Chapter 4

Two-species Bose-Einstein Condensate

4.1 Introduction

In this chapter we describe the utility of the NLSE, which is known to be GP equation, in aptly describing the dynamics of the Bose-Einstein condensate (BEC) as a mean field theoretic approximation. First we describe briefly the single component BEC, by deriving various interesting properties of the condensates. Then we explore some interesting physical features of TBEC. We also present the numerical simulations whenever, the GP equation is intractable for the analytical methods, for various coupling strengths. However, we have obtained exact solutions for various parameter values of the low density limit.

The phenomenon of trapped bosons (mainly alkali atoms like H, Li, Na, Rb) undergoing BEC has been the subject of intense research in recent times. The dynamics of the condensed phase is effectively captured by the GP equation which has a cubic nonlinearity, originating from the effective point-interaction of the trapped atoms at ultra-cold temperatures. The fact that these type of nonlinear equations are ubiquitous in nonlinear optical systems allows one to borrow many results from the same and apply it
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Elongated BEC has also been realized in quasi-one dimensional regime, under a weaker longitudinal confinement. This quasi-one dimensional cylindrical BEC has been extensively studied, particularly for the possibility of observing solitary wave solutions. A number of experiments have observed dark solitons, in case of repulsive interactions. This has also led to the possibility of observing the Lieb mode as compared to the Bogoliubov modes, which are analogous to sound modes in elastic media. The use of the Feshbach resonance on these systems has led to the possibility of tuning the interaction strength from repulsive to attractive, with desired strength. Recently, solitons and soliton trains have been seen in the attractive BEC. Although the presence of traps is necessary to stabilize the BEC in higher dimensions; the delicate balance of the nonlinearity and the dispersion can produce stable localized structures in quasi-ID geometry. Recently such solitons and soliton trains have been observed in the single component BEC of $^7$Li. The soliton trains particularly show interesting behavior. In the weakly attractive regime, the pulse trains have been observed with the neighboring ones having a phase difference of $\pi$. The mechanism of production of these type of nonlinear waves has also received considerable attention in recent times, since the known procedures like the modulation instability have been found to be inadequate.

TBEC have been recently produced and have been studied in both quasi-1D and higher dimensions. Although soliton solutions have been studied quite extensively, using the connection of the GP equation with the Manakov system [1], the solitary train solutions have not been investigated in detail. Hence, a good part of this chapter attempts to analyze exhaustively the solitary wave solutions of the TBEC in the quasi-1D, in particular the low density regime, and strong coupling limit. The former is studied analytically, whereas the latter is studied numerically.
4.2 Bose-Einstein condensation

Christened in the name of its discoverer, Albert Einstein in 1925, BEC is a paradigm of quantum statistical phase transitions. It occurs when the mean particle separation is comparable with the de Broglie wavelength and is manifested by an abrupt growth in the population of the ground state of the potential confining the Bose gas. Thus, the experimental realization of BEC requires creating a sample of Bosonic particles at extremely low temperatures and high densities was a long standing problem. Finally in 1995, the physicists from JILA and MIT, created BEC in $^{87}$Rb, and $^{23}$Na, by combining the laser cooling and evaporative cooling techniques.

Consider an ideal Bose gas consisting of $N$ noninteracting particles of mass $m$, contained in a box of volume $V$, characterized by the de Broglie wavelength $\lambda_{dB} = (2\pi^2\hbar^2/m\kappa_B T)^{1/2}$, where $T$ is the temperature and $\kappa_B$ is the Boltzmann constant. For $\lambda_{dB} \ll d$, where $d$ is the mean inter particle separation; the quantum effects are negligible and the particles behave classically. At high temperatures, therefore the momenta $(p)$ of the particles are distributed according to the classical Maxwell-Boltzmann distribution. For low temperatures, $\lambda_{dB}$ begins to approach $d$ and the quantum effects become evident. At critical $T$, $\lambda_{dB} = d$. In the case of Bosons, since there is no restriction on the number of particles that can stay in any state, the quantum behavior is manifested by an increase in the occupation of states at small $p$. As the temperature is lowered further to zero, the quantum effects are dominant, and a macroscopic quantum state is formed with $p = 0$. This is referred to as Bose-Einstein condensation.

4.3 Single-component BEC

To describe the dynamics of the weakly interacting BEC, the GP equation [2] has been shown to be an appropriate theoretical frame work [3, 4, 5, 6,
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7, 8, 9, 10]. At zero temperature this equation can be written as

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V + U_0 |\psi|^2 \right] \psi,$$

(4.1)

where $U_0 = fofra/rn$, is the effective atom-atom interaction term. For $a < 0$, a bright soliton will emerge as the solution, while for the case $a > 0$, a dark soliton is a solution. Dark solitons in cigar-shaped BEC of $RB^{87}$ were created by phase imprinting method [11]. Also, the NIST group has investigated soliton like states in nearly spherical BEC’s in Na$^{23}$ [12]. And, bright solitons were reported in BEC of Lithium atoms [13]. However, the GP equation could not be applied to describe BEC in liquid He$^4$, because of the large interaction and depletion.

Below, we study the GP equation to describe some interesting properties of single component BEC, following the derivation given in Refs. [14, 15, 16]. We study in detail, both the low density limit as well as high density limit using TF approximation. We also present the exact periodic solutions in attractive case.

We assume that a Bose-Einstein condensed atoms are confined in a cylindrical harmonic potential $V = m\omega_1^2(x^2 + y^2)/2$, and the wave motion is along $z$ axis. There is no confinement along $z$ axis. Further, we assume that the transverse dimension of the cloud is sufficiently small, such that the problem is reduced to a one dimensional one, and the solitary pulse can be described by a local velocity, $v(z)$, and a local density of particles per unit length, $\sigma(z)$, $\sigma(z) = \int dx dy |\Psi(x,y,z)|^2$. Thus, the wavefunction can be written as $\psi(r,t) = f(z,t)g(x,y,\sigma)$, where $g$ is the equilibrium wavefunction for the transverse motion which depends on time implicitly through the time dependence of $\sigma$. Choosing $g$ to be normalized as $\int |g|^2 dxdy = 1$, and thus from the above equations, $|f|^2 = \sigma$.

**Weak coupling limit:** In this limit one can neglect the nonlinear term and solve the Schrodinger equation. This results in $|g|^2 = (\pi a_1^2)^{-1} e^{-\rho/a}$ and
noticing that \( a \pm = (\hbar/m\omega_\perp)^{1/2} \). Thus the equation for \( \varphi \) can be written as

\[
i\hbar \partial_t \varphi = -\frac{\hbar^2}{2m} \partial_x^2 \varphi + \hbar \omega_\perp (1 + 2a_{sc} |\varphi|^2) \varphi. \tag{4.2}
\]

By applying one more transformation

\[
w = f e^{i\omega_\perp(1+2a_{\sigma_0})t}, \tag{4.3}
\]

\[
i\hbar \partial_t w = -\frac{\hbar^2}{2m} \partial_x^2 w + \hbar \omega_\perp 2a( |w|^2 - \sigma_0)w, \tag{4.4}
\]

where \( a \) is the scattering length, and the chemical potential is \( \mu = \sigma_0 \hbar \omega_\perp 2a \). By writing \( w = \sqrt{\sigma} e^{i(\phi + \omega t)} \), we can separate the real and imaginary parts of Eq. (4.4) and obtain

\[
v = u + \frac{c}{\sigma}, \tag{4.5}
\]

where \( c \) is the integration constant, and

\[
2\sigma \sigma'' - \sigma'^2 + \Gamma \sigma^2 + g \sigma^3 - \delta = 0, \tag{4.6}
\]

where the effective 1D coupling \( \hbar \omega_\perp a \) has been taken to be weakly attractive; \( g = -16m\omega_\perp |a|/\hbar\Gamma = 4m^2 a^2/\hbar^2 - 8m\omega/\hbar + 4\mu m/\hbar^2 \). Furthermore, we have assumed that both \( \sigma \), and \( \phi \) are functions of \((z - ut)\). In arriving at Eq. (4.6), use was made of the superfluid velocity \( v = (\hbar/m) \partial_z \phi \).

One finds both chirped and non-chirped solutions of the above equation. These are in the form of elliptic functions. The repulsive and attractive cases show distinct behavior for certain values of the modulus parameter \( \tilde{m} \). Particularly in the repulsive case, for the soliton solution, the Lieb mode has been reported [16]. For the attractive case, the solution can be written as

\[
\sigma = A + Bcn^2(z/\alpha \tilde{m}). \tag{4.7}
\]

In the limit \( m = 1 \), we find that \( B = 8/ga^2 \), and \( A = -(8/a^2 + 2\Gamma)/3g \).
This equation is solved numerically for the attractive case. The same has been depicted in Fig 4.1.

Figure 4.1: Plot depicting the amplitude of single component BEC (attractive case) for \( u = 0.8, g = 1.0, \) and \( \mu = 0.2. \)

**Strong coupling limit:**

For large nonlinearities \( U_0, \) we use TF approximation. Then it is observed that for the harmonic trap, the population density goes to zero at \( r = \pm 2/\alpha \sqrt{\mu}, \) where \( \alpha^2 = 2m \omega \). Consequently, the normalization condition

\[
\int |g|^2 \, dx \, dy = 1, \tag{4.8}
\]

implies that \( \mu = 4\hbar \omega \alpha^{3/2} |f|. \) In light of these results we can show that

\[
\hat{h} \partial_t f = -\frac{\hbar}{2m} \partial_z^2 f + (2\hbar \omega \alpha^{1/2} |f|) f. \tag{4.9}
\]

By applying another transformation \( f = \psi e^{2i\omega t}, \) we obtain the GP equation for strong coupling case

\[
\hat{h} \partial_t w = -\frac{\hbar^2}{2m} \partial_z^2 w + \hbar \omega \alpha |w| - |w_0| w.
\]

We again write \( w = \sqrt{\sigma} e^{i\phi + \omega}, \) and separate the real and imaginary parts. The real part is given by

\[
\frac{d^2 \sigma}{dz^2} = \frac{1}{2\sigma} \left( \frac{d \sigma}{dz} \right)^2 + 2m^2 c^2 \sigma - 2m^2 u^2 - g\sigma^{3/2} + g\sigma_0^{1/2} + \frac{4m \omega}{\hbar} \sigma. \tag{4.11}
\]

This equation is solved numerically for the attractive case. The same has been depicted in Fig 4.1.
4.4 Two-component BEC

We study the solitary waves of TBEC in a quasi-1D geometry in particular, the soliton trains are investigated in detail, in light of their recent observation in one-component BEC. We study repulsive and attractive nonlinearities, both in the weak and strong coupling regimes, analytically and numerically. Chirped and nonchirped soliton trains, respectively, with or without a constant background are obtained analytically for the weak coupling case. A number of interesting features of the periodic and localized solutions are observed. In the attractive strong coupling case, for certain parameter values, we find that one of the species shows autocorrelation traces due to modulational instability, akin to a two laser system (Nd:YAG, and InGaAsP).

Much theoretical work has already gone into studying the ground state solutions of the coupled GP equations describing multi-component BECs [17, 18]. TBEC has been observed experimentally [19], where the two hyperfine levels of the $^{87}\text{Rb}$ act as the two components. In this case, a fortuitous coincidence in the triplet and singlet scattering lengths has led to the suppression of exoergic spin-exchange collisions, which lead to heating and resultant loss of atoms. A number of interesting features, like the preservation of the total density profile and coherence for a characteristic long time, in the face of the phase-diffusing couplings to the environment and the complex relative motions, point to the extremely interesting dynamics of the TBEC. The effects of spatial inhomogeneity, three-dimensional geometry, and dissipation are examined, in light of the dark-bright solitons in repulsively interacting TBEC in $^{23}\text{Na}$ and $^{87}\text{Rb}$ [20]. Dark solitons in TBEC for a miscible case have also been analyzed recently [21]. TBEC in two different atoms has been produced in a system of $^{41}\text{K}$ and $^{87}\text{Rb}$, in which sympathetic cooling of the Rb atoms was used to condense the K atoms [22]; as also in $^{7}\text{Li}-^{133}\text{Cs}$ [23]. Keeping in mind the
importance of the matter wave solitons for Bose-Einstein condensates [2], we present exact solutions of the generic TBEC model, for the low density limit. However, for the strong coupling limit, we solved the coupled GP equations numerically for both attractive and repulsive cases.

**Derivation of the TBEC coupled GP equations:** To derive the coupled GP equations of TBEC we assume that both the components of are confined in a cylindrical harmonic potential \( V = m \omega_r (x^2 + y^2)/2 \). By assuming that there is no confinement along \( z \) axis, we can reduce the problem to a quasi-1D one. Thus the double component wavefunction can be written as \( \Psi_a = f_a(z, t)g(x, y, \sigma(z, t)), \Psi_b = f_b(z, t)g(x, y, \sigma(z, t)) \), such that

\[
\sigma(z) = \int dx dy [|\Psi_a|^2 + |\Psi_b|^2] = |f_a|^2 + |f_b|^2. \tag{4.12}
\]

In Eq. (4.12), we have used the normalization condition on \( g \). The action for the TBEC can be written as:

\[
S = \int \left[ i \hbar \psi_a \frac{\partial \psi_a}{\partial t} + i \hbar \psi_b \frac{\partial \psi_b}{\partial t} - \frac{\hbar^2}{2m} (\nabla \psi_a, \nabla \psi_a) - \frac{U_0}{2} (|\psi_a|^2 + |\psi_b|^2)^2 - \frac{\hbar^2}{2m} (\nabla \psi_a, \nabla \psi_a) - V(x, y)(|\psi_a|^2 + |\psi_b|^2) \right] dt. \tag{4.13}
\]

Minimizing Eq. (4.13) w.r.t \( g^* \), we find that

\[
\frac{\hbar^2}{2m} \nabla^2 g + V g + U_0 \left[ (|f_a|^2 + |f_b|^2)|g|^2 \right] g + \frac{\hbar^2}{2m(|f_a|^2 + |f_b|^2)} \left[ |\frac{\partial f_a}{\partial z}|^2 + |\frac{\partial f_b}{\partial z}|^2 \right] g = \mu(\sigma)g \tag{4.14}
\]

where \( \mu \) is the chemical potential. We neglect the last term in Eq. (4.14) since the characteristic length of the pulses is sufficiently long that it is negligible in all cases of interest. For the same reason the term \( f_a \nabla g \) is equal to

\[
f_a \nabla g + \dot{z} f_a \frac{\partial g}{\partial z} \approx f_a \nabla g, \tag{4.15}
\]

where

\[
\nabla \perp = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}. \tag{4.16}
\]
with $x, y$ and $z$ being the unit vectors in the directions $x, y$ and $z$ respectively. Thus, Eq. (4.14) reduces to

$$-\frac{\hbar^2}{2m} \nabla_x^2 g + V g + U_0[|f_a|^2 + |f_b|^2] |g|^2 g = \mu(\sigma) g$$  \hspace{1cm} (4.17)$$

Now minimizing Eq. (4.13) w.r.t $f^*_a$ we find that

$$i\hbar \frac{\partial f_a}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 f_a}{\partial z^2} + \left( \frac{\hbar^2}{2m} \int |\nabla_y g|^2 \, dxdy \right) f_a + U_0 \left( \int |g|^4 \, dxdy \right) (|f_a|^2 + |f_b|^2) f + \left( \int |g|^2 V \, dxdy \right) f_a.$$  \hspace{1cm} (4.18)$$

Similarly, one can obtain the equation for $f_b$.

For the homogeneous medium assuming $f_a = \sigma_a^{1/2} \exp(i\phi_a)$, and $f_b = \sigma_b^{1/2} \exp(i\phi_b)$ and defining the velocity field associated with $f_a$, and $f_b$ as $v_a = h/m \partial_z \phi_a$, and $v_b = h/m \partial_z \phi_b$ we obtain the following equation for $\mu$:

$$\mu(\sigma) = \left( \frac{\hbar^2}{2m} \int |\nabla_y g|^2 \, dxdy \right) + \left( \int |g|^2 V \, dxdy \right) + U_0 \left( \int |g|^4 \, dxdy \right) (|f_a|^2 + |f_b|^2).$$  \hspace{1cm} (4.19)$$

**Weak coupling limit:**

To derive the coupled GP equations in the low density limit, we assume that the nonlinearity is weak, and hence only concentrate on the linear part of the Schrodinger equation satisfied by $g$, in a harmonic trap

$$-\frac{\hbar^2}{2m} \nabla_x^2 g + V g = \mu g$$

This implies

$$|g| \propto \exp \left( -\frac{x^2 + y^2}{2a_{\perp}^2} \right).$$

From Eq. (4.19) we find that

$$\mu = \left( \frac{\hbar^2}{2m} \int |\nabla_y g|^2 \, dxdy \right) + \left( \int |g|^2 V \, dxdy \right);$$

$$|g|^2 = \exp \left( -\frac{x^2 + y^2}{2a_{\perp}^2} \right).$$
We also note that \( a \pm = (\hbar/m\omega_\perp)^{1/2} \). After performing the integrals, we obtain

\[
\mu = (1 + 2a\sigma)\hbar\omega;
\]

where the first term corresponds to ground state energy and the second term correspond to the interaction energy. Thus the equation for \( f_a \) can be written as

\[
i\hbar \partial_t f_a = -\frac{\hbar^2}{2m} \partial_x^2 f_a + \hbar\omega_\perp \left[ 1 + 2a_{sc}(|f_a|^2 + |f_b|^2) \right] f_a.
\]

By performing these transformations

\[
f_a = w_a e^{i\omega_\perp (1+2a\sigma_0)t},
\]

and

\[
f_b = w_b e^{-i\omega_\perp (1+2a\sigma_0)t},
\]

we obtain the GP equations for one species of the TBEC, in the low density limit as

\[
i\hbar \partial_t w_a = -\frac{\hbar^2}{2m} \partial_x^2 w_a + \varepsilon(|w_a^2| + |w_b^2|)w_a - \epsilon_a w_a.
\]

By implementing the same procedure we can obtain the GP equation for the second component also as

\[
i\hbar \partial_t w_b = -\frac{\hbar^2}{2m} \partial_x^2 w_b + \varepsilon(|w_a^2| + |w_b^2|)w_b - \epsilon_b w_b.
\]

Here, \( \varepsilon = 2\hbar\omega_\perp a, \ \epsilon_a \) and \( \epsilon_b \) are the chemical potentials of the two species respectively. This scenario is similar to the much studied Manakov system [1, 24], which is integrable system. By writing \( w_a = \sqrt{\sigma_a}e^{i(\phi_a + \omega_\perp t)}, \) and \( w_b = \sqrt{\sigma_b}e^{i(\phi_b + \omega_\perp t)}; \) we can separate the real and imaginary parts of Eqs. (4.20) and (4.21), and obtain the two equations

\[
v_a = u + \frac{c_1}{\sigma_a},
\]

and

\[
v_b = u + \frac{c_2}{\sigma_b}
\]
where \( c_1 \) and \( c_2 \) are the integration constants. We assumed that both \( \sigma \), and \( (p \) are functions of \((z - ut)\). Use was also made of the superfluid velocity \( v_a = \frac{\hbar}{m} \partial_z \phi_a \) similarly, for \( v_b \). Plugging the above results into the real parts one obtains

\[
\frac{d^2 \sigma_a}{dz^2} = \frac{1}{2 \sigma_a} \left( \frac{d \sigma_a}{dz} \right)^2 + \frac{4M\varepsilon}{h^2} (\sigma_a + \sigma_b) \sigma_a - \frac{4M}{h^2} \epsilon_a \sigma_a - \frac{2M^2u^2}{h^2} \sigma_a + \frac{2M^2c_1}{h^2 \sigma_a} - \frac{4M\omega_a}{h} \sigma_a, \tag{4.22}
\]

\[
\frac{d^2 \sigma_b}{dz^2} = \frac{1}{2 \sigma_b} \left( \frac{d \sigma_b}{dz} \right)^2 + \frac{4M\varepsilon}{h^2} (\sigma_a + \sigma_b) \sigma_b - \frac{4M}{h^2} \epsilon_b \sigma_b - \frac{2M^2u^2}{h^2} \sigma_b + \frac{2M^2c_1}{h^2 \sigma_b} - \frac{4M\omega_b}{h} \sigma_b. \tag{4.23}
\]

One finds both chirped and non-chirped \((c_i \neq 0)\) solutions of the above coupled equations. These are in the form of elliptic functions. The repulsive and attractive cases show distinct behavior for certain values of the modulus parameter \( fn \).

### 4.5 Exact solutions

In this section we present the exact periodic solutions of the coupled equations (4.20) and (4.22) in terms of Jacobi elliptic functions with appropriate modulus parameter. It is well-known that these elliptic functions interpolate between the trigonometric and the hyperbolic functions for \( fn = 0 \), and \( m = 1 \) respectively. We start with a more general solution of Eqs. (4.22) and (4.22), which, for various limiting conditions of the parameter values, yields traveling solutions:

\[
\sigma_a = A + B\gamma^2 \text{cn}^2(\alpha \gamma(z - ut), \tilde{m}),
\]

\[
\sigma_b = C + D\gamma^2 \text{sn}^2(\alpha \gamma(z - ut), fn).
\]

Then it is clear that the coefficients of \( \text{cn}^n(z - ut, \tilde{m}) \), and \( \text{sn}^n(z - ut, \tilde{m}) \) for \( n = 0, 2, 4, 6 \) respectively, can be set to zero to reduce the problem to a set of eight algebraic equations, and obtain the solution. The identities satisfied by these cnoidal functions are handy in finding the solutions.
The consistency conditions for both $\sigma_a$, and $\sigma_b$ are:

$$2AB\alpha^2\gamma^4(1 - m) + A^2\alpha_2 + \frac{2M^2u^2}{h^2} - A^2 + \frac{4M\omega_a}{h} A^2 - \Gamma A^2 C - \Gamma A^3 - \Gamma A D \gamma^2 - \delta = 0,$$

$$4B\alpha^2\gamma^2(2m - 1) + \Gamma AD + 2\alpha_2 B + \frac{4M^2u^2}{h^2} - B + \frac{8M\omega_a}{h} B - 2\Gamma BC - 2\Gamma BD \gamma^2 - 3\Gamma AB = 0,$$

$$6A\alpha^2m - 2B\alpha^2\gamma^2(2m - 1) - 2\Gamma AD - \frac{2M^2u^2}{h^2} B - B\alpha_2 - \frac{4M\omega_a}{h} B + \Gamma BC + \Gamma BD \gamma^2 + 3\Gamma AB = 0,$$

$$4\alpha^2m - \Gamma D + \Gamma B = 0.$$

$$2CD\alpha^2\gamma^4 + C^2\alpha_2 + \frac{2M^2u^2}{h^2} - C^2 + \frac{4M\omega_b}{h} C^2 - \Gamma C^2 A - \Gamma C^3 - \Gamma BC^2 \gamma^2 - \delta_1 = 0,$$

$$-4D\alpha^2\gamma^2(m + 1) + \Gamma BC + 2\alpha_2 D + \frac{4M^2u^2}{h^2} D + \frac{8M\omega_a}{h} D - 2\Gamma AD - 2\Gamma BD \gamma^2 - 3\Gamma CD = 0,$$

$$6C\alpha^2m - 2D\alpha^2\gamma^2(m + 1) + 2\Gamma BC + \frac{2M^2u^2}{h^2} D + D\alpha_1 + \frac{4M\omega_b}{h} D - \Gamma AD - \Gamma BD \gamma^2 - 3\Gamma DC = 0,$$

$$4\alpha^2m - \Gamma D + \Gamma B = 0.$$

Here, $\Gamma = \frac{4M}{h^2}$, $\alpha_2 = \frac{4M\omega_a}{h^2}$, $\alpha_1 = \frac{4M\omega_b}{h^2}$, $\delta = \frac{2M^2\gamma^2}{h^2}$, and $\delta_1 = \frac{2M^2\gamma^2}{h^2}$.

**Without chirping.**

Case(i): Hyperbolic solutions.

For $A = 0$, and $C = 0$ imply $c_1 = 0$, and $c_2 = 0$. For $m = 1$ one obtains

$$\sigma_a = B\gamma^2 \text{sech}^2(\alpha \gamma(z - ut)),$$

$$\sigma_b = D\gamma^2 \tanh^2(\alpha \gamma(z - ut)),$$

where

$$D = \frac{2\alpha^2\gamma^2 + \frac{4M\omega_a}{h^2} + \frac{4M\omega_b}{h}}{\Gamma \gamma^2},$$
and

\[ B = \frac{-4\alpha^2 \gamma^2 + \frac{4M \epsilon_b}{h^2} + \frac{4M \omega_b}{h}}{\Gamma \gamma^2}. \]

The width of the solitons is constrained by the condition

\[ \alpha \gamma = \sqrt{\frac{2M}{h^2} (\epsilon_a - \epsilon_b) + \frac{2M}{h} (\omega_a - \omega_b)}. \]

Case(ii): Trigonometric solution.

For \( m = 0 \), it is observed that both the fields have same amplitudes. Furthermore, the solutions are constrained by the condition \( \epsilon_a + h \omega_a = \epsilon_b + h \omega_b \).

\[ \sigma_a = B \gamma^2 \cos^2(\alpha \gamma (z - ut)), \]
\[ \sigma_b = D \gamma^2 \sin^2(\alpha \gamma (z - ut)), \]

where

\[ B = D = \frac{-2\alpha^2 \gamma^2 + \frac{4M \epsilon_b}{h^2} + \frac{4M \omega_b}{h}}{\Gamma \gamma^2}. \]

Case(iii): Pure cnoidal function.

For any value of \( m \), one obtains the pure cnoidal solutions as

\[ \sigma_a = B \gamma^2 \cosh^2(\alpha \gamma (z - ut), \tilde{m}), \]
\[ \sigma_b = D \gamma^2 \sinh^2(\alpha \gamma (z - ut), \tilde{m}), \]

where

\[ B = \frac{-2\alpha^2 \gamma^2 (1 + m) + \frac{4M \epsilon_b}{h^2} + \frac{2M^2 u^2}{h^2} \frac{4M \omega_b}{h}}{\Gamma \gamma^2}, \]
\[ D = \frac{2\alpha^2 \gamma^2 (2m - 1) + \frac{4M \epsilon_b}{h^2} + \frac{2M^2 u^2}{h^2} \frac{4M \omega_b}{h}}{\Gamma \gamma^2}. \]

In the static case \( (u = 0) \), under the limits \( (\tilde{m} = 0, 1) \), the respective amplitudes of the solitary waves get modified leaving the relationship of the chemical potentials with the frequencies and the widths unaffected.
Figure 4.2: Plot depicting the amplitudes of TBEC (weak coupling) for $E = 0.4$, $\epsilon_a = 0.8$, $\epsilon_h = 0.02$, $\sigma_a^{0} = 1.0$, $\sigma_b^{0} = 0.5$, and $u = 0.8$

Chirping pulses.

Case(i): Hyperbolic case.

We assume that, there is no chirping of the $\sigma_b$ field, hence, $C = 0$. This case yields a hyperbolic solution with chirping in $\sigma_a$ field for $D = 1$.

$$\sigma_a = A + B\gamma^2 \text{sech}^2(\alpha\gamma(z - ut)),$$

$$\sigma_b = \gamma^2 \tanh^2(\alpha\gamma(z - ut)),$$

where, $A = (1 - \frac{4a^2}{\Gamma})(1 - \frac{4a^2}{\Gamma} - P)/Q$, with $P = -2a^2\gamma^2 + \frac{4M_a}{\hbar^2} + \frac{4M_a}{\hbar} + \frac{2M_a^2}{\hbar^2}$, $Q = 6a^2 - \Gamma$, and $B = 1 - \frac{4a^2}{\Gamma}$. 
Figure 4.3: Plot depicting the amplitudes of TBEC (weak coupling) for $E = -1.0$, $\epsilon_a = 0.8$, $\epsilon_b = 0.02$, $\sigma_a^0 = 1.0$, $\sigma_b^0 = 0.5$, and $u = 0.8$

Figure 4.4: Plot depicting the amplitudes of TBEC (weak coupling) for $E = -1.0$, $\epsilon_a = 0.8$, $\epsilon_b = 0.02$, $\sigma_a^0 = 1.0$, $rf = 1.0$, and $u = 0.8$
Strong coupling limit:

We take the TF limit. For large nonlinearities $U_0$, and thus for high particle density, and high self energy in real experiments for BEC, the kinetic term becomes small. Hence it can be neglected. We then get a simple analytic solution for $\Psi(r,t)$. It is observed that for the harmonic trap, the population density goes to zero at $r = \pm 2/\alpha \sqrt{\mu}$, where $\alpha^2 = 2m\omega_1$. Consequently, the normalization condition is

$$\int |g|^2 \, dx \, dy = 1.$$ 

In certain cases, for the eigenstates of a harmonic trap, the wavefunction can be written as

$$\Psi_a(r,t) = \Psi_a(r)e^{-i\mu t/\hbar}$$

with eigenvalue $\mu$ representing the chemical potential at zero temperature. Substituting this into the GP equation, we get

$$\mu \Psi_a = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V + U_0(|\Psi_a^2| + |\Psi_b|^2) \right] \Psi_a.$$ 

Thus, for harmonic trap, in the TF approximation we get

$$\mu = \alpha^2 \frac{r^2}{4} + U_0[|f_a|^2 + |f_b|^2]g.$$ 

The normalization condition implies that

$$2\pi \int_0^{2/\alpha \sqrt{\mu}} dr \frac{r}{\sigma U_0}[\mu - \alpha^2 \frac{r^2}{4}] = 1.$$ 

This integral is solved to give

$$\mu = 2\hbar \omega_1 (\sigma a)^{1/2}.$$ 

Substituting this into Eq. (4.18) yields

$$i\hbar \partial_t f_a = -\frac{\hbar}{2m} \partial_x^2 f_a + 2\hbar \omega_1 \alpha^{1/2}[|f_a|^2 + |f_b|^2]^{1/2} f_a.$$
4.6 Numerical results

The numerical evolution of these solutions reveal different behavior for attractive \((e < 0)\) and repulsive cases. In Fig. 4.2 the amplitudes for the repulsive case are plotted. On the other hand the attractive case is very interesting. As depicted in Fig. 4.3 and Fig. 4.4 for suitable values of the parameters, one finds oscillatory solutions and a chirped oscillatory solutions. It should be noted that for the aforementioned cases, the integration constants are nonzero.

In the high density limit, the attractive case is very interesting. As depicted in Fig. 4.5, for appropriate values of the parameters, one finds that \(\sigma_a\) is an oscillatory solution whereas \(\sigma_b\) shows auto-correlation traces of pulse trains generated by the induced modulational instability [25]. It is indeed, illuminating to note that these autocorrelation traces are observed when the pumping Nd:YAG laser and side-band InGaAsP semiconductor are coupled to single-mode fiber having zero-dispersion wavelength.

By applying another transformation \(f_a = w_a e^{2i\omega_1 \sigma_a t}\), and \(f_b = w_b e^{2i\omega_1 \sigma_b t}\) we obtain the GP equation for strong coupling case

\[
\frac{i\hbar}{2m} \frac{\partial^2}{\partial t^2} a + \epsilon [ |a|^2 + |b|^2 ]^{1/2} a - \epsilon_a a. \]

By extending the same procedure, we also derive the GP equation for \(w_b\):

\[
\frac{i\hbar}{2m} \frac{\partial^2}{\partial t^2} b + \epsilon [ |a|^2 + |b|^2 ]^{1/2} b - \epsilon_b b. \]
Figure 4.5: The upper panel depicts the oscillatory pulse trains, and the lower panel depicts the autocorrelation traces of pulse trains of TBEC generated by the induced modulational instability; for \( e = -1.0, \epsilon_a = 5.0, \epsilon_b = 0.02, \sigma_a^0 = 1.0, \sigma_b^0 = 1.0, \) and \( u = 0.8. \)

Figure 4.6: Plot depicting the amplitudes of TBEC for \( E = 0.4, \epsilon_a = 5.0, \epsilon_b = 0.02, \sigma_a^0 = 1.0, \sigma_b^0 = 0.4, \) and \( u = 0.9. \)
at $\lambda_0 = 1.275 \mu m$. Furthermore, for repulsive interaction in a suitable range of parameter values, $\sigma_a, \sigma_b$ are still oscillatory and the same has been depicted in Fig. 4.6.
References


