumping but with the cavity mode initially in a coherent state and the
atom in the excited state. They discover collapse and revivals ("super revivals") which
appear at time scales larger than their counterparts in the standard JCM. They also
report sub-Poissonian statistics for certain interaction times of the external field and the
atoms.

Our model dispels the difficulty of the micromaser experiments involved in studying
the fundamental atom and field interaction. There need be no ambiguity in the photon
statistics of the cavity mode. In the way suggested by our model the external pump
can prepare the cavity mode initially in vacuum state, to be in the coherent state. The
complexity of the external field entering into the cavity can be dealt with as in Kimble's
experiment [3] which uses an extra field to study the JCM.
REFERENCES


