Chapter VII

LAMINAR FLOW MODEL OF BLOOD IN A TUBE WITH AN IDEALIZED PLASMA LAYER AT WALL.

7.1 INTRODUCTION

Flow of blood in vessels of small diameter demands consideration of blood as a non-homogeneous, two-phase fluid, whereas this is not necessary for analysis of blood flow in large diameter vessels. Copley et al [1], [2], [3] studied bloodflow in the microvessels and reported the presence of cell-free marginal layer called plasma layer near the wall. They found that thickness of layer decreases as the velocity of the fluid decreases. Bugliarello et al. [4]. Studied the velocity distribution and other characteristics of steady and pulsatile blood flow in fine glass tubes. They found that the velocity distributions for both steady and pulsating conditions fell generally between those described by models which assume the erythrocytes to be either uniformly distributed in the cross section or concentrated in a cen-
tral region of uniform radius. Chatkirani, et. al., [5-6] studied the flow of blood taking help of casson and power-law fluid models. Kiani et. al. [7] developed a semi-empirical model to describe the dependence of apparent viscosity of blood on vessel diameter (2.7 to 500 micron) and vessel discharge hematocrat (5 \% to 60 \%). The blood flow is modeled as a cell rich core region near the axis of the vessel and a cell free plasma layer near the wall. An equation is derived in which apparent viscosity is a function of vessel diameter, core viscosity and width of marginal layer. Secomb [8], Reinke et. al [9], Singh [10] studied the flow of blood in small vessels.

In the present analysis, we have used casson's equation for the constitutive relation. We have obtained the flow profile and the flow rate. It is shown that a region of plug flow exists. This may extend to all of the tube radius. Two-layer fluid model with cell-rich region in the centre and a cell-free region near the wall is assumed. Both the fluids obey casson constitutive relation. We assume that the motion is steady so that inertia terms are neglected we also assume that the flow is fully
developed and the velocity in the axial direction is a function of radial component only. Thus, with usual cylindrical co-ordinate system \((r, \theta, z)\) the axial velocity \(v\) and the shear stress \(\tau_{r z}\) can be written as

\[
v = v(r) \tag{1}
\]

\[
\tau_{r z} = \tau(r) \tag{2}
\]

Flow is considered in a circular tube of radius \(R\). A plasma layer of thickness \(R - R_0\) (region) exist around a core fluid of radius \(R_0\) (Region 2).

7.2 **EQUATION OF MOTION**

Equation of motion for steady incompressible laminar flow in cylindrical polar coordinates are

\[
\phi = - \frac{\partial \psi}{\partial \phi} \tag{3}
\]

\[
0 = - \frac{\partial \psi}{\partial z} + \frac{1}{\sigma} \frac{\partial}{\partial \phi} \left( \sigma \tau_{\phi \phi} \right) \tag{4}
\]
Where $p$ is the pressure, $z$ the axial coordinate, $T_{\sigma z}$ the shear stress normal to $r$ in the direction of $z$. Equation (3) shows that pressure is the function of only axial co-ordinate $z$. Two terms of right hand side of equation (4) added together yields zero and is a function of $z$ and the other is a function of $r$, therefore we can obtain $\frac{\partial p}{\partial z}$ as constant.

An additional assumption that the velocity of the fluid at the tube wall is not zero. Thus, we have considered a slip un fluid velocity at the inner surface of the wall of the vessel. Blood is a mixture of various kinds of cells and other particulate matters which are suspended in plasma. We are interested in average velocity of fluid.
The velocity of the plasma at the rigid boundary may be zero, but there is no region to expect that the velocity of the suspended cells near the boundary may also zero. We suppose that the velocity of the fluid at the wall is proportional to the shear stress at the wall. That is

\[ v_R = k \tau_R \]  

(5)

Where \( k \) is a constant of proportionality (a function of the properties of the wall and the fluid), \( v_R \) the slip velocity at the wall and \( \tau_R \) the shear stress all.

7.3 **CONSTITUTIVE RELATION**

Merrile et al (ii) in their experiment on the flow of blood showed that for a wide range of hematocrit, the Casson equation represent the behaviour of blood. Since blood possesses a yield stress, e.g. [12], plug flow must exist wherever shear stress does no exceed yield stress. Thus, at low flow rates casson's constitutive relation will serve the purpose of theoretical study with more accuracy. Casson's relation may be written as
The quantity \( \tau_0 \) is the yield stress (the stress below which no flow can be observed under the conditions of experimentation, \( \tau \) the shear stress, \( \dot{\varepsilon} \) the strain rate and \( \mu \) is a coefficient of aiscosity.

### 7.4 SOLUTION

Integration of (4) yields,

\[
( \sigma_{r z} ) = \frac{\partial p}{\partial z} \frac{\gamma^2}{2} + C
\]

Constant C is zero because the stress is assumed to be finite at the axis of the tube. Hence

\[
\tau_{r z} = \frac{\partial p}{\partial z} \frac{\gamma^2}{2} = \frac{\Delta P}{2L} \gamma
\]

Where \( \Delta P \) is a pressure drop over a length \( L \) of the tube.

At \( \dot{\varepsilon} = R \)
Equation (8) with the help of (6) becomes,
\[ \frac{\Delta P}{2L} \gamma = \left[ \frac{\gamma}{2} + \mu \gamma \varepsilon \gamma \right] \frac{2\gamma}{\gamma} \] (10)

At the interfacial boundary of regions (1) and (2) it is assumed that
\[ \frac{\gamma_z}{\gamma} \omega = \frac{\gamma_z}{\gamma} \omega , \quad \gamma_z = \gamma_1 \left( \omega \neq \gamma = R \right) \] (11)

While at the tube wall ( \( \gamma = R \)), \( \gamma_1 = \gamma_R \). (12)

Assuming, both the fluids of regions (1) and (2) obeying Casson's constitutive relation we have the equation (9) in the form as
\[ \frac{\Delta P}{2L} \gamma = \left[ \frac{\gamma}{2} + \mu \gamma \varepsilon \gamma \left( \frac{d \gamma_1}{d \omega} \right) \frac{1}{2} \right] ; R_0 \leq \omega \leq R \] (13)

\[ \text{and} \quad \frac{\Delta P}{2L} \gamma = \left[ \frac{\gamma}{2} + \mu \gamma \varepsilon \gamma \left( \frac{d \gamma_2}{d \omega} \right) \frac{1}{2} \right] ; \gamma \leq R_0 \] (14)
Quantities with subscripts 1 and 2 represent their usual measuring in regions 1 and 2, respectively.

Equation (13) can be written in the form

\[
\frac{1}{\mu_1} \left[ \left( \frac{\Delta p}{2 L} \right)^{\frac{1}{2}} \tau_1^{\frac{1}{2}} \right]^2 = \left| \frac{d v_1}{d \sigma} \right| = - \frac{d v_1}{d \gamma} \quad (15)
\]

which on integration with respect to boundary condition (12) gives

\[
v_1 = v_R + \frac{\Delta p}{\eta \mu_1 L} \left( R^2 - \sigma^2 \right) - \eta \tau_1 \left( \frac{\Delta p}{2 L} \right) \left( R^2 - \sigma^2 \right) \quad (16)
\]

Integrating equation (14) and using the boundary condition (11) we have

\[
v_2 = v_R + \frac{\Delta p}{\eta \mu_2 L} \left( R_0^2 - \sigma^2 \right) - \eta \tau_2 \left( \frac{\Delta p}{2 L} \right) \left( R_0^2 - \sigma^2 \right) + \frac{\tau_2}{\mu_2} \left( R_0 - \sigma \right) + v_0 \quad (17)
\]

Where,

\[
v_0 = \frac{\Delta p}{\eta \mu_1 L} \left( R^2 - R_0^2 \right) - \eta \tau_1 \left( \frac{\Delta p}{2 L} \right) \left( R^2 - R_0^2 \right) + \frac{\tau_1}{\mu_1} \left( R - R_0 \right) \quad (18)
\]
Using equation (5) & (9) the equation (16) becomes:

\[ v_1 = \frac{R^2 \tau_R}{2\mu_1} \left[ 2 \frac{k \mu_1}{R^2} + 1 - \left( \frac{x}{R} \right)^2 - \frac{2}{3} \beta \frac{\chi}{\mu_1} \left( 1 - \frac{3 \chi}{R^2} \right) \right] \]

Where \( \beta_1 = \frac{T_1}{\mu R} \)

Similarly we find \( v_2 \), as

\[ v_2 = \frac{R^2 \tau_R}{2\mu_2} \left[ 2 \frac{k \mu_2}{R^2} + 1 - \left( \frac{x}{R} \right)^2 - \frac{2}{3} \beta_2 \frac{\chi}{\mu_2} \left( 1 - \frac{3 \chi}{R^2} \right) \right] \]

\[ + \beta_2 \left( 1 - \frac{R^2}{R} \right) \]

where \( \beta_2 = \frac{T_2}{\mu_2} \)

If \( Q \) and \( Q \) denotes the flow rate in region 1, and region 2, respectively, then total flow rate \( Q \) is determined as

\[ Q = Q_1 + Q_2 = \int_{R_0}^{R} 2\pi x v_1 \, dx + \int_{0}^{R_0} 2\pi x v_2 \, dx \]

\[ = \frac{R^3 \tau_R}{4\mu_1} \left[ \frac{k \mu_1}{R^2} + 1 - \frac{2}{3} \beta \frac{\chi}{\mu_1} \right] + \frac{R^2 \tau_R}{\mu_1} \left[ \frac{k \mu_1}{R^2} + \left( 2 - \frac{R_1^2}{R^2} \right) - \frac{4}{3} \beta_1 \left( 1 - \frac{3 \chi}{R^2} \right) \right] \]

\[ + \beta_1 \left( 1 - \frac{2}{3} \frac{R_1^2}{R^2} \right) \]

\[ (22) \]
Flow rate in marginal plasma layer (23) can be obtained from equation (22).

When the plasma fluid in the marginal layer is represented by purely Newtonian constitutive relation \( \beta_1 = \alpha \) then the flow rate increases due to slip effect by the amount \( 2 \kappa \mu \delta R^2 \lambda \) as compared to the result of charmental [12] \( \delta = R - R_0 \) is the width of the layer. As low shear rate a constant plug flow exist near the axis in the core region and can be obtained from equation (17).

For only one phase flow of the casson fluid \( R_0 = R, \beta_1 = \beta_2 \), equation (20) takes the form,

\[
\frac{\nu_c}{\mu} = \frac{R \tau_R}{2 \mu} \left[ \frac{\kappa}{R^2} + \left( 1 - \frac{\sigma^2}{R^2} \right) - \frac{8 \beta \chi_c}{3} \left( 1 - \frac{\sigma^2}{R^2} \right) + \frac{2}{3} \beta \right] (24)
\]

Where \( \chi_c \) denotes the velocity of core fluid homogeneously.

The constant plug velocity \( \nu_p \) in this case can be obtained by putting plug radius \( \sigma = \beta R \) in above equation.

\[
\nu_p = \frac{R \tau_R}{2 \mu} \left[ \frac{\sigma}{2} + 1 + 2 \beta - \frac{1}{3} \beta \sigma^2 - \frac{8}{3} \beta \chi_c \right]
\]
The flow rate is obtained as

\[ Q = \int_{2\pi}^{\beta R} \nu_p \, d\gamma + \int_{2\pi}^{\beta R} \nu_c \, d\gamma \tag{25} \]

or

\[ Q = \pi R^3 \tau R \left[ \alpha + 1 - \frac{16}{7} \beta \frac{\lambda_2}{2} + \frac{7}{3} \beta - \frac{1}{2} \beta \frac{\lambda_1}{2} \right] \tag{26} \]

Which gives the result of OKa[13] for no slip condition at the wall. We can discuss the effect of slip on \( Q \) with the help of equation (26).

For two-phase flow with plug near the axis, the flow rate is obtained as

\[ Q = 2\pi \int_{0}^{R_f} \nu \, \lambda_1 \, d\gamma + \int_{0}^{R_e} \nu \, \lambda_2 \, d\gamma \tag{27} \]

or

\[ Q = \pi R^3 k \tau R + \pi R^3 \tau R \left[ 1 - \frac{16}{7} \beta \frac{\lambda_2}{2} + \frac{7}{3} \beta - \frac{1}{2} \beta \frac{\lambda_1}{2} \right] \]

\[ - \pi R^3 \tau R \cdot \frac{y}{R^2} \left( 1 + \beta \frac{\lambda_2}{2} \right) \]

\[ + \pi R^3 \tau R \cdot \frac{y}{R^2} \left( 1 + \beta \frac{\lambda_1}{2} \right) \]
If $\mu_\alpha$ represent the apparent viscosity of the fluid then

$$\frac{1}{\mu_\alpha} = \frac{4}{n} \frac{Q}{R^3} \tau_r + \frac{1}{\mu_2} f_1(\beta_2) - \frac{1}{\mu_2} \frac{\gamma \delta f_2(\beta_2)}{R} + \frac{1}{\mu_1} \frac{\gamma \delta f_3(\beta_1)}{R}$$

(28)

Where

$$f_1(\beta_2) = 1 - \frac{16}{7} \beta_2 \frac{1}{2} + \frac{4}{3} \beta_2^2 - \frac{1}{2} \beta_2^4$$

$$f_2(\beta_2) = 1 + \beta_2 - 2 \beta_2 \frac{1}{2}$$

$$f_3(\beta_1) = 1 + \beta_1 - 2 \beta_1 \frac{1}{2}$$

We may assume that the yield values $\tau_1$ and $\tau_2$ is so small that $\beta_1, \beta_2 << 1$. Under there assumption we discuss the effect of slip on apparent viscosity $\mu_\alpha$. From (28) we observe that the loss in apparent viscosity is caused due to increase in $\delta$ as $\frac{\mu_2}{\mu_1} \frac{\beta_2}{\gamma} \frac{R^3}{Q} > \frac{1 + \beta_2 - 2 \beta_2 \frac{1}{2}}{1 + \beta - 2 \beta \frac{1}{2}}$ for a constant slip parameter.
References


