CHAPTER VI

Study of dispersion processes in blood flow in narrow vessels

6.1 Introduction:

In biological sciences, the study of diffusivity of nutrients, metabolic products, drugs and other solutes is of utmost importance. Specially, many life giving materials mixed in the blood reach to the different parts of the body by the process of diffusion. Taylor (1953, 54) studied the dispersion process in Newtonian flow and discussed the effective dispersion coefficient with respect to the average speed of the flow, the radius of the tube and molecular diffusion coefficient. Many authors as Patel and Sirs (1983), Federspial and Fredberg (1988), Rudraiah et. al (1986) have studied dispersion processes by taking different flow models. Shukla and Gupta (1981) studied the Taylor dispersion in Bingham Plastic fluid surrounded by Newtonian peripheral layer. Scottblair and spanner (1974), Charm and Kurland (1968), Bugliarello and Sevilla (1970) and many others have proposed that casson model has an edge over the other fluid models for the purpose of discussing blood behaviour.

In our problem we have proposed the flow model with casson fluid in the core region surrounded by a Newtonian plasma layer near the wall. Taylor's limiting condition and Ficks law of diffusion are used for finding out the solution of the problem. The effective dispersion coefficient with which the solute disperses across a plane moving with mean speed of the medium is found to be decreased with respect to the yield stress and molecular diffusion coefficient whereas a reciprocal behaviour is observed with respect to the viscosity of the casson fluid.
Mathematical analysis:

Consider an incompressible steady viscous flow of casson fluid through a circular tube of radius $R$. During motion, the viscosity and yield stress of fluid vary along the radial direction. The constitutive equation in one dimensional form of casson fluid is:

$$\tau = \tau_y K(r) + \eta \frac{\partial}{\partial r} \gamma; \quad \tau > \tau_y K(r)$$

$$\dot{\gamma} = 0; \quad \tau \leq \tau_y K(r)$$

where $\eta(r)$ is the viscosity, $\tau_y K(r)$ is the yield stress, $\gamma$ is the strain rate. $\eta(r)$ and $K(r)$ are assumed to be decreasing function of $r$. The region $0 < r < R$ is divided into a central region $0 \leq r < R_0$ of casson fluid with a plug radius $R_0$, $0 < r < R_0$ and by a peripheral region of another casson fluid ($R_1 < r < R$). The function $\eta(r)$ and $K(r)$ are assumed to be

$$\eta(r) = \eta_1; \quad K(r) = 1 \text{ when } 0 \leq r < R_1$$

$$\eta(r) = \eta_2; \quad 0 \leq K(r) = K < 1 \text{ when } R_1 < r < R$$

Where $\eta_1$ and $\eta_2$ are the viscosities of the central and the peripheral fluids. In case of blood, the peripheral layer is a Newtonian fluid and thus the function $K(r) = k = 0$ in $R_1 < r < R$.

For one dimensional steady laminar flow in cylindrical co-ordinate system $(r, x, \theta)$ whose origin lies on the axis of the vessel, the equation of motion can be written as

$$\frac{\partial p}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r \tau) = 0$$

Integrating equation (3) and using the boundary condition $\tau = 0$ at $r=0$, we obtain,

$$\tau = \frac{\partial p}{\partial x} \frac{r}{2}$$

For plug boundary equation (4) gives

$$\tau_y K(r) = \frac{\partial p}{\partial x} \frac{R_0}{2}$$
In view of equations (4), (5) and boundary condition \(V_x=0\) at \(r=R\), the integration of equation (1) gives

\[
V_x = \left( \frac{-R^2 \frac{dp}{dx}}{2\eta_1} \right) \left[ \int_y y \frac{dy}{\eta(y)} + \frac{y_0}{\eta(y)} \int_y y \frac{K(y) \frac{dy}{y}}{\eta(y)} + \frac{2y_0}{\eta(y)} \int_y \frac{K(y) \frac{dy}{y}}{\eta(y)} \int_y y \frac{dy}{\eta(y)} \right] \quad ; \quad y_0 \leq y \leq 1
\]  
(6)

Where \(y = \frac{r}{R} \), \(y_0 = \frac{R_0}{R} \), \(\eta(y) = \eta(y) \right)

For \(y = y_0\), the plug velocity \(V_x\) is calculated from equation (6) as

\[
V_x = \left( \frac{-R^2 \frac{dp}{dx}}{2\eta_1} \right) \left[ \int_y y \frac{dy}{\eta(y)} - \frac{y_0}{\eta(y)} \int_y y \frac{K(y) \frac{dy}{y}}{\eta(y)} - \frac{2y_0}{\eta(y)} \int_y \frac{K(y) \frac{dy}{y}}{\eta(y)} \int_y y \frac{dy}{\eta(y)} \right] \quad ; \quad 0 \leq y \leq y_0
\]  
(8)

The average velocity \(\bar{V}\), of the fluid is obtained as

\[
\bar{V} = \frac{1}{\pi R^2} \int_0^R 2\pi r V_x \, dr = -\int_0^{y_0} \frac{dy}{y} \frac{dV_x}{dy}
\]

Using equations (6) and (8) in above equation, we get

\[
\bar{V} = \left( \frac{R^2 \frac{dp}{dx}}{2\eta_1} \right) \left[ \int_y y^3 \frac{dy}{\eta(y)} + \frac{y_0}{\eta(y)} \int_y y^2 \frac{K(y) \frac{dy}{y}}{\eta(y)} - \frac{2y_0}{\eta(y)} \int_y y \frac{K(y) \frac{dy}{y}}{\eta(y)} \int_y y^2 \frac{dy}{\eta(y)} \right] \quad ; \quad 0 \leq y \leq y_0
\]  
(9)

Now, the equation for concentration is

\[
\frac{\partial c}{\partial t} + V_x \frac{\partial c}{\partial x} = D(r) \frac{\partial}{\partial r} \left[ r \frac{\partial c}{\partial r} \right] \quad \text{(10)}
\]

Where \(D(r)\) is the molecular diffusion coefficient assumed to vary symmetrically along \(r\) direction.

Now introducing the non-dimensional quantities

\[
\theta = t, \quad \bar{t} = L, \quad \xi = x - \bar{V} t, \quad y = r, \quad \bar{D} = \frac{D}{D_1}
\]

Where \(L\) and \(D\) are the typical values of the length along the tube and molecular diffusion coefficient, respectively. Equation (10) is transformed to give flow of solute relative to a plane moving with the mean speed of the flow as
\[ \frac{1}{l} \frac{\partial c}{\partial \theta} + (V_x - \bar{V}) \frac{\partial c}{\partial y} = D_1 \frac{\partial}{\partial y} \left( y \frac{\partial c}{\partial y} \right) \]

Applying Taylor's limiting conditions (\( \frac{\partial c}{\partial y} \) is independent of \( y \) and \( \frac{\partial c}{\partial \xi} = 0 \)), equation (11) gives

\[ \frac{1}{y} \frac{\partial}{\partial y} \left( \bar{D} y \frac{\partial c}{\partial y} \right) = F \left( g(y) \right) \quad (12) \]

Where

\[ g(y) = \int_{y_0}^{1} y dy \left( \frac{y}{\bar{D}y} \right) K(y) \frac{dy}{\eta(y)} - \frac{2y_0^{0.5}}{K_{y}(y_0)} \int_{y_0}^{1} y^{0.5} K_y(y) \frac{dy}{\eta(y)} \quad (13) \]

\[ G = \frac{y^2 K(y) \frac{dy}{\eta(y)}}{2y_0^{0.5} K_{y}(y_0) \frac{dy}{\eta(y)}} \quad (14) \]

\[ F = \int_{-1}^{1} \frac{1}{2n_1 \frac{R}{D_1 \lambda x}} \frac{\partial c}{\partial \xi} \frac{dx}{dy} \quad (15) \]

Integrating equation (12) and using conditions \( c = 0 \) at \( \beta = 0; \frac{\partial c}{\partial \beta} = 0 \) at \( \beta = 1 \), we obtain

\[ C = C_0 - F M(y) \quad (16) \]

Where

\[ M(y) = \int_{y_0}^{1} y g(y) \frac{dy}{y} \quad (17) \]

The average volumetric flow rate \( \bar{Q} \) is given by

\[ \bar{Q} = 2 \int_{0}^{1} y (V_x - \bar{V}) c \frac{dy}{dy} \]

Using equations (6), (8), (9) and (16) in the above equation we get

\[ \bar{Q} = -2\sqrt{V} R^2 \sqrt{2} \int_{0}^{1} M(y) g(y) \frac{dy}{dy} \]

\[ \frac{D_1 L}{\int_{y_0}^{1} y^3 K(y) \frac{dy}{\eta(y)} - 2y_0^{0.5} \int_{y_0}^{1} y^{0.5} K_{y}(y) \frac{dy}{\eta(y)}} \left( \frac{1}{2y_0^{0.5} K_{y}(y_0) \frac{dy}{\eta(y)}} \right)^2 \]

\[ (18) \]
Compare equation (18) with the Fick's law of diffusion

\[ J^* = -D^*_c \frac{\partial c}{\partial x} \]

The solute disperses relative to a plane moving with the mean speed of flow with an effective dispersion coefficient \( D^* \), given by

\[ D^* = 2R^2V \int_0^1 y M(y) g(y) \, dy \]

\[ D^*_1 \left[ \int_0^1 y^3 \, dy + y_0 \int_0^1 y^2 \kappa(y) \, dy + 2y_0 \int_0^1 y^{5/2} \kappa^3(y) \, dy \right] \]

The expression for \( D^* \) is applicable for any general functions \( \tilde{\eta}(y), \bar{D}(y) \) and \( K(y) \).

Now, for the effect of a peripheral layer around a Casson fluid, the boundary condition for the functions \( \tilde{\eta}(y), \bar{D}(y) \) and \( K(y) \) may be assumed as follows:

\[ \tilde{\eta}(y) = \begin{cases} 1 & ; 0 \leq y \leq y_1 = \frac{R_1}{R} \\ \eta_2 = \eta_1 & ; y_1 \leq y \leq 1 \end{cases} \]

\[ \bar{D}(y) = \begin{cases} 1 & ; 0 \leq y \leq y_1 \\ D_2 = D_1 & ; y_1 \leq y \leq 1 \end{cases} \]

\[ K(y) = \begin{cases} 1 & ; 0 \leq y \leq y_1 \\ 0 & ; y_1 \leq y \leq 1 \end{cases} \]

(20)

(21)

(22)

Where \( y_1, \eta_1 \) and \( D_1 \) are the radius, viscosity and molecular diffusion coefficient of Casson fluid respectively and \( \eta_2, D_2 \) are peripheral layer fluid.

Using equations (20), (21) and (22) in equations (13), (14), (17) and (19) we obtain

\[ D^* = \frac{R^2 \nabla^2}{D_1} \]

(23)
Where

\[
H = \frac{S_0 (S_1 - S_2)}{\left[ 1 - (1-n) y_1^4 - y_0^{\mu y} \{y_0^{7/2} - 4y_1^3 (7y_0^{1/4} - 12 y_1^{1/4})\} \right]}
\]

\[
S_0 = 1 - (1-n) (2-y_1^2) y_1^2 - y_0^{2\mu} (14-y_0^2) + 4y_0 y_1^3 (3-y_1^2)
\]

\[
S_1 = y_1^2 \log_2 y_1 - y_1^2 - y_1^4
\]

\[
S_2 = S_1 + 3
\]

6.3 Discussion

For fixed mean speed of the flow, the effects of \(\mu\), \(y_0\) and \(\nu\), namely the viscosity, yield stress and molecular diffusion coefficient on \(H\) are seen from the tables (I), (II) and (III). We find that the magnitude of \(H\) decreases as the yield stress and molecular diffusion coefficient increase but increases with respect to the coefficient of viscosity.

### Table 7

Effects on \(\mu\), \(\nu\) and \(y_0\) on \(H\), (\(\nu = 1.1, y_1 = 0.90\))

<table>
<thead>
<tr>
<th>(y_0)</th>
<th>(\mu = 0.20)</th>
<th>(0.40)</th>
<th>(0.60)</th>
<th>(0.80)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>-0.0480</td>
<td>0.0873</td>
<td>-0.1316</td>
<td>-0.1710</td>
</tr>
<tr>
<td>0.05</td>
<td>-0.0321</td>
<td>0.0680</td>
<td>-0.1034</td>
<td>-0.1360</td>
</tr>
<tr>
<td>0.09</td>
<td>-0.0219</td>
<td>-0.0450</td>
<td>-0.0689</td>
<td>-0.0928</td>
</tr>
</tbody>
</table>
\textbf{Table III}

\begin{tabular}{|c|c|c|c|c|}
\hline
\multicolumn{2}{|c|}{H} & \hline
\hline
\textit{y}_0 & \textit{\mu}=0.20 & 0.40 & 0.60 & 0.80 \\
\hline
0.03 & 0.0401 & 0.0857 & -0.1292 & -0.1679 \\
0.05 & 0.0315 & 0.0668 & -0.1015 & -0.1336 \\
0.09 & 0.0215 & 0.0442 & -0.0677 & -0.0904 \\
\hline
\end{tabular}

\textbf{Table IV}

\begin{tabular}{|c|c|c|c|c|}
\hline
\multicolumn{2}{|c|}{H} & \hline
\hline
\textit{y}_0 & \textit{\mu}=0.20 & 0.40 & 0.60 & 0.80 \\
\hline
0.03 & 0.0395 & 0.0846 & -0.1275 & -0.1657 \\
0.05 & 0.0311 & 0.0659 & -0.1002 & -0.1318 \\
0.09 & 0.0212 & 0.0436 & -0.0668 & -0.0892 \\
\hline
\end{tabular}

Hence in view of equation (23) and above discussion it can be seen that for a given mean speed of the flow, the effective dispersion coefficient \( D' \) decreases with increasing \( y_0 \) and \( \delta \) while it increases with respect to \( \mu \).