We observe that the $\Gamma$-semigroup $T(A,B)$ is regular. In this chapter we study regular $\Gamma$-semigroups in general. In section one we give some examples of regular $\Gamma$-semigroups and prove a representation theorem. In section two we study some properties of regular $\Gamma$-semigroups. In the third section $\Gamma$-group congruences on a regular $\Gamma$-semigroup are discussed. In the last section the idea of idempotent separating congruence on a regular $\Gamma$-semigroup is introduced and we characterize the maximum idempotent separating congruence on a regular $\Gamma$-semigroup.

1. EXAMPLES AND A REPRESENTATION OF REGULAR $\Gamma$-SEMIGROUPS

A $\Gamma$-semigroup $(S, \Gamma)$ is said to be regular if for every element $a \in S$ we have $a \in a \Gamma S \Gamma a$ where $a \Gamma S \Gamma a = \{a \alpha \beta a : b \in S \text{ and } \alpha, \beta \in \Gamma \}$. An element $b \in S$ is said to be an $(\alpha, \beta)$-inverse of $a$ if $a = a \alpha \beta a$ and $b = b \beta a \alpha b$ ($\alpha, \beta \in \Gamma$) and in this case we write $b \in \mathcal{V}_{\alpha}^\beta(a)$. Consideration of several examples serves to bring the above idea into focus.

EXAMPLE 1.1. Let $S$ be the set of all $3 \times 2$ matrices over the
field of rational numbers and \( \Gamma \) be the set of all \( 2 \times 3 \) matrices over the ring of integers. Let \( A, B \in S \) and \( D \in \Gamma \). Then with respect to usual matrix product \( ADB \in S \) and \( (S, \Gamma) \) is a \( \Gamma \)-semigroup. We show that \( (S, \Gamma) \) is a regular \( \Gamma \)-semigroup.

Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S. \)

**Case (1).** Let \( ad - bc \neq 0 \). Then

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ 2(ad-bc) & 2(ad-bc) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

**Case (2).** \( af - be \neq 0 \). Then

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} f & -b \\ 2(af-be) & 2(af-be) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

**Case (3).** \( cf - de \neq 0 \). Then

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} f & -d \\ 2(cf-de) & 2(cf-de) \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
Case (4). When the submatrices are singular. Then

either \( ad-bc = 0 \) or \( ad-be = 0 \)

or \( cf-de = 0 \) or \( af-be = 0 \)

If all the elements of \( A \) are 0 then the case is trivial.

Next we consider at least one of the elements of \( A \) is non-zero say \( a \neq 0 \).

Then

\[
\begin{pmatrix}
    a & b \\
    c & d \\
    e & f
\end{pmatrix}
\begin{pmatrix}
    2 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    1/2a & 0 \\
    0 & 0 \\
    0 & 0
\end{pmatrix}
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}

= 
\begin{pmatrix}
    a & b \\
    c & d \\
    e & f
\end{pmatrix}
\begin{pmatrix}
    a & b \\
    c & d \\
    e & f
\end{pmatrix}
\]

Thus \( A \) is regular. Hence \((S, \Gamma)\) is a regular \( \Gamma \)-semigroup.

EXAMPLE 1.2. Let \( A = \{1, 2, 3\} \) and \( B = \{4, 5\} \) be two nonempty sets.

Let \( S = \{x, y, z\}, \Gamma = \{\alpha, \beta, \gamma, \delta, \theta, \phi\} \) where \( x, y, z \) are mappings from the set \( A \) to the set \( B \) and \( \alpha, \beta, \gamma, \delta, \theta, \phi \) are mappings from the set \( B \) to the set \( A \). They are defined by

\[
x = \begin{pmatrix}
    1 & 2 & 3 \\
    4 & 4 & 4
\end{pmatrix}, \quad y = \begin{pmatrix}
    1 & 2 & 3 \\
    5 & 5 & 5
\end{pmatrix}, \quad z = \begin{pmatrix}
    1 & 2 & 3 \\
    4 & 4 & 5
\end{pmatrix},
\]

\[
\alpha = \begin{pmatrix}
    4 & 5 \\
    1 & 1
\end{pmatrix}, \quad \beta = \begin{pmatrix}
    4 & 5 \\
    2 & 2
\end{pmatrix}, \quad \gamma = \begin{pmatrix}
    4 & 5 \\
    3 & 3
\end{pmatrix}, \quad \delta = \begin{pmatrix}
    4 & 5 \\
    2 & 1
\end{pmatrix}, \quad \theta = \begin{pmatrix}
    4 & 5 \\
    2 & 3
\end{pmatrix}, \quad \phi = \begin{pmatrix}
    4 & 5 \\
    1 & 3
\end{pmatrix}
\]

where \( x = \begin{pmatrix}
    1 & 2 & 3 \\
    4 & 4 & 4
\end{pmatrix} \) means that \( 1x = 4, 2x = 4, 3x = 4 \) and similarly others. The composition table for the \( \Gamma \)-semigroup \( S \) is given by the following table where \( xy \) means the usual composition
Here $xx\delta x = x$, $yvyvy = y$, $z\theta z\phi z = z$. Therefore $(S,\Gamma)$ is a regular $\Gamma$-semigroup. But we note that though $(S,\Gamma)$ is a regular $\Gamma$-semigroup, $S_\alpha$ is not a regular semigroup, for there is no element $p$ in $S$ for which $z\alpha p\alpha z = z$ holds. Also we note that $S_\Theta$ is a regular semigroup.

**Example 1.3.** The $\Gamma$-semigroup $(LR(M,N), LR(N,M))$ (Example 1.9 of Chapter 1) of all linear relations from a vector space $M$ into another vector space $N$ over the same field $F$ is a regular $\Gamma$-semigroup. Indeed for $A \in LR(M,N)$ we define $A^{-1} = \{(n,m) \in N \times M : (m,n) \in A\}$. We now show that $A^{-1} \in LR(N,M)$. 

- 52 -
For this let \((n_1, m_1) \in A^{-1}, (n_2, m_2) \in A^{-1}\) and \(a \in \Gamma\), then
\((m_1, n_1) \in A, (m_2, n_2) \in A\) and consequently \((m_1 + m_2, n_1 + n_2) \in A\) and \((am_1, an_1) \in A\). Hence \((n_1 + n_2, m_1 + m_2) \in A^{-1}\) and \((an_1, am_1) \in A^{-1}\) that is \(a(n_1, m_1) \in A^{-1}\). Therefore \(A^{-1} \subseteq LR(N, M)\). Now we prove that \(AA^{-1}A = A\). As \((m, n) \in A\) implies that \((m, n) \in A, (n, m) \in A^{-1}, (m, n) \in A\) we have \((m, n) \in AA^{-1}A\). That is \(A \subseteq AA^{-1}A\). Conversely \((m, n) \in AA^{-1}A\) implies that there exist \(m_1 \in M, n_1 \in N\) such that \((m, n) \in A, (n_1, m_1) \in A^{-1}\) and \((m_1, n) \in A\). So we have \((m, n_1) \in A, (-m_1, n) \in A\) and \((m_1, n) \in A\) implying \((m - m_1 + m_1, n_1 - n_1 + n) \in A\).

That is \((m, n) \in A\). Consequently \(AA^{-1}A \subseteq A\). Thus \(AA^{-1}A = A\).

This implies that \(A\) is regular. Thus the \(\Gamma\)-semigroup

\((LR(M, N), LR(N, M))\)

is a regular \(\Gamma\)-semigroup.

Let \((S, \Gamma, \mu)\) be a regular \(\Gamma\)-semigroup which we denote by \((S, \Gamma)\). Let us define \(* : S \times \Gamma \times S \to S\) by \((a, \alpha, b)^* = b\alpha a\) for all \(a, b \in S, \alpha \in \Gamma\) where \(b\alpha a = (b, a, a)\mu\). We now show that \((S, \Gamma, *)\) is also a regular \(\Gamma\)-semigroup. For this let \(a, b, c \in S\) and \(\alpha, \beta \in \Gamma\). Then

\((a\alpha \beta)^*, \beta, \alpha)^* = (b\alpha a, \beta, \alpha)^* = c\beta (b\alpha a) = (c\beta b) \alpha a = (a, \alpha, c\beta b)^* = (a, \alpha, (b, \beta, c)^*)^*\).

Thus \((S, \Gamma, *)\) is a \(\Gamma\)-semigroup. Now for any \(a \in S\), we have \(b \in S\) and \(\alpha, \beta \in \Gamma\) such that \(a = a\alpha \beta a\) since \((S, \Gamma)\) is a regular \(\Gamma\)-semigroup. Now

\(((a, \alpha, b)^*, \alpha, a)^* = (b\alpha a, \alpha, a)^* = a\alpha \beta a = a\).

So \(a\) is regular in \((S, \Gamma, *)\). Consequently \((S, \Gamma, *)\) is a regular \(\Gamma\)-semigroup. We
denote this $\Gamma'$-semigroup by $(S,\Gamma')$.

Let $(S,\Gamma')$ and $(S',\Gamma')$ be two regular $\Gamma'$-semigroups. Let us define $\mu : (S \times S') \times (\Gamma \times \Gamma') \times (S \times S') \to S \times S'$ by

$\mu((s,s'),(a,a'),(t,t')) = (s\alpha t, s'\alpha' t')$ for all $(s,s'), (t,t') \in S \times S'$ and $(a,a') \in \Gamma \times \Gamma'$. Then $(S \times S', \Gamma \times \Gamma', \mu)$ is a regular $\Gamma'$-semigroup. To prove this let us write

$\mu((s,s'),(a,a'),(t,t'))$ simply as $(s,s')(a,a')(t,t')$. Then for $(s,s'), (t,t'), (u,u') \in S \times S'$ and $(a,a'), (\beta,\beta') \in \Gamma \times \Gamma'$ we have

$[(s,s')(a,a')(t,t')](\beta,\beta')(u,u') = (s\alpha t, s'\alpha' t')(\beta,\beta')(u,u') = (s\alpha(t\beta u), s'\alpha'(t'\beta' u')) = (s,s')(a,a')(t\beta u, t'\beta' u')$.$\Gamma'$-semigroup. To prove that it is a regular $\Gamma'$-semigroup let $(a,a') \in S \times S'$. Since both $(S,\Gamma')$ and $(S',\Gamma')$ are regular $\Gamma'$-semigroups, we have $b \in S$, $b' \in S'$, $a, \beta \in \Gamma$, $a', \beta' \in \Gamma'$ such that $a = a\beta b a$ and $a' = a'\beta' b' a'$. Now we have $(a,a')$, $(b,b') \in S \times S'$, $(\alpha,\alpha'), (\beta,\beta') \in \Gamma \times \Gamma'$ and $(a,a')(\alpha,\alpha')(b,b')(\beta,\beta')(a,a') = (a\alpha b a, a'\alpha' b' a') = (a,a')$. Thus $(a,a')$ is a regular element. Consequently $(S \times S', \Gamma \times \Gamma', \mu)$ is a regular $\Gamma'$-semigroup. We denote this $\Gamma'$-semigroup simply by $(S,\Gamma') \times (S',\Gamma')$.

Let $S = \mathcal{P}^T(A,B)$ denote the set of all partial mappings including
the empty mapping from a nonempty set \( A \) into the set \( B \) and \( \Gamma = \emptyset T(B,A) \). Then it is easy to see that \( (\emptyset T(A,B), \emptyset T(B,A)) \)
is a regular \( \Gamma \)-semigroup if we define for \( f, g \in S \) and \( \alpha \in \Gamma \),
\[
fg = \{(a,b) \in A \times B: \text{there exists } b_1 \in B, a_1 \in A \text{ for which } (a,b_1) \in f, (b_1,a_1) \in \alpha \text{ and } (a_1,b) \in g\}.
\]
As \( (\emptyset T(A,B), \emptyset T(B,A)) \) is regular \( \Gamma \)-semigroup by our above discussion \( (\emptyset T(A,B), \emptyset T(B,A)) \) * is also a regular \( \Gamma \)-semigroup. Consequently \( (\emptyset T(A,B), \emptyset T(B,A)) \times (\emptyset T(A,B), \emptyset T(B,A)) * \) is a regular \( \Gamma \)-semigroup.

A faithful representation of regular semigroup was given by Lallement in [7]. Here we give a faithful representation of a regular \( \Gamma \)-semigroup.

**Theorem 1.1** [35] Let \( (S, \Gamma) \) be a regular \( \Gamma \)-semigroup. Let \( A = S \times \Gamma \) and \( B = S \). For each \( a \in S \) and each \( \theta \in \Gamma \) define
\[
\gamma_a = \{((x,\alpha), y) \in A \times B: y = a\alpha x \text{ and } (x,y) \in \mathcal{R}\},
\]
\[
\delta_a = \{((x,\beta), y) \in A \times B: y = x\beta a \text{ and } (x,y) \in \mathcal{R}\},
\]
\[
\lambda_\theta = \{(x, (x, \theta)) \in B \times A\}
\]
Then \( (\delta, \psi) : (S, \Gamma) \rightarrow (\emptyset T(A,B), \emptyset T(B,A)) \times (\emptyset T(A,B), \emptyset T(B,A)) * \)
defined by \( a\delta = (\delta_a, \gamma_a) \) and \( \theta\psi = (\lambda_\theta, \lambda_\theta) \) is a monomorphism.

**Proof.** Let \( a, b \in S \) and \( \theta \in \Gamma \) we prove that \( \delta_a \lambda_\theta \delta_b = \delta_{a\theta b} \).

For this let \( (x, \alpha) \in \text{dom}(\delta_a \lambda_\theta \delta_b) \). Then we have \( (x, x\alpha a) \in \mathcal{R} \)
and \((x^a, x^a \theta b) \in R\), which implies that \((x, x^a \theta b) \in R\) since \(R\) is an equivalence relation. Consequently \((x, a) \in \text{dom}(S_{a \theta b})\).

Conversely if \((x, a) \in \text{dom}(S_{a \theta b})\) then \((x, x^a \theta b) \in R\).

Consequently \(\bigcap S = x^a \theta b \bigcap S \subseteq x^a \bigcap S \subseteq x \bigcap S\). So we have \(\bigcap S = x^a \bigcap S\). Therefore \((x, x^a) \in R\), since \(R\) is an equivalence relation it now follows that \((x^a, x^a \theta b) \in R\).

Here \((x, a) \in \text{dom}(S_a \gamma \theta S_b)\). Therefore \(\text{dom}(S_a \gamma \theta S_b) = \text{dom}(S_{a \theta b})\). Also we have \((x, a) \in S_a \gamma \theta S_b = (x, a) \in S_{a \theta b}\) for all \((x, a) \in \text{dom}(S_a \gamma \theta S_b)\). Therefore \(S_a \gamma \theta S_b = S_{a \theta b}\). Next we prove that \(Y_a \gamma \theta Y_b = Y_b \theta \alpha a\). For this let \((x, a) \in \text{dom}(Y_a \gamma \theta Y_b)\), then \((x, a \alpha x) \in L\) and \((a \alpha x, b \theta a \alpha x) \in L\). So we have

\((x, b \theta a \alpha x) \in L\), since \(L\) is an equivalence relation. Consequently \((x, a) \in \text{dom}(Y_{b \theta \alpha a})\). Conversely, let \((x, a) \in \text{dom}(Y_{b \theta \alpha a})\) then \((x, b \theta a \alpha x) \in L\). Consequently \(S \bigcap x = S \bigcap b \theta a \alpha x \subseteq S \bigcap a \alpha x \subseteq S \bigcap x\).

So \(S \bigcap x = S \bigcap a \alpha x\) implying that \((x, a \alpha x) \in L\). Since \(L\) is an equivalence relation, we have \((a \alpha x, b \theta a \alpha x) \in L\). Hence

\((x, a) \in \text{dom}(Y_a \gamma \theta Y_b)\). Therefore \(\text{dom}(Y_{b \theta \alpha a}) = \text{dom}(Y_a \gamma \theta Y_b)\).

Thus \(Y_a \gamma \theta Y_b = Y_b \theta \alpha a\). Now \((a \theta b) \Phi = (S_{a \theta b}, Y_{a \theta b}) = (S_a \gamma \theta S_b, Y_b \gamma \theta Y_a) = (S_a \gamma \theta S_b, (Y_a \gamma \theta Y_b)^x) = (S_a \gamma \theta S_b, Y_a \gamma \theta Y_b) = a \theta b \psi \theta b\). Therefore \((\Phi, \Psi)\) is a homomorphism from \((S, \bigcap)\) into \((\Theta T(A, B), \Theta T(B, A)) \times (\Theta T(A, B), \Theta T(B, A))^x\). We now show that both \(\Phi\) and \(\Psi\) are one to one. For
this let $a_0 = b_0$ then $(S_a, Y_a) = (S_b, Y_b)$. Consequently we have $S_a = S_b$ and $Y_a = Y_b$. Let $a' \in V_\alpha(a)$ for some $\alpha, \beta \in \Gamma$. Then $(a' \beta a, a) \in \mathcal{L}$ so $((a' \beta a, \alpha), a) \in Y_a = Y_b$. Thus $a = b_0 a' \beta a$.

Again $(a \alpha a', a) \in \mathcal{R}$ so $((a \alpha a', \beta), a) \in S_a = S_b$. Therefore $a = a \alpha a' \beta b \in S \Gamma b$. Now for $b' \in V_\theta(b)$ for some $\theta, \mu \in \Gamma$.

$(b \theta b', b) \in \mathcal{R}$ so $((b \theta b', \mu), b) \in S_b = S_a$. Thus $b = b \theta b' \mu a \in S \Gamma a$.

Consequently we have $S \Gamma b = S \Gamma a$, so $(a, b) \in \mathcal{L}$. Hence there exist $u \in S$, $\gamma \in \Gamma$ such that $b = u \gamma a = u \gamma a \alpha a' \beta a = b_0 a' \beta a = a$.

Hence $\phi$ is one to one. Next let $\theta_1 \psi = \theta_2 \psi$ for $\theta_1, \theta_2 \in \Gamma$.

Then $(\lambda_{\theta_1}, \lambda_{\theta_1}) = (\lambda_{\theta_2}, \lambda_{\theta_2})$. Hence $x \lambda_{\theta_1} = x \lambda_{\theta_2}$ for all $x \in B$. Therefore we have $(x, \theta_1) = (x, \theta_2)$ implying $\theta_1 = \theta_2$.

Thus $\psi$ is one to one. Hence $(\phi, \psi)$ is a monomorphism from the regular $\Gamma$-semigroup $(S, \Gamma)$ into the regular $\Gamma$-semigroup $(\hat{\Theta}(A, B), \hat{\Theta}(B, A)) x (\hat{\Theta}(A, B), \hat{\Theta}(B, A))^*$. 

2. **SOME PROPERTIES OF REGULAR $\Gamma$-SEMIGROUPS**

In this section we study those regular $\Gamma$-semigroups in which any two $\alpha$-idempotents are $\alpha$-commutative. We also find out a necessary and sufficient condition for a regular $\Gamma$-semigroup to be simple.

**LEMMA 2.1.** Every $\alpha$-idempotent $e$ in a $\Gamma$-semigroup $(S, \Gamma)$ is
such that for any $a$ in $R_e$ we have $e a a = a$ and for any $b$ in $L_e$ we have $b a e = b$ where $R_e$ and $L_e$ are respectively $\mathcal{Q}$-class and $\mathcal{L}$-class containing $e$.

**Proof.** Let $a \in R_e$. Then $a \in e \mathcal{L} S$. So $a = e s \beta$ for some $\beta \in \mathcal{L}$ and $s \in S$. So $e a a = e \epsilon e \beta s = e \beta s = a$. Next let $b \in L_e$. Then $b \in S \mathcal{L} e$, so $b = t \gamma e$ for some $t \in S$ and $\gamma \in \mathcal{L}$. Therefore $b a e = t \gamma e \epsilon e = t \gamma e = b$.

**Theorem 2.2.** In a regular $\mathcal{L}$-semigroup $(S, \mathcal{L})$ the following conditions are equivalent.

(i) For all $a$ belonging to $S$, $|\mathcal{V}_\alpha(a)| = 1$ if $\mathcal{V}_\alpha(a) \neq \emptyset$.

(ii) If $e$ and $f$ are two $a$-idempotents then $e a f = f a e$ where $a \in \mathcal{L}$.

(iii) Each $\mathcal{Q}$-class and each $\mathcal{R}$-class of $(S, \mathcal{L})$ contains a unique $a$-idempotent for some $a \in \mathcal{L}$.

(iv) Each principal left $\mathcal{L}$-ideal and each principal right $\mathcal{L}$-ideal of $(S, \mathcal{L})$ contains a unique $a$-idempotent generator for some $a \in \mathcal{L}$.

**Proof.** Suppose (i) holds. Let $a \in \mathcal{V}_\beta^\gamma(e a f)$ for some $\beta, \gamma \in \mathcal{L}$. Then $e a f = (e a f) \beta a \gamma(e a f)$ and $a = a \gamma(e a f) \beta \ a$. Now 

$$(f \beta a \gamma e) a (f \beta a \gamma e) = f \beta(a \gamma e a f \beta a) \gamma e = f \beta a \gamma e.$$ 

Therefore $f \beta a \gamma e$ is an $a$-idempotent. Also $(e a f) a (f \beta a \gamma e) a (e a f) = (e a f) \beta a \gamma(e a f) = e a f$.
and $(f^a \beta e \gamma e) \alpha(a(eaf) \alpha(\beta \gamma e)) = f \beta (a(eaf \beta e) \gamma e) = f \beta \gamma e$. Hence $eaf \in V^\alpha(\beta \gamma e)$. But $f \beta \gamma e$ being an $\alpha$-idempotent, belongs to $V^\alpha(\beta \gamma e)$. Therefore $eaf = f \beta \gamma e$ by (i). Hence $eaf$ is an $\alpha$-idempotent. Also $fae$ is an $\alpha$-idempotent of $(S, \Gamma')$.

Indeed, if $b \in V^\beta(\gamma e)$, then $(\gamma e) \beta f(\gamma e) = \gamma e$ and $b \beta \gamma e \in b = b$.

Then $(e \beta \theta f) \alpha(e \beta \theta f) = e \beta (b \beta \gamma e \in b) \theta f = e \beta \theta f$. Thus $e \beta \theta f$ is an $\alpha$-idempotent. Also, $(\gamma e) \alpha(e \beta \theta f) \alpha(\gamma e) = (\gamma e) \beta \theta (\gamma e) = \gamma e$ and $(e \beta \theta f) \alpha(e \beta \theta f) \alpha(e \beta \theta f) = (e \beta \theta f) \alpha(e \beta \theta f) = e \beta \theta f$. Hence $fae \in V^\alpha(e \beta \theta f)$. But $e \beta \theta f$ being $\alpha$-idempotent belongs to $V^\alpha(e \beta \theta f)$. Hence $fae = e \beta \theta f$. Thus $fae$ is an $\alpha$-idempotent of $(S, \Gamma')$. Now $(eaf) \alpha(\beta \gamma e) \alpha(eaf) = (eaf) \alpha(fae)$ $= eaf$ and $(fae) \alpha(eaf) \alpha(fae) = (fae) \alpha(fae) = fae$. So $fae \in V^\alpha(fae)$.

But $eaf \in V^\alpha(fae)$. Therefore by (i) $eaf = fae$ which is (ii).

Thus (i) implies (ii).

Next let us assume (ii).

Since $(S, \Gamma')$ is a regular $\Gamma'$-semigroup, every $\mathcal{L}$-class contains atleast one $\alpha$-idempotent for some $\alpha \in \Gamma$, by Lemma I.5.1.

We now show that this $\alpha$-idempotent is unique. If $e$ and $f$ are $\mathcal{L}$-equivalent $\alpha$-idempotents, then by Lemma 2.1 $f = f \alpha e = eaf = e$.

Similarly if $e$ and $f$ are $\mathcal{R}$-equivalent $\alpha$-idempotents, then $f = eaf = f \alpha e = e$. Thus (ii) implies (iii).

Let us now assume (iii).
Let \((a)_1\) be any principal left \(\Gamma\)-ideal generated by \(a\) in \(S\). Then by (iii) \(L_a\), the \(\mathcal{L}\)-class containing \(a\), contains a unique \(\alpha\)-idempotent for atleast one \(\alpha \in \Gamma\). Let \(e\) be an \(\alpha\)-idempotent belonging to \(L_a\). Then \((a)_1 = (e)_1\). So each principal left \(\Gamma\)-ideal contains a unique \(\alpha\)-idempotent generator for atleast one \(\alpha \in \Gamma\). Similarly each principal right \(\Gamma\)-ideal contains a unique \(\alpha\)-idempotent generator for atleast one \(\alpha \in \Gamma\). Thus (iii) implies (iv).

Finally we assume (iv).

Let \(a', a'' \in V_{\beta}(a)\) for some \(\alpha, \beta \in \Gamma\) and \(a \in S\). Then
\[a = \alpha a' \beta a, \quad a' = \alpha' \beta a' a, \quad a = \alpha a'' \beta a \text{ and } a'' = \alpha'' \beta a'' a'.\]

Now \(\alpha a'\) and \(\alpha a''\) are \(\beta\)-idempotents of \((S, \Gamma)\) and
\[(a)_r = (\alpha a')_r = (\alpha a'')_r.\]
Consequently \(\alpha a' = \alpha a''\).

Similarly \(\alpha' \beta a\) and \(\alpha'' \beta a\) are \(\alpha\)-idempotents of \((S, \Gamma)\) and
\[(a)_1 = (\alpha' \beta a)_1 = (\alpha'' \beta a)_1.\]
Consequently \(\alpha' \beta a = \alpha'' \beta a\). So
\[a' = \alpha' \beta a a' = \alpha' \beta a a' = \alpha'' \beta a a'' = a''.\]
Thus (iv) implies (i).

This completes the proof of the theorem.

In [23] we call a regular \(\Gamma\)-semigroup which satisfies the conditions given in the above theorem, an inverse \(\Gamma\)-semigroup. We now give an example of inverse \(\Gamma\)-semigroup.

**DEFINITION**: Let \(A\) and \(B\) are two nonempty sets. Then any
subset of $A \times B$ is called a binary relation from the set $A$ into the set $B$. A binary relation $f$ from the set $A$ into the set $B$ is called a one to one partial mapping from the set $A$ into the set $B$ if the following conditions hold:

1. $(x, y) \in f$ and $(x, y') \in f$ imply that $y = y'$
2. $(x', y) \in f$ and $(x, y) \in f$ imply that $x = x'$.

**Example 2.1.** Let $S = \mathcal{P}(A, B)$ denote the set of all one to one partial mappings from the set $A$ into the set $B$ including the empty mapping and $\Gamma = \mathcal{P}(B, A)$ denote the set of all one to one partial mappings from the set $B$ into the set $A$. Let $f, g, h \in S$ and $\alpha, \beta, \gamma \in \Gamma$. We define

$$f \circ g = \{(a, b) \in A \times B : \text{there exist } a_1 \in A, b_1 \in B \text{ for which } (a, b_1) \in f, (b_1, a_1) \in \alpha, (a_1, b) \in g\}.$$ 

It can be shown easily that $f \circ g$ is a one to one partial mapping and $(f \circ g) \circ h = f \circ (g \circ h)$ for all $f, g, h \in S$ and $\alpha, \beta \in \Gamma$. Hence $(S, \circ)$ is a $\Gamma$-semigroup.

We now show that $(S, \circ)$ is a regular $\Gamma$-semigroup. Let $f \in S$ then $f : \text{dom } f \to \text{ran } f$ is one to one. Let $f^{-1} : \text{ran } f \to \text{dom } f$ be such that $ff^{-1} = 1_{\text{dom } f}$ and $f^{-1}f = 1_{\text{ran } f}$ where $1_{\text{dom } f}$ denotes the identity mapping on $\text{dom } f$ ($\text{dom } f = \text{domain of } f$, $\text{ran } f = \text{range of } f$). It is immediate that $f^{-1} \in \Gamma$. Now

$$f = ff^{-1}ff^{-1} \in f \Gamma S \Gamma f.$$ 

Thus $(S, \circ)$ is a regular $\Gamma$-semigroup.
Next we show that for any two \( \alpha \)-idempotents \( f, h \) of \((S, \Gamma)\) we have \( fah = haf \) where \( \alpha \in \Gamma \). If an element \( f \in S \) be an \( \alpha \)-idempotent then \( f\alpha = 1_X \) where \( X = \text{dom} f \) and conversely if \( f\alpha = 1_X \), where \( X = \text{dom} f \), then \( f \) is an \( \alpha \)-idempotent. Now if \( f, h \) are any two \( \alpha \)-idempotents in \((S, \Gamma)\). We prove that \( fah = haf \).

Now \( \alpha = 1_X \) where \( X = \text{dom} f \) and \( h\alpha = 1_Y \) where \( Y = \text{dom} h \). Then \( \text{dom}(fah) = \text{dom}(1_Xh) = (X \cap Y) 1_X^{-1} = X \cap Y \), \( \text{ran}(fah) = \text{ran}(1_Xh) = (X \cap Y) h \), \( \text{dom}(haf) = \text{dom}(1_Yf) = (Y \cap X) 1_Y^{-1} = Y \cap X = X \cap Y \) and \( \text{ran}(haf) = \text{ran}(1_Yf) = (Y \cap X) f = (X \cap Y) f \).

We now show that \((X \cap Y) f = (X \cap Y) h\). If possible let \((X \cap Y) f \neq (X \cap Y) h\). Then \((X \cap Y) f \neq (X \cap Y) h\). That is \((X \cap Y) 1_X \neq (X \cap Y) 1_Y \). Then \( X \cap Y \neq X \cap Y \), which is absurd. Hence \( fah \) and \( haf \) have same domain and range. Let \( \alpha \in X \cap Y \). Then \( a1_Y = a1_X \) implies that \( ah\alpha = a\alpha f \) that is \( ah = af \) (since \( \alpha \) is one to one). Then \((a1_X) h = (a1_Y) f\), that is \((a) fah = (a) haf \). Consequently \( fah = haf \). Thus the \( \Gamma \)-semigroup \((S, \Gamma)\) is an inverse \( \Gamma \)-semigroup.

**Theorem 2.3.** If in a regular \( \Gamma \)-semigroup \((S, \Gamma)\) no \( \alpha \)-idempotent has \((\alpha, \alpha) \) inverse other than itself then the \( \Gamma \)-semigroup \((S, \Gamma)\) is an inverse \( \Gamma \)-semigroup and conversely.
PROOF. Suppose that the given condition holds in \((S, \Gamma')\) We prove that \((S, \Gamma')\) is an inverse \(\Gamma\)-semigroup. Since \((S, \Gamma)\) is a regular \(\Gamma\)-semigroup, by Lemma 1.5.1, each \(L\)-class has at least one \(\alpha\)-idempotent for some \(\alpha \in \Gamma\). If possible let one of the \(L\)-classes contains two \(\alpha\)-idempotents \(e\) and \(f\). Then by Lemma 2.1 \(eaf = e\) and \(fae = f\). So we get \(eafae = e\) and \(faeaf = f\). Therefore \(e, f \in V^\alpha(e)\). By our assumption \(e = f\). So \((S, \Gamma)\) is an inverse \(\Gamma\)-semigroup. Converse is obvious.

**Lemma 2.4**. Let \((S, \Gamma)\) be a regular \(\Gamma\)-semigroup and \((T, \Gamma')\) be another \(\Gamma\)-semigroup. Let \((f, g)\) be a homomorphism from \((S, \Gamma)\) onto \((T, \Gamma')\). Let \(e'\) be an \(\alpha'\)-idempotent of \((T, \Gamma')\) then \(e'f^{-1}\) contains an idempotent of \((S, \Gamma)\).

**Proof**. Let \(a \in S\) be such that \(af = e' = e'\alpha'e'\), where \(\alpha' \in \Gamma'\). Let \(a \in \Gamma\) be such that \(ag = \alpha'\). Now let us consider the element \(aa\). As \((S, \Gamma)\) is a regular \(\Gamma\)-semigroup there exist \(b, \beta, \gamma \in S\) and \(\beta, \gamma \in \Gamma\) such that \((a\alpha\beta)\beta\gamma(a\alpha\beta) = a\alpha\beta\gamma(a\alpha\beta)\beta\gamma = b\). Now \((a\beta\gamma\alpha)(a\beta\gamma\alpha) = a\beta(b\alpha\gamma(a\alpha\beta)\gamma\alpha = a\beta\gamma\alpha\). So \(a\beta\gamma\alpha\) is an \(\alpha\)-idempotent in \((S, \Gamma)\). Also
\[
(a\beta\gamma\alpha)f = (af)(\beta\gamma)(bf)(\gamma)(af) = e'\alpha'e'(\beta\gamma)(bf)(\gamma)e'\alpha'e' = (a\alpha\beta)(\beta\gamma)(bf)(\gamma)(a\alpha\beta)f = (a\alpha\beta)(b\alpha\gamma(a\alpha\beta)\gamma)(a\alpha\beta)f = (a\alpha\beta)f = e'.
\]
Hence \(e'f^{-1}\) contains an idempotent of \((S, \Gamma)\).
LEMMA 2.5. Let \((S, \Gamma')\) be a regular \(\Gamma^\prime\)-semigroup and \((S', \Gamma'')\) be a \(\Gamma^\prime\)-semigroup. Let \((f, g)\) be a homomorphism from \((S, \Gamma')\) onto \((S', \Gamma'')\). Then \((S', \Gamma'')\) is also a regular \(\Gamma^\prime\)-semigroup.

PROOF. Let \(a^\prime = af\) be an element of \(S'\) where \(a \in S\). Since \((S, \Gamma')\) is a regular \(\Gamma^\prime\)-semigroup there exist \(b \in S\) and \(\alpha, \beta \in \Gamma\) such that \(a = \alpha a b \beta a\). Then \(a^\prime = af = (a \alpha a b \beta a) f = (af)(\alpha g)(bf)(\beta g)(af) = a^\prime (\alpha g)(bf)(\beta g) a^\prime\). Thus \(a^\prime\) is regular. Hence \((S', \Gamma'')\) is a regular \(\Gamma^\prime\)-semigroup.

THEOREM 2.6. Let \((S, \Gamma')\) be an inverse \(\Gamma^\prime\)-semigroup and \((S', \Gamma'')\) be another \(\Gamma^\prime\)-semigroup. Let \((f, g)\) be a homomorphism from \((S, \Gamma')\) onto \((S', \Gamma'')\). Then \((S', \Gamma'')\) is also an inverse \(\Gamma^\prime\)-semigroup.

PROOF. By Lemma 2.5 \((S', \Gamma'')\) is a regular \(\Gamma^\prime\)-semigroup. Now we show that for any two \(\alpha^\prime\)-idempotents \(e_1', e_2'\) of \((S', \Gamma'')\) we have \(e_1' \alpha^\prime e_2' = e_2' \alpha^\prime e_1'\). Since \((f, g)\) is onto homomorphism there exist (by Lemma 2.4) \(e_1, e_2 \in S\) and \(\alpha \in \Gamma\) such that \(e_1\) and \(e_2\) are \(\alpha\)-idempotents and \(e_1 f = e_1', \alpha g = \alpha'\) (\(i = 1, 2\)).

As \((S, \Gamma')\) is an inverse \(\Gamma^\prime\)-semigroup we have \(e_1 \alpha e_2 = e_2 \alpha e_1\).

Therefore \((e_1 \alpha e_2) f = (e_2 \alpha e_1) f\) that is \(e_1' \alpha' e_2' = e_2' \alpha' e_1'\)

Hence \((S', \Gamma'')\) is an inverse \(\Gamma^\prime\)-semigroup.
We define a \( \Gamma \)-semigroup to be simple if it has no proper \( \Gamma \)-ideal. Now we find out a necessary and sufficient con­dition for a regular \( \Gamma \)-semigroup to be simple.

**DEFINITION** Let \((S, \Gamma)\) be a regular \( \Gamma \)-semigroup and \( E \) be the set of all idempotents of \((S, \Gamma)\). If \( e, f \in E \) be such that \( eae = e, f \beta f = f \) where \( \alpha, \beta \in \Gamma \), then we define "\( \leq \)" on \( E \) by \( e \leq f \) if and only if \( e = e\alpha f = f\beta e \).

**LEMMA 2.7.** A \( \Gamma \)-semigroup \((S, \Gamma)\) is simple if and only if \( S = S \Gamma a \Gamma S \) for all \( a \in S \).

**PROOF.** If \((S, \Gamma)\) is a simple \( \Gamma \)-semigroup then obviously \( S = S \Gamma a \Gamma S \) for all \( a \in S \). Conversely let \( S \Gamma a \Gamma S = S \) for all \( a \in S \). If \( I \) is a \( \Gamma \)-ideal of \((S, \Gamma)\) and \( a \in I \) then \( S \Gamma a \Gamma S \subseteq I \) that is \( S \subseteq I \). Consequently \( S = I \). Thus \((S, \Gamma)\) is simple \( \Gamma \)-semigroup.

**THEOREM 2.8.** A regular \( \Gamma \)-semigroup \((S, \Gamma)\) is simple if and only if for any two idempotents \( e \beta e = e \) and \( f \delta f = f \) of \((S, \Gamma)\), there exists an element \( a \) in \( S \) with \( a' \in \mathcal{V}_e(a) \) such that \( a \gamma a' = e \) and \( a' \beta a \leq f \).

**PROOF.** Suppose that the given condition is satisfied. Let \( a, b \) be any two elements of \( S \). Let \( a' \in \mathcal{V}_e(a) \) and \( b' \in \mathcal{V}_f(b) \).
Let $aa' = e$ and $b' = f$. Then $e$ is a $\beta$-idempotent and $f$ is a $\gamma$-idempotent. So there exists an element $u$ with $u' \in V(\gamma)(u)$ in $S$ such that $u \in u' = e$ and $u' \beta u \subseteq f$. Now

$$ u \in b' \subseteq u' \beta a = (u \in u' \beta u) \subseteq (b' \subseteq (b' \subseteq u' \beta a = u \in u' \beta u \subseteq u' \beta a = e \beta e a = e \beta a = aa' \beta a = a. $$

Thus there exist $c = u$ and $d = b' \subseteq u' \beta a$ in $S$ such that $c \subseteq b \subseteq d = a$. Therefore $(S, \Gamma)$ is simple.

Conversely let $(S, \Gamma)$ be simple and $e \subseteq e = e$ and $f \subseteq f = f$ be two idempotents of $(S, \Gamma)$. Then there exist $a', b' \in S$ such that $e = a' \subseteq f \subseteq b'$. That is $e = (e \subseteq f \subseteq f \subseteq f \subseteq b' \subseteq e) = a \subseteq b$ (where $a = e \beta a \in \subseteq f \subseteq b' \subseteq e$ and $b = f \subseteq b' \subseteq e$) with $e \beta a = a = a \subseteq f$ and $f \subseteq b = b = b \subseteq e$. Therefore $a \subseteq b \subseteq a = (a \subseteq f \subseteq b) \subseteq a = e \beta a = a$ and $b \subseteq b = b \subseteq (a \subseteq f \subseteq b) = b \subseteq e = b$. Thus $b \in V(\gamma)(a)$. Also $a \subseteq b = a \subseteq f \subseteq b = e$, and $(b \beta a) \subseteq f = b \beta a$, $f \subseteq (b \beta a) = (f \subseteq b) \beta a = b \beta a$ that is $b \beta a \subseteq f$. This completes the proof of the theorem.

We complete this section with the following theorem which is exceedingly useful in the location of inverses of a regular element.

**Theorem 2.9.** Let $a$ be an element of a regular $\mathcal{J}$-class $D$ of a $\Gamma$-semigroup $(S, \Gamma)$.

(i) If $a' \in V(\gamma)(a)$ then $a', e \in D$ and the two $H$-classes $R_a \cap L_a, L_a \cap R_a$, contain respectively $\beta$-idempotent $aa'$ and
(ii) If \( b \in D \) be such that \( R_a \cap R_b \) contain respectively \( \alpha \)-idempotent \( e \) and \( \beta \)-idempotent \( f \) then \( H_b \) contains a 
\((\beta, \alpha)\) inverse \( a^* \) of \( a \) such that \( a\beta a^* = e \) and \( a^*\alpha a = f \).

**Proof.** (i) If \( a \in D \), then an \((\alpha, \beta)\) inverse \( a^* \) of \( a \) must belong

to \( D \). Because \( a^* \alpha a \in L_a \) imply \( a \in L_a \). Also \( a \alpha a \in R_a \),

\( a \alpha a \in L_a \), imply \( R_a \cap L_a \) contains the \( \beta \)-idempotent \( a \alpha a \).

Similarly \( a \beta a \beta a \in L_a \) imply \( a \beta a \in L_a \).

(ii) As \( a \in e \) by Lemma 2.1 \( e \alpha a = a \). Similarly \( a \notin f \) implies

that \( a \beta f = a \). Now \( a \in e \) implies that \( e = a \gamma t \) for some \( \gamma \in \Gamma \) and

\( t \in S \). Let \( a^* = f \gamma t a e \). Then \( a \beta a \alpha a = a \beta (f \gamma t a e) \alpha a = (a \beta f) \gamma t a e (e \alpha a) = a \gamma t a e = e \alpha a = a \) and \( a^* \alpha a \beta a^* = (f \gamma t a e) \alpha a \beta (f \gamma t a e) = f \gamma t a a \gamma t a e = f \gamma t a \alpha e = a^* \). Therefore \( a^* \in \forall \beta(a) \). Moreover

\( a \beta a^* = a \beta (f \gamma t a e) = (a \beta f) \gamma t a e = a \gamma t a e = e \alpha a = e \).

As \( a = a \beta a^* a \), we have \( (a) \downarrow = S \gamma a \). Since \( a \notin f \) we have \( f = S \gamma a \) for some \( s \in S \). Hence \( a^* \alpha a = (f \gamma t a e) \alpha a = f \gamma t a e = s \gamma a \gamma t a e = s \alpha e = s \gamma a = f \). Therefore \( a^* \in L_a R_f = L_a \cap R_f = H_f \).

This theorem allows us to locate the inverses of a regular

element provided that we know where the idempotents are. For

example in a finite \( \Gamma \)-semigroup, we can say immediately that

the number of \((\alpha, \beta)\)-inverses of a regular element \( a \) is the number
of α-idempotents in $L_a$ multiplied by the number of β-idempotents in $R_a$.

3. $\Gamma$-GROUP CONGRUENCES ON REGULAR $\Gamma$-SEMGROUPS

We know that $\Gamma$-groups are regular $\Gamma$-semigroups. But the converse is not true. In this section we study those congruences $\wp$ on a regular $\Gamma$-semigroup $(S, \Gamma)$ for which $(S/\wp, \Gamma)$ become $\Gamma$-groups. We call such a congruence $\wp$ on $(S, \Gamma)$ a $\Gamma$-group congruence. Here we give some equivalent expressions for any $\Gamma$-group congruence on a regular $\Gamma$-semigroup. We define closed normal family on a regular $\Gamma$-semigroup and finally establish one to one order preserving correspondence between the set of all closed normal families and the set of all $\Gamma$-group congruences on a regular $\Gamma$-semigroup. The group congruences on regular semigroups was studied by LaTorre in [9]. The results of this section are from [33] and actually generalise some of the results of [9].

First let us recall an important theorem from [24].

**THEOREM 3.1.** A regular $\Gamma$-semigroup $(S, \Gamma)$ is a $\Gamma$-group if and only if for all $\alpha, \beta \in \Gamma$, $e\alpha f = f\alpha e = f$ and $e\beta f = f\beta e = e$ for any two idempotents $e = e\alpha e$ and $f = f\beta f$ of $(S, \Gamma)$. 
DEFINITION: An equivalence relation $\mathcal{P}$ on $S$ of a $\Gamma$-semigroup $(S, \Gamma)$ is called a left (right) congruence if $(a, b) \in \mathcal{P}$ implies $(c a a, c a b) \in \mathcal{P}$ for all $c \in S$ and all $a \in \Gamma$. If $\mathcal{P}$ is both left and right congruence on $(S, \Gamma)$ then $\mathcal{P}$ is called a congruence on $(S, \Gamma)$.

Let $\mathcal{P}$ be a congruence on $(S, \Gamma)$ and $S/\mathcal{P}$ be the set of all equivalence classes of $S$. If $a^\mathcal{P}$ and $b^\mathcal{P}$ be any two elements of $S/\mathcal{P}$ and $a \in \Gamma$ then we define $(a^\mathcal{P}) a (b^\mathcal{P}) = (a a b)^\mathcal{P}$. Now it is easy to verify that $(S/\mathcal{P}, \Gamma)$ is a $\Gamma$-semigroup.

DEFINITION: A congruence $\mathcal{P}$ on a regular $\Gamma$-semigroup $(S, \Gamma)$ is called a $\Gamma$-group congruence if $(S/\mathcal{P}, \Gamma)$ is a $\Gamma$-group.

Henceforth in this section, we assume $(S, \Gamma)$ to be a regular $\Gamma$-semigroup and $E_\alpha$ to be the set of all $\alpha$-idempotents of $(S, \Gamma)$.

DEFINITION: A family $\{K_\alpha : \alpha \in \Gamma\}$ of subsets of $S$ is said to be a normal family if the following hold.

(i) $E_\alpha \subseteq K_\alpha$ for all $\alpha \in \Gamma$.

(ii) For each $a \in K_\alpha$ and $b \in K_\beta$, $a b a \in K_\alpha$ and $a b \in K_\beta$.

(iii) For each $a' \in V_{\alpha}^{\beta}(a)$ and $c \in K_\gamma$, $a c y a'$ and $a y c a' \in K_\beta$.

Now let $e \in E_\alpha$, $f \in E_\beta$ and $\mu \in \Gamma$. Let $x \in V_{\beta}^{\alpha}(e y f)$. Then $f y x y e \in E_\mu$. Thus $E_\mu \neq \emptyset$ for all $\mu \in \Gamma$ and so $K_\mu \neq \emptyset$ for all
\[ \mu \in \Gamma. \] We further note that in an orthodox \( \Gamma \)-semigroup \( (S, \Gamma) \)
\[ [25] \{ E_\alpha : \alpha \in \Gamma \} \] is a normal family of \( (S, \Gamma) \).

Let \( N \) be the collection of all normal families \( K_i \) of \( (S, \Gamma) \)
\[ (i \in \Lambda), \] where \( K_i = \{ K_\alpha^i : \alpha \in \Gamma \} \). Let \( U = \bigcap_{i \in \Lambda} K_i \) and
\[ U = \{ U_\alpha : \alpha \in \Gamma \}. \] Then obviously \( E_\alpha \subseteq U_\alpha \). Also if \( a \in U_\alpha \),
\[ b \in U_\beta, \] then \( a \in K_i^\alpha \) for all \( i \in \Lambda \), \( b \in K_i^\beta \) for all \( i \in \Lambda \).

Thus \( a \circ b \in K_i^\alpha \) and \( a \circ b \in K_i^\beta \) for all \( i \in \Lambda \) implying \( a \circ b \in U_\beta \) and
\[ a \circ b \in U_\alpha. \] Similarly we can show that if \( a' \in V_\alpha^\beta (a) \) and \( c \in U_\gamma \)
then \( a \circ c \gamma a', a \gamma c \circ a' \in U_\beta. \) Thus \( U \) is a normal family of subsets of \( S \) and \( U \) is the least member in \( N \) if we define a partial order in \( N \) by \( K_i \leq K_j \) if and only if \( K_i^\alpha \subseteq K_j^\alpha \) for all \( \alpha \in \Gamma \).

We also observe that when \( (S, \Gamma) \) is an orthodox \( \Gamma \)-semigroup,
\[ U = \{ E_\alpha : \alpha \in \Gamma \}. \]

**Theorem 3.2.** Let \( (S, \Gamma) \) be a regular \( \Gamma \)-semigroup. Then for each normal family \( K = \{ K_\alpha : \alpha \in \Gamma \} \), \( P_K = \{(a, b) \in S \times S : a \circ e = f \circ b \text{ for some } \alpha, \beta \in \Gamma \text{ and some } e \in K_\alpha, f \in K_\beta \} \) is a \( \Gamma \)-group congruence in \( (S, \Gamma) \).

**Proof.** Let \( a \in S \) and \( a' \in V_\alpha^\beta (a) \). Then \( a \circ (a' \circ b) = (a \circ a') \circ b \)
implies \( (a, a) \in P_K \). Next let \( (a, b) \in P_K \). Then there exist
\[ e \in K_\alpha, f \in K_\beta \] for some \( \alpha, \beta \in \Gamma \) such that \( a \circ e = f \circ b. \) Let \( a' \in V_\alpha^\gamma (a) \)
and \( b' \in \mathcal{V}^b(b) \) then \( \theta(b' \varphi(b)) \varphi(a' \sigma a) = ((b \varphi(b')) \varphi(a \varphi(a'))) \sigma a \).

But \( b' \varphi(b) \in \mathcal{K}_\theta \), \( a' \varphi(a) \in \mathcal{K}_\gamma \) and so \( (b' \varphi(b)) \varphi(a' \sigma a) \in \mathcal{K}_\theta \) and \( b \varphi(b) \in \mathcal{K}_\theta \), \( a \varphi(a) \in \mathcal{K}_\gamma \) and so \( (b \varphi(b')) \varphi(a \varphi(a')) \in \mathcal{K}_\theta \). Consequently, \( (b, a) \in \mathcal{P}_K \).

Now let \((a, b) \in \mathcal{P}_K \), \((b, c) \in \mathcal{P}_K \). Then there exist \( a, \beta, \gamma, \delta \in \Gamma \), \( e \in \mathcal{K}_\alpha \), \( f \in \mathcal{K}_\beta \), \( g \in \mathcal{K}_\gamma \), \( h \in \mathcal{K}_\delta \) such that \( a \varphi a = f \psi b \) and \( b \psi g = h \psi c \). But \( a \psi (e \psi g) = (a \psi e) \psi g = (f \psi b) \psi g = f \psi (b \psi g) = f \psi (h \psi c) = (f \psi h) \psi c \) where \( e \psi g \in \mathcal{K}_\alpha \) and \( f \psi h \in \mathcal{K}_\delta \). Thus \((a, c) \in \mathcal{P}_K \) and consequently \( \mathcal{P}_K \) is an equivalence relation. Let \((a, b) \in \mathcal{P}_K \), \( \theta \in \Gamma \), \( c \in S \). Then \( a \varphi a = f \psi b \) for some \( a, \beta \in \Gamma \) and some \( e \in \mathcal{K}_\alpha \), \( f \in \mathcal{K}_\beta \). Let \( c' \in \mathcal{V}_\gamma^c(c) \), \( \gamma \in \mathcal{V}_\gamma^1(b \psi c) \), \( x \in \mathcal{V}_\gamma^2(a \psi c) \).

Now \((a \psi c) \psi y_2x \in \mathcal{E}_y \in \mathcal{K}_\gamma^1 \) and consequently \( \mathcal{P}_K \) is a congruence on \((S, \Gamma)\). Also as \((S, \Gamma)\) is regular, \((S/\mathcal{P}_K, \Gamma)\) is a regular \( \Gamma \)-semigroup (by Lemma 2.5). Let \( e \in \mathcal{K}_\alpha \), \( f \in \mathcal{K}_\beta \). Then \( e \psi f, f \psi e \in \mathcal{K}_\beta \), \( e \psi f, f \psi e \in \mathcal{K}_\alpha \).

Now \((e \psi f) \psi f = (e \psi f) \psi f \) shows that \((e \psi f, f) \in \mathcal{P}_K \) and \((f \psi e) \psi f = (f \psi e) \psi f \) implies that \((f \psi e, f) \in \mathcal{P}_K \). Thus \((f \psi e, f) \in \mathcal{P}_K \).
and \((f \circ K) \alpha (e \circ K) = f \circ K\). Similarly we can show that
\((e \circ K) \beta (f \circ K) = e \circ K\) and \((f \circ K) \beta (e \circ K) = e \circ K\). So it follows from Theorem 3.1 that \((S/\mathcal{K}, \Gamma)\) is a \(\Gamma\)-group. Thus \(\mathcal{K}\) is a \(\Gamma\)-group congruence on \((S, \Gamma)\).

For any normal family \(K = \{K_\alpha : \alpha \in \Gamma\}\) of a \(\Gamma\)-semigroup \((S, \Gamma)\), the closure \(K_w\) of \(K\) is the family defined by
\[K_w = \{ (K_w)_{\gamma} \mid \gamma \in \Gamma \}\]
where \((K_w)_{\gamma} = \{x \in S : e x \in K_{\gamma}\} \text{ for some } \alpha \in \Gamma \text{ and } e \in K_\alpha\}.

\(K\) is called a closed normal family if \(K = K_w\). The following theorem gives an alternate characterisation of \(\mathcal{K}\).

**THEOREM 3.3.** Let \((S, \Gamma)\) be a regular \(\Gamma\)-semigroup. For each normal family \(K\), \(\mathcal{K} = \{ (a, b) \in S \times S : a \gamma b' \in (K_w)_\gamma \text{ for some } b' \in V^S_\gamma (b) \}\).

**PROOF.** Let \((a, b) \in \mathcal{K}\). Then \(f \beta \alpha = bae\) for some \(\alpha, \beta \in \Gamma\) and some \(e \in K_\alpha\), \(f \in K_\beta\). Then \(f \beta (a \gamma b') = bae \gamma b' \in K_\gamma \) for some \(b' \in V^S_\gamma (b)\). Consequently \(a \gamma b' \in (K_w)_\gamma\). Conversely, let \(a \gamma b' \in (K_w)_\gamma\) for some \(b' \in V^S_\gamma (b)\). Then \(e a \gamma b' \in K_\gamma\) for some \(a \in \Gamma\) and some \(e \in K_\alpha\). Therefore \(e a \gamma b' = f\) where \(f \in K_\gamma\). So
\[(b \theta (a' \theta e a) \gamma b') \in e a = b \theta (a' \theta f \theta e a)\], for some \(a' \in V^S_\theta (a)\) where \(b \theta (a' \theta e a) \gamma b' \in K_\gamma\) and \(a' \theta f \theta e a \in K_\theta\). Consequently \((a, b) \in \mathcal{K}\).

For any congruence \(\mathcal{P}\) on \((S, \Gamma)\) let \(\text{Ker} \mathcal{P} = \{ (\text{Ker} \mathcal{P})_\alpha : \alpha \in \Gamma \}\)
where \((\text{Ker} \mathcal{P})_\alpha = \{ x \in S : (e, x) \in \mathcal{P} \text{ for some } e \in E_\alpha \}\).
LEMMA 3.4. For any $K \in N$, $\text{Ker } \phi_K = K_w$.

PROOF. To prove that $\text{Ker } \phi_K = K_w$, we are to show that 

$$(\text{Ker } \phi_K)_a = (K_w)_a$$

for all $a \in \Gamma$. For this let $x \in (\text{Ker } \phi_K)_a$ for some $a \in \Gamma$. Then $(e, x) \in \phi_K$ for some $e \in E_a$ that is $e \beta f = g \gamma x$ for some $\beta, \gamma \in \Gamma$, $f \in K_\beta$, $g \in K_\gamma$. So $g \gamma x \in K_a$ as $e \beta f \in K_a$. Thus $x \in (K_w)_a$.

Next let $x \in (K_w)_a$. Then $g \gamma x \in K_a$ for some $\gamma \in \Gamma$ and $g \in K_\gamma$. Now for some $e \in E_a$, $e \alpha (g \gamma x) = (e \alpha g) \gamma x$ where $g \gamma x \in K_a$ and $e \alpha g \in K_\gamma$. Thus $(e, x) \in \phi_K$. Consequently $x \in (\text{Ker } \phi_K)_a$.

So $(\text{Ker } \phi_K)_a = (K_w)_a$ for all $a \in \Gamma$. Thus $\text{Ker } \phi_K = K_w$.

Let $K$ be a normal family and suppose $a' \gamma b' \in (K_w)_a$ for some $b' \in V_{\gamma}(b)$. Then $e \alpha a' \gamma b' \in K_\xi$ for some $a \in \Gamma$ and $e \in K_a$. Then for any $a' \in V_{\theta}^\xi(a)$, $a' \phi (e \alpha a' \gamma b') \xi_a \in K_\theta$ and $(a' \xi \phi a) \gamma b' \xi_a \theta a' \phi b = (a' \xi \phi a) \gamma b' \xi_a (a' \phi a) \phi b \in K_\phi$. Thus $a' \phi b \in (K_w)_\phi$. Conversely suppose $a' \phi b \in (K_w)_\phi$ for some $a' \in V_{\theta}^\xi(a)$. Then $f \beta (a' \phi b) \in K_\theta$ for some $\beta \in \Gamma$ and $f \in K_\beta$ and $a \beta (f \beta a' \phi b) \theta a' \in K_\phi$. Therefore for some $b' \in V_{\gamma}^\xi(b)$, $(a \beta \xi a' \phi b) \phi (a' \gamma b') = (a \beta \xi a' \phi b) \phi (a' \phi a) \gamma b' \in K_\xi$.

Therefore $a \gamma b' \in (K_w)_\xi$. Thus $a \gamma b' \in (K_w)_\xi$ for some (all) $b' \in V_{\gamma}^\xi(b)$ if and only if $a' \phi b \in (K_w)_\xi$ for some (all) $a' \in V_{\theta}^\xi(a)$.

Interchanging roles of $a$ and $b$ we see that $b \theta a' \in (K_w)_\xi$ for some (all) $a' \in V_{\theta}^\xi(a)$ if and only if $b' \xi a \in (K_w)_\gamma$ for some (all) $b' \in V_{\gamma}(b)$. Moreover, the symmetric property of $\phi_K$ shows that
THEOREM 3.5. For each \( K \in \mathbb{N} \), \( (a, b) \in \mathcal{P}_K \) if and only if one of the following equivalent conditions hold.

(i) \( a \iff b \in (Kw)_a \) for some (all) \( b \in V_\gamma(b) \).

(ii) \( b \iff a \in (Kw)_a \) for some (all) \( b \in V_\gamma(b) \).

(iii) \( a \iff b \in (Kw)_a \) for some (all) \( a \in V_\gamma(a) \).

(iv) \( b \iff a \in (Kw)_a \) for some (all) \( a \in V_\gamma(a) \).

THEOREM 3.6. The mapping \( K \to \mathcal{P}_K = \{ (a, b) \in S \times S : a \iff b \in K \} \) is a one to one order preserving mapping from \( \mathbb{N} \), the collection of all closed normal families in \( \mathbb{N} \), onto the set of all \( \Gamma \)-group congruences on \( (S, \Gamma) \).

PROOF. Let \( \mathcal{P} \) be a \( \Gamma \)-group congruence on \( (S, \Gamma) \). Let us denote \( \ker \mathcal{P} \) by \( K \) and \( (\ker \mathcal{P})_a \) by \( K_a \). Then \( K_a = \{ x \in S : (x, e) \in \mathcal{P} \text{ where } e \in E_a \} \). Then \( E_a \subseteq K_a \). Let \( a \in K_a \), \( b \in K_b \) then

\[ (a, e) \in \mathcal{P} \text{ and } (b, f) \in \mathcal{P} \text{ where } e \in E_a \text{ and } f \in E_b \].

Now \( a \mathcal{P} b \mathcal{P} f = (a \mathcal{P}) a(b \mathcal{P}) = (e \mathcal{P}) a(f \mathcal{P}) = e \mathcal{P} \). Thus \( a \mathcal{P} b \mathcal{P} f \in \mathcal{P} \), where \( f \in E_b \). Thus \( a \mathcal{P} b \mathcal{P} K_b \). Similarly \( a \mathcal{P} b \mathcal{P} K_a \). Next let \( a' \in V_\beta(a) \) and \( c \in K_\gamma \). Then \( (c, g) \in \mathcal{P} \text{ where } g \in E_\gamma \). Then

\[ (a \mathcal{P} c \mathcal{P} a') \mathcal{P} = (a \mathcal{P}) a(c \mathcal{P}) \mathcal{P} (a' \mathcal{P}) = (a \mathcal{P}) a((g \mathcal{P}) \mathcal{P} (a' \mathcal{P})) = (a \mathcal{P}) a(a' \mathcal{P}) = (aa \mathcal{P} a') \mathcal{P} \]. Thus \( (a \mathcal{P} c \mathcal{P} a', aa \mathcal{P} a') \in \mathcal{P} \text{ where } aa \mathcal{P} a' \in E_\beta \).
Hence $aa'yS' \in K_\beta$. Similarly $a'yca'a' \in K_\beta$. Therefore $K$ is a normal family of subsets of $S$. Next $(Kw)_\gamma = \{x \in S : eax \in K_\gamma$ where $e \in K_\alpha$ for some $\alpha \in \Gamma\}$. Then $K_\gamma \subseteq (Kw)_\gamma$. To show $(Kw)_\gamma \subseteq K_\gamma$, let $x \in (Kw)_\gamma$. Then $eax \in K_\gamma$ for some $\alpha \in \Gamma$ and $e \in K_\alpha$. Consequently $(eax)_\gamma = g_\gamma$ where $g \in E_\gamma$. So $(e_\gamma a)(x_\gamma) = g_\gamma$ that is, $x_\gamma = g_\gamma$. Therefore $x \in K_\gamma$. Thus $(Kw)_\gamma \subseteq K_\gamma$. Therefore $K = Kw$ and so $K = \text{Ker} \rho \in \overline{N}$. Thus if $\rho$ is a $\Gamma$-group congruence, then $\text{Ker} \rho = K \in \overline{N}$. We now prove that $\rho_K = \rho$. If $(a,b) \in \rho_K$ then $a'b' \in K_S$ for some $b' \in V_\gamma(b)$. Thus $(a'b', h) \in \rho$ for some $h \in E_\beta$ and $a_\phi = (a_\phi)(h'((b'_\phi b)'_\phi)) = (h'_\phi) \in (b'_\phi) = b'_\phi$. Thus $\rho_K \subseteq \rho$. Conversely if $(a,b) \in \rho$ and $b' \in V_\gamma(b)$, then $(a'b', b'yb') \in \rho$ where $b'yb' \in E_\beta$ and so $(a,b) \in \rho_K$. Therefore $\rho = \rho_K$. Thus from above and Lemma 3.4 for any $K \in \overline{N}$, $K \mapsto \rho_K$ is a one-to-one mapping from $\overline{N}$ onto the set of all $\Gamma$-group congruences on $(S,\Gamma)$. Also it is easy to see that $K \mapsto \rho_K$ is an order preserving mapping.

Let $\tau$ be a $\Gamma$-group congruence on $(S,\Gamma)$. Theorem 3.6 implies that $\tau = \rho_K$, where $K = \text{Ker} \tau \in \overline{N}$. Thus each $\Gamma$-group congruence is of the form $\rho_K$ for some $K \in \overline{N} \subseteq N$. Hence by Lemma 3.4 we have the following.

**THEOREM 3.7.** The least $\Gamma$-group congruence $\rho$ on $(S,\Gamma)$ is given
by \( \delta = \mathcal{P}_{U} \) and \( \text{Ker} \delta = U_w \).

**THEOREM 3.8.** For any \( \Gamma \)-group congruence \( \mathcal{P}_{K} \), with \( K \) in \( N \), on a regular \( \Gamma \)-semigroup the following are equivalent.

1. \((a, b) \in \mathcal{P}_{K}\).
2. \(\alpha \mu \gamma \beta' \in K_\delta \) for some \( x \in K_\mu \) (\( \mu \in \Gamma \)) and some (all) \( b' \in V_\gamma(b) \).
3. \(\alpha' \phi \gamma \mu b \in K_\theta \) for some \( x \in K_\mu \) (\( \mu \in \Gamma \)) and some (all) \( a' \in V_\theta(a) \).
4. \(\beta \mu \theta \alpha' \epsilon K_\phi \) for some \( x \in K_\mu \) (\( \mu \in \Gamma \)) and some (all) \( a' \in V_\phi(a) \).
5. \( b' \in V_\gamma a \in K_\gamma \) for some \( x \in K_\mu \) (\( \mu \in \Gamma \)) and some (all) \( b' \in V_\gamma(b) \).
6. \(\alpha \epsilon e = f \beta b \) for some \( \alpha, \beta \in \Gamma \) and some \( e \in K_\alpha, f \in K_\beta \).
7. \(\epsilon e a = b \beta f \) for some \( \alpha, \beta \in \Gamma \) and some \( e \in K_\alpha, f \in K_\beta \).
8. \(K_\beta \beta \alpha \epsilon K_\alpha \cap K_\beta \beta \beta \epsilon K_\alpha \neq \emptyset \) for some \( \alpha, \beta \in \Gamma \).

**Proof.** (ii) \( \Rightarrow \) (iii) Suppose \( \alpha \mu \gamma \beta' \in K_\delta \), for some \( x \in K_\mu \) and \( b' \in V_\gamma(b) \). Then for any \( a' \in V_\theta(a) \), \( a' \phi (\alpha \mu \gamma \beta') S b = (a' \phi a) \mu (x \gamma (b' \gamma b)) \in K_\theta \) as \( a' \phi a \in K_\theta \) and \( x \gamma b' \in K_\mu \).

(iii) \( \Rightarrow \) (vi) Let \( a' \phi \gamma \mu b \in K_\theta \) for some \( a' \in V_\theta(a) \) and \( x \in K_\mu \).

Then \( a \theta (a' \phi \gamma \mu b) = (a \theta a' \phi x) \mu b \) which is (vi) as \( a' \phi \gamma \mu b \in K_\theta \) and \( a \theta a' \phi x \in K_\mu \).

(vi) \( \Rightarrow \) (viii) Let \( a \epsilon e = f \beta b \) for some \( \alpha, \beta \in \Gamma \) and \( e \in K_\alpha, f \in K_\beta \).

Then we have \( f \beta \alpha \epsilon e = f \beta f \beta a \epsilon e \) implying \( K_\beta \beta \alpha \epsilon K_\alpha \cap K_\beta \beta \beta \epsilon K_\alpha \neq \emptyset \).

(viii) \( \Rightarrow \) (ii) Let \( K_\beta \beta \alpha \epsilon K_\alpha \cap K_\beta \beta \beta \epsilon K_\alpha \neq \emptyset \). Thus \( x \beta \alpha y = x_1 \beta \beta \alpha y_1 \) for some \( x, x_1 \in K_\beta, y, y_1 \in K_\alpha \). If \( a' \in V_\theta(a), b' \in V_\gamma(b) \). Then
a'σxβa / K_θ , (a'σxβa) ay / K_θ and we have, aθ(a'σxβa ay) γb' =
(aθa')σ(a'σxβa ay) γb' = (aθa')σx_1 β(bαγγ_1 γb') / K_θ
as bαγγ_1 γb' / K_θ , x_1 β(bαγγ_1 γb') / K_θ and aθa' / K_θ . Thus (ii),
(iii), (vi) and (viii) are equivalent. Interchanging the roles
of a and b we see that (iv), (v), (vii) and (viii) are equiva­
lent. Also (i) and (vi) are equivalent by Theorem 3.2. Thus
all the conditions (i) - (viii) are equivalent.

COROLLARY 3.9. Let \( \mathcal{G} \) denote the least \( \Gamma \)-group congruence on a
regular \( \Gamma \)-semigroup \((S, \Gamma)\). Then the following conditions are
equivalent.

(i) \((a, b) \in \mathcal{G}\)

(ii) \(auxy b' \in U_\mathcal{G}\) for some \(x \in U_\mu (\mu \in \Gamma)\) and some (all) \(b' \in V_\mathcal{G}(b)\).

(iii) \(a'\sigma x\beta b \in U_\mathcal{G}\) for some \(x \in U_\mu (\mu \in \Gamma)\) and some (all) \(a' \in V_\mathcal{G}(a)\).

(iv) \(buxy a' \in U_\mathcal{G}\) for some \(x \in U_\mu (\mu \in \Gamma)\) and some (all) \(a' \in V_\mathcal{G}(a)\).

(v) \(b' \in x^\alpha a \in V_\mathcal{G}\) for some \(x \in U_\mu (\mu \in \Gamma)\) and some (all) \(b' \in V_\mathcal{G}(b)\).

(vi) \(a\sigma a = f\beta b\) for some \(a, \beta \in \Gamma\) and \(e \in U_\alpha , f \in U_\beta\).

(vii) \(e\sigma a = b\beta f\) for some \(a, \beta \in \Gamma\) and \(e \in U_\alpha , f \in U_\beta\).

(viii) \(U_\beta b\sigma a U_\alpha \cap U_\beta b\sigma a U_\alpha \neq \emptyset\) for some \(a, \beta \in \Gamma\).
4. THE MAXIMUM IDEMPOTENT SEPARATING CONGRUENCE ON A REGULAR \( \Gamma \)-SEMIGROUP

In the discussion for \( \Gamma \)-group congruences on regular \( \Gamma \)-semigroups we see that a \( \Gamma \)-group congruence on a regular \( \Gamma \)-semigroup is such that all \( \alpha \)-idempotents \((\alpha \in \Gamma)\) is contained in a single congruence class. So just opposite to the idea of \( \Gamma \)-group congruence is the congruence in which no two \( \alpha \)-idempotents belong to the same congruence class. We define such a congruence on a regular \( \Gamma \)-semigroup as idempotent separating congruence. In this section we characterize the maximum idempotent separating congruence on a regular \( \Gamma \)-semigroup \([27]\). This actually generalises the work of John Meakin \([14]\) regarding the maximum idempotent separating congruence on a regular semigroup.

**DEFINITION**: A congruence \( \mathcal{P} \) on a \( \Gamma \)-semigroup \((S, \Gamma)\) is said to be idempotent separating if \( eae = e, faf = f \) belong to \( S \) and \((e, f) \in \mathcal{P} \) imply \( e = f \).

From the definition it follows that if \( \mathcal{P} \) is an idempotent separating congruence on a \( \Gamma \)-semigroup \((S, \Gamma)\) then \( \mathcal{P} \) is an idempotent separating congruence on each semigroup \( S_\alpha \), \( \alpha \in \Gamma \). But the following example shows that a congruence \( \mathcal{P} \) on a \( \Gamma \)-semigroup
(S, \Gamma) may be an idempotent separating congruence on some semi-
group S_\alpha, \alpha \in \Gamma, but may not be an idempotent separating con-
gruence for the \Gamma-semigroup (S, \Gamma). So it is necessary to
study idempotent separating congruences on a \Gamma-semigroup (S, \Gamma).

**Example 4.1.** Let S = \mathcal{J}/(6) = the set of all residue classes
modulo 6 = \{0, 1, 2, 3, 4, 5\} and \Gamma = \{1, \bar{3}\}. If we define
\mu : S \times \Gamma \times S \rightarrow S by (\bar{a}, \bar{b}, \bar{c})_\mu = \bar{a} \bar{b} \bar{c} = \bar{a} \bar{b} \bar{c} where \bar{a}, \bar{b}, \bar{c} \in S and
\bar{a} \in \Gamma. Then (S, \Gamma) is a \Gamma-semigroup. Furthermore (S, \Gamma) is a
regular \Gamma-semigroup. But S_\beta is not a regular semigroup where
\beta = \bar{3} as there is no p in S such that I\bar{3}p3I = I holds. Now we
see that 0, \bar{1}, \bar{3}, 4 are I-idempotents and \bar{0} and \bar{3} are 3-idempotents.
Let us define a binary relation \rho on S by (\bar{a}, \bar{b}) \in \rho if and only
if \bar{a} - \bar{b} = n where n is even. Then \rho is a congruence. Now in
S_\beta, where \beta = \bar{3}, 3 and \bar{0} are idempotents and \bar{3} - \bar{0} = \bar{3}. Thus \bar{3}
is not \rho-related to \bar{0}. Consequently \rho is an idempotent
separating congruence in S_\beta. But in S_\alpha, where \alpha = \bar{1}, (4, 0) \in \rho
and (\bar{3}, \bar{1}) \in \rho. Consequently \rho is not an idempotent separating
congruence in S_\alpha.

We define partial orders among \mathcal{L}, \mathcal{R} and \mathcal{J}-classes of a
\Gamma-semigroup (S, \Gamma) as follows: \mathcal{L}_a \leq \mathcal{L}_b if (a)_1 \subseteq (b)_1 ,
\mathcal{R}_a \leq \mathcal{R}_b if (a)_r \subseteq (b)_r and \mathcal{J}_a \leq \mathcal{J}_b if (a) \subseteq (b) .
[where \((a)_l = S \cap a \cup a\), \((a)_r = a \cap S \cup \{a\}, (a) = \{a\} \cup S \cap a \cap S\]. We now introduce the following notation: If in a \(\Gamma\)-semigroup \((S, \Gamma)\) \(a \in S\) then we define

\[EL(a) = \{e \in S : L_e \subseteq L_a \text{ and } (\exists \theta \in \Gamma)(e \theta e = e)\}\] and

\[ER(a) = \{e \in S : R_e \subseteq R_a \text{ and } (\exists \theta \in \Gamma)(e \theta e = e)\}.\]

If \((S, \Gamma)\) is a regular \(\Gamma\)-semigroup, then for any \(a \in S\), \(EL(a) \neq \emptyset\) and \(ER(a) \neq \emptyset\), also we note that if \((a, b) \in \mathcal{H}\) then \(EL(a) = EL(b)\) and \(ER(a) = ER(b)\). We can prove the following lemma easily.

**Lemma 4.1.** Let \((S, \Gamma)\) be a \(\Gamma\)-semigroup then \(\mathcal{R}\) is a left congruence and \(\mathcal{L}\) is a right congruence on \((S, \Gamma)\).

**Lemma 4.2.** Let \((S, \Gamma)\) be a regular \(\Gamma\)-semigroup and \(a, b\) be any two elements of \(S\). Then \(a \mathcal{R} b\) if and only if there exist \(a' \in V^\beta_\alpha(a)\) and \(b' \in V^\beta_\alpha(b)\) such that \(a' \beta a = b' \beta b\) and \(a \mathcal{L} a' = b \mathcal{L} b'\) for some \(\alpha, \beta \in \Gamma\).

(In fact we prove that if \((a, b) \in \mathcal{H}\) then for every \(a' \in V^\beta_\alpha(a)\) there exists \(b' \in V^\beta_\alpha(b)\) such that \(a \mathcal{L} a' = b \mathcal{L} b'\) and \(a' \beta a = b' \beta b\)).

**Proof.** Suppose \(a \mathcal{R} b\) and that \(a' \in V^\beta_\alpha(a)\). Then \(a' \beta a \mathcal{L} b \mathcal{R} a \mathcal{L} a'\).

Then by Theorem 2.9 there exists a unique \(b' \in V^\beta_\alpha(b)\) such that \(b' \beta b = a \mathcal{L} a'\) and \(b' \beta b = a' \beta a\). Conversely if for some \(a' \in V^\beta_\alpha(a)\), \(b' \in V^\beta_\alpha(b)\) we have \(a' \beta a = b' \beta b\) and \(a \mathcal{L} a' = b \mathcal{L} b'\) then \(a \mathcal{R} a' \beta a = b' \beta b \mathcal{R} b\) and \(a \mathcal{R} a' = b' \beta b \mathcal{L} b\). Hence \(a \mathcal{R} b\).
LEMMA 4.3. Let \( \rho \) be a congruence on a regular \( \Gamma \)-semigroup \((S, \Gamma)\). If \( a \rho \) is an \( a \)-idempotent in \((S/\rho , \Gamma)\) then there exists an \( a \)-idempotent \( e \) in \( S \) such that \( a \rho = e \rho \) and \( R_e \leq R_a \), \( L_e \leq L_a \).

PROOF. If \( a \rho \) is an \( a \)-idempotent in \((S/\rho , \Gamma)\), then \( a \rho = (a \rho) a(a \rho) = (aca) \rho \). So \((a, aca) \in \rho \). Let \( b \in V_\beta (aca) \).

Let \( e = a \beta b \gamma a \) then \( e \) is an \( a \)-idempotent in \((S, \Gamma)\). Also \((a, aca) \in \rho \) implies \((a \beta b \gamma a, (aca) \beta b \gamma a) \in \rho \) and \((aca) \beta b \gamma (aca) \in \rho \). That is \( (e, aca) \in \rho \). Consequently \((e, a) \in \rho \). Thus \( e \rho = a \rho \). Now \( (e)_r = e a s = a \beta b \gamma a \rho s \leq a \Gamma s = (a)_r \) implies \( R_e \leq R_a \). Similarly \( L_e \leq L_a \).

LEMMA 4.4. If \((S, \Gamma)\) is a regular \( \Gamma \)-semigroup then a congruence \( \rho \) on \((S, \Gamma)\) is idempotent separating if and only if \( \rho \subseteq \mathcal{H} \).

PROOF. Let \( \rho \subseteq \mathcal{H} \). Then \( \rho \) is idempotent separating because if \( e \rho = e \), \( f \rho = f \) and \((e, f) \in \rho \) then \((e, f) \in \mathcal{H} \). Then \( e \rho f \) and \( e \rho f \rho = f \rho \) and so \( S e \rho = S a \rho = a \rho s \). Consequently \( e = e \rho f = f \).

Conversely let \( \rho \) be an idempotent separating congruence. Let \((a, b) \in \rho \). Then for any \( a' \in V_\alpha ^\beta (a) \) where \( \alpha, \beta \in \Gamma \), we have \((aca', bca') \in \rho \). Therefore \((bca') \rho = (aca') \rho \) is a \( \beta \)-idempotent in \((S/\rho , \Gamma)\). Hence by Lemma 4.3 there exists a \( \beta \)-idempotent
e ∈ S such that e Φ = (baa') Φ and Ra ⊆ R_baa'. But then
e Φ = (aca') Φ implies that e = aca' (since Φ is an idempotent
separating congruence). Thus Ra = R_a = Re ⊆ R_baa' ∈ R_b.
Similarly we can prove that R_b ⊆ R_a. Thus a ∼ b. Similar
argument prove that a ∼ b. Consequently a ∼ b.

Now we give a characterisation of the maximum idempotent sepa­
rating congruence on a regular Γ-semigroup.

THEOREM 4.5. The maximum idempotent separating congruence on a
regular Γ-semigroup (S, Γ) is given by μ = \{(a, b) ∈ S x S :
there exist a, b ∈ Γ, a' ∈ VA(a), b' ∈ VA(b) such that aθa' =
bθb' for all e ∈ EL(a) U EL(b), a'βθa = b'βθb for all
fθf = f ∈ ER(a) U ER(b)\}.

PROOF. It is obvious that μ is reflexive and symmetric. We now
show that μ ∈ Υ. Let (a, b) ∈ μ and let a' ∈ VA(a), b' ∈ VA(b) be
such that the conditions given in the definition of μ are
satisfied. Now aca' ∈ R_a and aca' is a β-idempotent, so
a'βa = a'β(aca')βa = b'β(aca')βb = (b'βb)β(aca')βb
Hence aca' = a(a'βa)ca' = a(b'βb)ca' = b(a'βb)ca' = bca'βb =
(bb')β(aca')β(bb'). Hence (bb')β(aca') = aca' = (aca')β(bb').
As (a, b) ∈ μ implies that (b, a) ∈ μ, proceeding as above we can
show that (bb')β(aca') = bb' = (aca')β(bb'). Consequently
Therefore from (1) $a' \beta a = b' \beta (aca') \beta b = b' \beta (bcb') \beta b = b' \beta b$ and so by Lemma 4.2 $(a,b) \in J$. Thus $\mu \subseteq J$. We now prove that the relation $\mu$ is transitive. Let $(a,b) \in \mu$, $(b,c) \in \mu$.

Then $(a,b) \in J$, $(b,c) \in J$ and there exist $\alpha, \beta, \gamma, \epsilon \in \Gamma$ and $a' \in V^\alpha(a)$, $b' \in V^\beta(b)$, $b^* \in V^\delta(b)$, $c' \in V^\epsilon(c)$ such that $a \theta e a' = b \theta e b'$, $b \theta e c^* = c \theta e c^*$ for all $e \theta e = e \in EL(a) = EL(b) = EL(c)$ and $a' \beta \theta a' = b' \beta \theta b$, $b^* \epsilon \theta b = c^* \epsilon \theta c$ for all $f \epsilon \theta f = f \in ER(a)$ $= ER(b) = ER(c)$. Then $a \theta a' = b \theta b'$, $a' \beta a = b' \beta b$, $b \gamma c^* = c \gamma c^*$, $b^* \epsilon b = c^* \epsilon c$. Also as $(a,c) \in J$ by Lemma 4.2 there exist $a' \in V^\alpha(a)$, $c' \in V^\alpha(c)$ such that $a \theta a' = c \theta c'$, $a' \beta a = c' \beta c$, $a \gamma a' = c \gamma c'$, $a \epsilon a = c \epsilon c$. Then for each $e \theta e = e \in EL(a) = EL(b) = EL(c) = EL(a' \beta a) = EL(b' \beta b)$, we have $a \theta e a' = a \theta (e \theta a' \beta a) \gamma a' = (a \theta e a') \beta (a \gamma a') = (b \theta e b') \beta (b \gamma b') = b \theta (e \theta b' \beta b) \gamma b' = b \theta e b' = c \theta e c'$. Also for each $f \theta f = f \in ER(a) = ER(b) = ER(c) = ER(a \theta a') = ER(b \theta b')$ we have $a^* \epsilon \theta a^* = a^* \epsilon (a \theta a' \beta f) \theta a = (a^* \epsilon a) \alpha (a' \beta \theta a) = (b^* \epsilon b) \alpha (b' \beta \theta b) = b^* \epsilon (b \epsilon b' \beta f) \theta b = b^* \epsilon \theta b = c^* \epsilon \theta c$. Hence $(a,c) \in \mu$ and so $\mu$ is transitive. Next we show that if $(a,b) \in \mu$ then $(c \gamma a, c \gamma b) \in J$ for each $c \in S$, $\gamma \in \Gamma$. Let $(a,b) \in \mu$, $c \in S$, $\gamma \in \Gamma$ and $a' \in V^\alpha(a)$, $b' \in V^\beta(b)$ be such that the conditions in the definition of $\mu$ are satisfied. Now $(a,b) \in \mu \subseteq J \subseteq R$ and $R$ is a left congruence (by Lemma 4.1) so $(c \gamma a, c \gamma b) \in R$.
Now \( c^\lambda a = (c^\lambda a) \triangleleft (c^\lambda a)' \ \mu \ (c^\lambda a) \) (where \( (c^\lambda a)' \in V^\mu(c^\lambda a) \))

\[
= (c^\lambda a) \sigma(a' \beta a) \triangleleft (c^\lambda a)' \ \mu \ (c^\lambda a) \sigma(a' \beta a)
\]

\[
= (c^\lambda a) \sigma(b' \beta b) \triangleleft (c^\lambda a)' \ \mu \ (c^\lambda a) \ \sigma(b' \beta b)
\]

\[
= (c^\lambda a \sigma b') \beta[b \in ((c^\lambda a)' \ \mu \ (c^\lambda a)) \sigma b']
\]

\[
= (c^\lambda a \sigma b') \beta[a \in ((c^\lambda a)' \ \mu \ (c^\lambda a)) \sigma a']b'
\]

\[
= ... \in EL(a) = EL(b).
\]

\[
= (c^\lambda a) \sigma(b' \beta a) \triangleleft (c^\lambda a)' \ \mu \ (c^\lambda a) \ \sigma(a' \beta b)
\]

\[
= (c^\lambda a) \sigma(b' \beta a) \triangleleft (c^\lambda a)' \ \mu \ c^\lambda (a \sigma a') \ \beta b
\]

\[
= (c^\lambda a) \sigma(b' \beta a) \triangleleft (c^\lambda a)' \ \mu \ c^\lambda (b \sigma b') \ \beta b.
\]

\[
\]

So \( L_{c^\lambda a} \subseteq L_{c^\lambda b} \). Similarly \( L_{c^\lambda b} \subseteq L_{c^\lambda a} \) and thus \( (c^\lambda a, c^\lambda b) \in \mathcal{L} \).

It follows that \( (c^\lambda a, c^\lambda b) \in \mathcal{F} \). Therefore by Lemma 4.2 if \( (c^\lambda a)' \in V^\mu(c^\lambda a), \) then there exists \( (c^\lambda b)' \in V^\mu(c^\lambda b) \) such that \( (c^\lambda a)' \mathcal{F} (c^\lambda b)' \). We now prove that \( (c^\lambda a, c^\lambda b) \in \mu \). Let \( (a, b) \in \mu, \ c \in S \) and \( \gamma \in \mathcal{F} \). Let \( e^\theta = e^\epsilon \in EL(c^\lambda a) = EL(c^\lambda b) \). We show that if \( (c^\lambda a)' \in V^\mu(c^\lambda a) \) then \( (c^\lambda a) \\theta e \subseteq (c^\lambda a)' = (c^\lambda b) \\theta e \subseteq (c^\lambda b)' \).

Let \( a' \) and \( b' \) denote the \( (\alpha, \beta) \) inverses of \( a \) and \( b \) respectively satisfying conditions given in the definition of \( \mu \). Now \( e^\epsilon \in EL(c^\lambda a) = EL((c^\lambda a)' \ \mu \ (c^\lambda a)) \) and so \( e \in (c^\lambda a)' \ \mu \ (c^\lambda a) = e \).

Also \( L(c^\lambda a)' \ \mu \ (c^\lambda a) \leq L_{a' \beta a} = L_a \), so \( e(a' \beta a) = e \) and \( L_e \subseteq L_a \).
Hence \((c\gamma a) \in (c\gamma a)\ ' = c\gamma a \theta \alpha (a' \beta a) \subseteq (c\gamma a)\ ' \)
\[ = c\gamma (a \theta \alpha a') \beta a \subseteq (c\gamma a)\ ' \]
\[ = c\gamma (b \theta \alpha b') \beta a \subseteq (c\gamma a)\ ' \]
\[ = (c\gamma b) \theta \alpha \beta (c\gamma a) \beta (c\gamma a) \subseteq (c\gamma a) \oplus (c\gamma b) \]
\[ = (c\gamma b) \theta \alpha \beta (c\gamma a) \beta (c\gamma a) \subseteq (c\gamma a) \oplus (c\gamma b) \]
\[ = (c\gamma b) \theta \alpha \beta (c\gamma a) \beta (c\gamma a) \subseteq (c\gamma a) \oplus (c\gamma b) \]
\[ = (c\gamma b) \theta \alpha \beta (c\gamma a) \beta (c\gamma a) \subseteq (c\gamma a) \oplus (c\gamma b) \]

(since \( L(c\gamma a) \mu (c\gamma a) \leq L_B = I_B \))
\[ = (c\gamma b) \theta \alpha \beta (c\gamma a) \beta (c\gamma a) \subseteq (c\gamma a) \oplus (c\gamma b) \]
\[ = (c\gamma b) \theta \alpha \beta (c\gamma a) \beta (c\gamma a) \subseteq (c\gamma a) \oplus (c\gamma b) \]
\[ = (c\gamma b) \theta \alpha \beta (c\gamma a) \beta (c\gamma a) \subseteq (c\gamma a) \oplus (c\gamma b) \]

Now let \( f \theta f = f \in ER(c\gamma a) = ER(c\gamma b) = ER[(c\gamma a) \subseteq (c\gamma a) \ '] \).

Then \((c\gamma a) \subseteq (c\gamma a) \ ' \mu f = f \) and it follows that \((c\gamma a) \ ' \mu f \theta (c\gamma a) \)
is a \( \subseteq \)-idempotent and \((c\gamma a) \ ' \mu f \theta (c\gamma a) \in EL(a) = EL(b) \). Hence
\[(c\gamma a) \mu f \theta (c\gamma a) = (c\gamma a) \ ' \mu (c\gamma a) \subseteq (c\gamma a) \ ' \mu f \theta (c\gamma a) \subseteq (c\gamma a) \ ' \mu (c\gamma a) \]
\[ = (c\gamma b) \ ' \mu (c\gamma b) \subseteq [(c\gamma a) \ ' \mu f \theta (c\gamma a) ] \subseteq (c\gamma b) \ ' \mu (c\gamma b) \]
\[ = (c\gamma b) \ ' \mu (c\gamma b) \subseteq [(c\gamma a) \ ' \mu f \theta (c\gamma a) ] \subseteq (c\gamma b) \ ' \mu (c\gamma b) \]
\[ = (c\gamma b) \ ' \mu (c\gamma b) \subseteq [(c\gamma a) \ ' \mu f \theta (c\gamma a) ] \subseteq (c\gamma b) \ ' \mu (c\gamma b) \]
\[ = (c\gamma b) \ ' \mu (c\gamma b) \subseteq [(c\gamma a) \ ' \mu f \theta (c\gamma a) ] \subseteq (c\gamma b) \ ' \mu (c\gamma b) \]
\[ = (c\gamma b) \ ' \mu (c\gamma b) \subseteq [(c\gamma a) \ ' \mu f \theta (c\gamma a) ] \subseteq (c\gamma b) \ ' \mu (c\gamma b) \]
Thus \( (c^a, c/b) \in \mu \). Similarly we can prove that \( (a, b) \in \mu \)
implies \( (a^c, b^c) \in \mu \) for all \( c \in \Gamma \) and \( c \in S \). Consequently \( \mu \)
is a congruence. Since \( \mu \subseteq \mathcal{H} \), \( \mu \) is an idempotent separating
congruence by Lemma 4.4.

Finally let \( \mathcal{F} \) be any idempotent separating congruence on \((S, \Gamma')\)
and let \( (a,b) \in \mathcal{F} \). Then \( (a,b) \in \mathcal{H} \) (by Lemma 4.4) and so by
Lemma 4.2 there exist \( a' \in \gamma^a(a), b' \in \gamma^b(b) \) for some \( a, b \in \Gamma \)
such that \( a^a a' = b^b b' \), \( a^b a = b^b b' \).

Let \( e e e = e \in EL(a) = EL(a^a a) = EL(b^b b) = EL(b) \).

Then \( e a' a a' = e \) and \( (a e e a') \beta(a e e a') = a \theta(e a' a a') \theta e e a' =
\theta a e e a' = a e e a' \). Thus \( a e e a' \) is a \( \beta \)-idempotent. Similarly
\( b e e a' \) is also a \( \beta \)-idempotent of \((S, \Gamma')\). Again \( b' = b^b b' =
b^b b a a' \) and \( a' = a^a a a' = b^b b a a' \). But \( (b^b b a a', b^b b a a') \in \mathcal{F} \)
impliciting thereby that \( (a', b') \in \mathcal{F} \). Hence \( (a e e a', b e e a') \in \mathcal{F} \).

But \( a e e a' \) and \( b e e a' \) are both \( \beta \)-idempotents and \( \mathcal{F} \) is an idempotent
separating congruence so \( a e e a' = b e e a' \). Similarly we can prove
that \( a^a a a' = b^b b b' \) for each \( a \in \mathcal{F} \). Thus \( a, b \in \mu \) which in turn implies that \( \mathcal{F} \subseteq \mu \). Consequently \( \mu \) is the
maximum idempotent separating congruence on \((S, \Gamma')\).