This chapter aspires for an investigation of fuzzy semiregularization spaces corresponding to given fts's. In Section 1.1 we describe such spaces which are inherently associated with given fts's and deduce certain characterization theorems for a fuzzy semiregular space. We also introduce the concepts of fuzzy ro-equivalent and fuzzy submaximal spaces, and ultimately establish the existence theorem for a fuzzy submaximal space as an expansion of a given fts. We find that there are certain properties which when possessed by an fts is also shared by its semiregularization space and vice-versa. We call such a property a fuzzy semiregular property. We see that fuzzy submaximal and semiregular spaces are closely associated with those spaces which are minimal or maximal with regard to fuzzy semiregular properties. We ultimately deduce, to this end, the interesting result that every maximal P (minimal P) fts is fuzzy submaximal (resp. fuzzy semiregular) if P is a fuzzy semiregular property. This result enables one to isolate and recognise certain fts's to be fuzzy semiregular or submaximal. Naturally, the above results demand a thorough investigation as to which properties are fuzzy semiregular ones. We take up such a study in Section 1.2, and examine certain properties for their inclusion in the class of fuzzy semiregular properties. Of course, there are various other such properties which can be tested along this line. Finally, in Section 1.3 our endeavour is to ascertain the behaviours of certain known mappings between fts's when their domains and/or codomains are restricted to the corresponding fuzzy semiregularization space. A similar study can obviously
be carried on with many other types of maps. Indeed, we shall take up some such cases in Chapter-VIII in connection with some stronger forms of fuzzy continuous functions.

§ 1.1. FUZZY SEMIREGULARIZATION AND FUZZY SUBMAXIMAL SPACES

DEFINITION 1.1.1. Let \((X,T)\) be an fts. Consider the set of all fuzzy regularly open sets in \(X\). Then it is easy to see that it forms a base for some fuzzy topology on \(X\). We call this topology the fuzzy semiregularization topology of \(T\), to be denoted by \(T_S\). Clearly \(T_S \subseteq T\). \((X,T_S)\) is called the fuzzy semiregularization space or simply the fuzzy semiregularization of \((X,T)\).

Azad [6] defined an fts \((X,T)\) to be fuzzy semiregular iff the fuzzy regularly open sets of \(X\) form a base for the fuzzy topology \(T\) on \(X\). Thus according to the above definition, \((X,T)\) is fuzzy semiregular iff \(T = T_S\).

A characterization of a fuzzy semiregular space in terms of fuzzy singletons is established as follows.

THEOREM 1.1.2. An fts \((X,T)\) is fuzzy semiregular iff for each fuzzy open set \(U\) and each fuzzy singleton \(x_\alpha\) with \(x_\alpha \notin U\), there exists a fuzzy open set \(V\) such that \(x_\alpha \in V \setminus \text{int} cIV \subseteq U\).

PROOF. Let \(X\) be fuzzy semiregular, and let \(x_\alpha\) be a fuzzy singleton in \(X\) and \(U\) any fuzzy open \(q\)-nbd of \(x_\alpha\). By hypothesis, there exists a collection \(\{U_i : i \in I\}\) of fuzzy regularly open sets such that \(U = \bigcup\{U_i : i \in I\}\). We claim that for some \(i\), \(U_i\) is a \(q\)-nbd of \(x_\alpha\). If not, then \(x_\alpha + U_i(x) \notin U_i\).
for all $i \in I$. Then $\alpha + \sup_{i \in I} \{ U_i(x) \} \not| 1$ so that $\alpha + U(x) \not| 1$ which is a contradiction. Hence for some $i_0 \in I$, $U_{i_0}$ is a fuzzy open q-nbd of $x_\alpha$. Also $\text{int} \, \text{cl} \, U_{i_0} \not| U$. Putting $V = U_{i_0}$, we have $\text{int} \, \text{cl} \, V \not| U$.

Conversely, let $U$ be any fuzzy open set in $X$. Let us start with an element $x$ of $\text{supp} \, U$ and keep it fixed for the time being, choose a positive integer $m_x$ such that $\frac{1}{2^{m_x}} \not| U(x)$. For any positive integer $n$ satisfying $n \not| m_x$, let us put $\alpha_n = l - U(x) + \frac{1}{2^n}$. Then $0 < \alpha_n \not| 1$. Since $U$ is a fuzzy open q-nbd of $x_\alpha$, there exists a fuzzy open set $V_n$ such that $x_\alpha \not| V_n \not| \text{int} \, \text{cl} \, V_n \not| U$. Also int cl $V_n$ + $\alpha_n$ = int cl $V_n$ + $1 - U(x) + \frac{1}{2^n} \not| 1$ so that int cl $V_n(x)$ > $U(x) - \frac{1}{2^n}$. Since int cl $V_n(x)$ > $U(x)$ for all positive integer $n \not| m_x$, it then follows that $\sup \{ \text{int} \, \text{cl} \, V_n \} = U(x)$. Thus $\{ \text{int} \, \text{cl} \, V_n : n \not| m_x \}$ is a collection of fuzzy regularly open sets in $X$ such that $\bigcup \{ \text{int} \, \text{cl} \, V_n : n \not| m_x \} = U(x)$. Now as $x$ runs over $\text{supp} \, U$, we see that $\bigcup \{ \text{int} \, \text{cl} \, V_n : x \in \text{supp} \, U \}$ is a family of fuzzy regularly open sets in $X$ such that $\bigcup \{ x \}$ is fuzzy regular set in $X$.

REMARK 1.1.3. By virtue of Result 0.9.37, Theorem 1.1.2 and Definition 0.9.38 it is clear that a fuzzy regular space is fuzzy almost regular as well as fuzzy semiregular.

In the next two theorems we like to further characterize a fuzzy semiregular space and ultimately prove that in the above theorem, fuzzy singleton $x_\alpha$ can be replaced by any arbitrary fuzzy set in $X$.

THEOREM 1.1.4. An fts $(X,T)$ is fuzzy semiregular iff for any fuzzy closed set $A$ and any fuzzy singleton $x_\alpha \nsubseteq A$, there exists a fuzzy regular closed set $B$ such that $A \not| B$ and $x_\alpha \nsubseteq B$. 
Proof. Let \( X \) be fuzzy semiregular. Suppose \( \alpha \) is any fuzzy singleton and \( A \) is any fuzzy closed set such that \( \alpha \not\subset A \). Then \( \alpha \not\subset q(1-A) \in \mathcal{T}. \) By Theorem 1.1.2, there exists a fuzzy regularly open set \( U \) such that \( \alpha \not\subset q U \cup (1-A) \).

If \( B = 1-U \), then \( B \) is a fuzzy regularly closed set such that \( \alpha \not\subset B \) and \( B \not\supset A \). Conversely, let the given condition hold. Let \( \alpha \) be any fuzzy singleton in \( X \) and \( U \) any fuzzy open \( q \)-nbd of \( \alpha \). Then \( \alpha \not\subset 1-U = A \) (say), where \( A \) is fuzzy closed. By hypothesis, there is a fuzzy regularly closed set \( B \) in \( X \) such that \( A \not\subset B \) and \( \alpha \not\subset B \), i.e., \( \alpha \not\subset q(1-B) \not\subset U \), where \( (1-B) \) is fuzzy regularly open. Hence \((X,\mathcal{T})\) is fuzzy semiregular.

Theorem 1.1.5. An fts \((X,\mathcal{T})\) is fuzzy semiregular iff for any fuzzy set \( A \) in \( X \) and any fuzzy open set \( B \) with \( A \not\subset B \), there is a fuzzy regularly open set \( U \) such that \( A \not\subset q U \not\subset B \).

Proof. If the given condition holds then by virtue of Theorem 1.1.2, \((X,\mathcal{T})\) is obviously fuzzy semiregular.

Conversely, let \( X \) be fuzzy semiregular. Suppose \( A \) is any fuzzy set in \( X \) and \( B \) is any fuzzy open set such that \( A \not\subset B \). Then \( A \not\subset 1-B \). Thus there is a fuzzy singleton \( \alpha \not\subset A \) such that \( \alpha \not\subset 1-B \). In view of Theorem 1.1.4, this implies the existence of a fuzzy regularly closed set \( V \) such that \( 1-B \not\subset V \) and \( \alpha \not\subset V \). Then \( U = 1-V \) is a fuzzy regularly open set with \( U \not\subset B \) and \( \alpha \not\subset q U \). Hence \( A \not\subset q U \not\subset B \).

Now, even if an fts \( X \) is not fuzzy semiregular, there are certain connections between the space and its semiregularization space with regard to interior and closure operations.
THEOREM 1.1.6. Let \((X,T_S)\) denote the fuzzy semiregularization of an fts \((X,T)\). Then for each fuzzy open set \(U\) in \(X\):

(a) \(T^-\text{c}1U = T^-\text{c}S\text{c}1U\) and (b) \(T^-\text{int}(T^-\text{c}1U) = T^-\text{int}(T^-\text{c}S\text{c}1U)\).

PROOF. (a) \(U\) being fuzzy open in \((X,T)\), \(T^-\text{c}1U\) is fuzzy regularly closed and hence \(1-T^-\text{c}1U\) is fuzzy regularly open in \((X,T)\). This implies that \(T^-\text{c}1U\) is \(T^-\text{c}S\)-closed and hence \(T^-\text{c}S\text{c}1U \subseteq T^-\text{c}1U\). Also, \(T\) being finer than \(T^-\), \(T^-\text{c}1U \subseteq T^-\text{c}S\text{c}1U\). Thus \(T^-\text{c}1U = T^-\text{c}S\text{c}1U\), for all \(U \in T\).

(b) For any \(U \in T\), using (a) we have \(T^-\text{int}(T^-\text{c}1U) \subseteq T^-\text{c}S\text{c}1U\). Since \(T^-\text{int}(T^-\text{c}1U) \in T^-\), \(T^-\text{int}(T^-\text{c}1U) \subseteq T^-\text{int}(T^-\text{c}S\text{c}1U)\). Again, \(T\) being finer than \(T^-\), \(T^-\text{int}(T^-\text{c}1U) \nsubseteq T^-\text{int}(T^-\text{c}S\text{c}1U)\). Thus \(T^-\text{int}(T^-\text{c}1U) = T^-\text{int}(T^-\text{c}S\text{c}1U)\).

COROLLARY 1.1.7. The collection of all fuzzy regularly open sets in \((X,T)\) coincides with that of all fuzzy regularly open sets in \((X,T_S)\). Hence fuzzy semiregularizations of \((X,T)\) and \((X,T_S)\) are the same.

COROLLARY 1.1.8. For any fts \((X,T),(X,T_S)\) is fuzzy semiregular.

THEOREM 1.1.9. Let \((X,T)\) be an fts and \((X,T_0)\) be any fuzzy semiregular space such that \(T_S \subseteq T_0 \subseteq T\). Then \(T_0 = T_S\).

PROOF. Let \(U\) be any fuzzy regularly open set in \((X,T_0)\). Since \(T_S \subseteq T_0 \subseteq T\) and since \(T^-\text{c}1U = T^-\text{c}S\text{c}1U\), we have \(T^-\text{c}S\text{c}1U = T^-\text{c}1U\). Again, \(T^-\text{int} T^-\text{c}S\text{c}1U = T^-\text{int} T^-\text{c}1U \Rightarrow T^-\text{int} T^-\text{c}1U = T^-\text{int} T^-\text{c}S\text{c}1U = U\). Hence \(U \in T_S\) and consequently, \(T_0 = T_S\).
COROLLARY 1.1.10. For an fts \((X,T)\), \((X,T_S)\) is a maximal element of the collection of all fuzzy semiregular spaces which are weaker than \((X,T)\).

DEFINITION 1.1.11. Two fts's \((X,T)\) and \((X,R)\) are said to be fuzzy ro-equivalent iff \(T_S = R_S\).

DEFINITION 1.1.12. An fts \((X,T)\) is said to be an expansion of an fts \((X,R)\) iff \(T\) is finer than \(R\) (i.e., \(R \subset T\)).

DEFINITION 1.1.13. A property \(P\) of an fts is called expansive iff whenever an fts \((X,T)\) possesses the property \(P\), every expansion of the space also shares the property.

THEOREM 1.1.14. An expansion \((X,R)\) of an fts \((X,T)\) is fuzzy ro-equivalent to \((X,T)\) iff \(T-clU = R-clU\), for all \(U \in \mathcal{R}\).

PROOF. Let \((X,T)\) be fuzzy ro-equivalent to \((X,R)\) so that \(T_S = R_S\). Let \(U \in \mathcal{R}\). Then \(R-clU \subseteq T-clU\). If \(V = 1 - R-clU\), then \(V = R-int R-clV\) and hence \(V \in R_S = T_S\). Thus \(V \subseteq T\) so that \(R-clU\) is \(T\)-closed. Consequently, \(T-clU \subseteq R-clU\).

We thus have \(T-clU = R-clU\).

Conversely, we start with an \(R\)-closed set \(F\). We have, \(R-intF = 1 - R-cl(1-F) = 1-T-cl(1-F)\) (since \((1-F) \in \mathcal{R}\)) = \(T-int F\). Thus for any \(R\)-closed set \(F\), \(R-int F = T-int F\) .... .... .... (1)

Now, let \(V\) be a fuzzy regularly open set in \((X,R)\). Then \(V = R-int R-clV = T-int R-clV\) (by (1)). Again, since \(V \in \mathcal{R}\) we have by hypothesis, \(T-clV = R-clV\). Then \(V = T-int T-clV\) so that \(V\) is fuzzy regularly open in \((X,T)\).
Hence $R \subseteq T$. Next let $U$ be fuzzy regularly open in $(X,T)$. Then $U \in TC (R)$ so that $T-cl U = R-cl U$. Using (1) we have, $R-int Rcl U = T-int T-cl U = U$. Hence $U$ is fuzzy regularly open in $(X,R)$ so that $T \subseteq R$. We thus conclude that $T = R$.

**Definition 1.1.15.** Let $(X,T)$ be a fuzzy semiregular space and let $\mathcal{U}$ be the collection of all fuzzy topologies $T_\alpha$'s on $X$ such that $(X,T_\alpha)$ is fuzzy ro-equivalent to $(X,T)$. Then $\mathcal{U}$ is partially ordered under the (fuzzy) set inclusion relation. Then $(X,T^*)$, where $T^*$ is a maximal element of $\mathcal{U}$, is defined to be a fuzzy submaximal space.

That the above definition is meaningful follows from the following existence theorem for a fuzzy submaximal space.

**Theorem 1.1.16.** For every fuzzy topological space, there exists a fuzzy submaximal space which is an expansion of the given fts.

**Proof.** Let $(X,T)$ be an fts. Let $\mathcal{F}$ denote the collection of all fuzzy topologies $T_\alpha$'s on $X$ such that each $(X,T_\alpha)$ is fuzzy ro-equivalent to $(X,T)$. Then $\mathcal{F}$ is a poset under set inclusion relation. Let $\mathcal{F}_i$ be the subcollection of $\mathcal{F}$ such that $T_\alpha \in \mathcal{F}_i$ iff $T \subseteq T_\alpha$. Then $\mathcal{F}_i$ is also a poset under the same relation as in $\mathcal{F}$. Let $\mathcal{F}_0$ be a chain in $\mathcal{F}_i$ and let $\mathcal{F}_0 = \bigcup \{ T_\alpha : T_\alpha \in \mathcal{F}_0 \}$. It is clear that $\mathcal{F}_0$ is a base for some fuzzy topology $R$ (say) on $X$ such that $T \subseteq R$. We claim that $T = R$. In fact, let $U \in R$. In view of Theorem 1.1.14, it suffices to show that $T-cl U = R-cl U$. Obviously, $T-cl U \supseteq R-cl U$. Next suppose $x_\alpha$ be a fuzzy singleton such that $x_\alpha \notin T-cl U$, and let $W$ be any $R$-open $q$-nbhd of $x_\alpha$. Then there exists $B \in \mathcal{F}_0$ such that $x_\alpha \notin B \subseteq W$.
Now, $B \in T_{a}$, for some $a$ for which $T_{a} \in \mathcal{L}_{o}$. Let $W^{*} = T_{a} - \text{int} T_{a} - \text{cl} B$. Then $W^{*} \in (T_{a})_{\mathcal{S}} = T_{S} \subseteq T$, i.e. $W^{*} \in T$. Thus $W^{*}$ is a $T$-open q-nbd of $x_{\lambda}$ and hence $W^{*} \cap U$. So there exists $y \in X$ such that $W^{*}(y) + U(y) > 1$. Let $\mu = W^{*}(y)$ so that $y \in U$, and hence there exists $U_{y} \in \mathcal{H}_{\mathcal{S}}$ such that $y \in U_{y} \subseteq U$. Now $U_{y} \subseteq T$, for some $T_{B} \in \mathcal{L}_{o}$. If possible, let $W \subseteq U$. Then $B \subseteq U$. Since $(X,T_{a})$ and $(X,T_{B})$ are comparable and each is fuzzy ro-equivalent to $(X,T_{S})$, by Theorem 1.1.14, $T_{a} - \text{cl} B = T_{B} - \text{cl} B$. Therefore by Result 0.9.26, $T_{a} - \text{cl} B \subseteq U_{y}$. Since $W^{*} \subseteq T_{a} - \text{cl} B$, we have $W^{*} \subseteq U_{y}$ and hence $y \in U_{y}$ which is a contradiction. Thus $x_{\lambda} \subseteq R - \text{cl} U$, and we finally conclude that $T - \text{cl} U = R - \text{cl} U$.

Hence $T_{S} = R_{S}$ so that $R \in \mathcal{L}_{o}$. Clearly $R$ is an upper bound of $\mathcal{L}_{o}$. By Zorn's lemma, $\mathcal{L}_{o}$ has a maximal element $T^{*}$ (say). This maximal element is also a maximal element of $\mathcal{L}_{o}$. Hence $(X,T^{*})$ is a fuzzy submaximal space which is an expansion of $(X,T)$.

DEFINITION 1.1.17. A property $P$ is said to be a fuzzy semiregular property provided that an fts $(X,T)$ possesses the property $P$ iff its fuzzy semiregularization space $(X,T_{S})$ possesses the property.

DEFINITION 1.1.18. An fts $(X,T)$ is said to be maximal (minimal) with respect to a property $P$ iff whenever an fts $(X,R)$ possesses the property $P$, one has $R \subseteq T$ (resp. $T \subseteq R$). For brevity, we shall call such a space maximal $P$ (resp. minimal $P$).

The following theorem gives an application of fuzzy semiregularity and the newly introduced concept of submaximality of an fts in the study of minimality and maximality of fts's with regard to fuzzy semiregular properties.
THEOREM 1.1.19. Let P be a fuzzy semiregular property. Then every maximal P (minimal P) fuzzy topological space is fuzzy submaximal (resp. fuzzy semiregular).

PROOF. Let \((X,T)\) be maximal P (minimal P). If it is not fuzzy submaximal (semiregular), there exists a fuzzy submaximal space \((X,T^*)\) which is strictly finer (resp. fuzzy semiregular space \((X,T_\sigma)\) which is strictly weaker) than \((X,T)\), and which possesses the property P. This contradicts the maximality (resp. minimality) of the space \((X,T)\) with respect to the property P.

§1.2. FUZZY SEMIREGULAR PROPERTIES

We have just seen in the last section that in order that an fts is maximal or minimal with respect to a fuzzy semiregular property, the space has to be necessarily submaximal or semiregular respectively. This motivates us to the investigation of different properties of fts's in order to ascertain whether these are fuzzy semiregular properties. We start with certain fuzzy separation axioms as defined in [41] in the following manner.

DEFINITION 1.2.1. An fts \((X,T)\) is said to be

(a) Fuzzy \(T_0\) iff for any two distinct fuzzy singletons \(x_\alpha\) and \(y_\beta\):

Case I when \(x \neq y\), either \(x_\alpha\) has a fuzzy open nbd which is not q-coincident with \(y_\beta\), or \(y_\beta\) has a fuzzy open nbd which is not q-coincident with \(x_\alpha\),

Case II when \(x = y\) and \(\alpha < \beta\) (say), there is a fuzzy open q-nbd of \(y_\beta\) which is not q-coincident with \(x_\alpha\);

(b) fuzzy \(T_1\) iff for every pair of distinct fuzzy singletons \(x_\alpha\) and \(y_\beta\):
Case I when $x \neq y$, $x_\alpha$ has a fuzzy open nbd which is not $q$-coincident with $y_\beta$, and $y_\beta$ has a fuzzy open nbd which is not $q$-coincident with $x_\alpha$.

Case II when $x = y$ and $\alpha < \beta$ (say), then there exists a fuzzy open $q$-nbd $V$ of $y_\beta$ such that $x_\alpha \notin V$.

(c) fuzzy $T_2$ or fuzzy Hausdorff iff for any two distinct fuzzy singleton $x_\alpha$ and $y_\beta$:

Case I when $x \neq y$, $x_\alpha$ and $y_\beta$ have fuzzy open nbds which are not $q$-coincident,

Case II when $x = y$ and $\alpha < \beta$ (say), then $y_\beta$ has a fuzzy open $q$-nbd $V$ and $x_\alpha$ has a fuzzy open nbd $U$ such that $V \notin U$.

It is clear that the property of a space to be fuzzy $T_o$ or fuzzy $T_1$ is an expansive property. Hence we have:

**Theorem 1.2.2.** An fts $(X,T)$ is fuzzy $T_o$ (fuzzy $T_1$) if $(X, T_S)$ is fuzzy $T_o$ (resp. fuzzy $T_1$).

**Theorem 1.2.3.** Fuzzy $T_2$ property is a fuzzy semiregular property.

**Proof.** Let $(X,T)$ be fuzzy $T_2$ and $x_\alpha$, $y_\beta$ be two distinct fuzzy singletons in $X$. We have to consider the following two cases:

Case I when $x \neq y$, $x_\alpha$ and $y_\beta$ have fuzzy open nbds $U$ and $V$ respectively in $(X,T)$ such that $U \notin V$. Then by Result 0.9.26, we have $T_{\text{cl}U} \notin V$, and hence $T_{\text{int}} T_{\text{cl}U} \notin V$. Again by the same result we have $T_{\text{int}} T_{\text{cl}V} \notin V$. Let us put $T_{\text{int}} T_{\text{cl}U} = U_S$ and $T_{\text{int}} T_{\text{cl}V} = V_S$. Then $U_S$ and $V_S$ are $T_S$-open nbds of $x_\alpha$ and $y_\beta$ respectively such that $U_S \notin V_S$.

Case II when $x = y$ and $\alpha < \beta$ (say), $x_\alpha$ has a fuzzy open nbd $U$ and $y_\beta$ has
a fuzzy open q-nbd \( \mathcal{V} \) in \((X, T)\) such that \( U \nsubseteq \mathcal{V} \). Then as in case I, \( U_S = T-\text{int} \ T-clU \nsubseteq T-\text{int} \ T-clV = V_S \), where \( U_S \) is a fuzzy open nbd of \( x_\alpha \) in \((X, T_S)\) and \( V_S \) is a \( T_S \)-open q-nbd of \( y_\beta \). Hence \((X, T_S)\) is fuzzy \( T_2 \).

Converse part follows at once since the fuzzy \( T_2 \) property is an expansive one.

**Theorem 1.2.4.** An fts \((X, T)\) is fuzzy almost regular iff \((X, T_S)\) is fuzzy regular.

**Proof.** (Necessity): Let \( x_\alpha \) be any fuzzy singleton in \( X \) and \( U \) be any \( T_S \)-open q-nbd of \( x_\alpha \). Then there exists a fuzzy regularly open set \( V \) in \((X, T)\) such that \( \chi_\alpha \cap V \not\subseteq U \). By fuzzy almost regularity of \((X, T)\), there exists a \( T \)-regularly open q-nbd \( W \) of \( x_\alpha \), i.e. a \( T_S \)-open q-nbd \( W \) of \( x_\alpha \) such that \( T_S-clW = T-clW \not\subseteq V \not\subseteq U \). Thus \((X, T_S)\) is fuzzy regular, by Result 0.9.37.

(Sufficiency): Let \( x_\alpha \) be any fuzzy singleton in \( X \) and \( U \) be any fuzzy regularly open q-nbd of \( x_\alpha \) in \((X, T)\). Then \( U \) is a \( T_S \)-open q-nbd of \( x_\alpha \). So by fuzzy regularity of \((X, T_S)\) there exists a \( T_S \)-open q-nbd \( V \) of \( x_\alpha \) such that \( T_S-clV \not\subseteq U \). Again there exists a \( T \) -regularly open q-nbd \( W \) of \( x_\alpha \) such that \( W \not\subseteq V \), and hence \( clW \not\subseteq T_S-clW \not\subseteq T_S-clV \not\subseteq U \). Hence \((X, T)\) is fuzzy almost regular.

In view of Remark 1.1.3 and Theorem 1.2.4, we obtain:

**Corollary 1.2.5.** An fts is fuzzy semiregular and fuzzy almost regular iff it is fuzzy regular.

**Corollary 1.2.6.** For any fts \((X, T)\), its fuzzy semiregularization space \((X, T_S)\) is fuzzy almost regular iff it is fuzzy regular.
COROLLARY 1.2.7. Fuzzy almost regularity is a fuzzy semiregular property.

In a manner similar to the definitions of fuzzy \( T_i \)-axiom \( (i = 0,1,2) \) as given above, one can define as follows.

DEFINITION 1.2.8. An fts \((X,T)\) is said to be fuzzy Urysohn iff for any two distinct fuzzy singletons \( x_\alpha \) and \( y_\beta \) in \( X \):

Case I When \( x \neq y \), \( x_\alpha \) and \( y_\beta \) have fuzzy open nbds \( U \) and \( V \) respectively such that \( \text{cl} U \nsubseteq \text{cl} V \);

Case II When \( x = y \) and \( \alpha \neq \beta \) (say), \( x_\alpha \) has a fuzzy open nbd \( U \) and \( y_\beta \) has a fuzzy open q-nbd \( V \) such that \( \text{cl} U \nsubseteq \text{cl} V \).

THEOREM 1.2.9. Fuzzy Urysohn property is a fuzzy semiregular property.

PROOF. Let \((X,T)\) be fuzzy Urysohn, and \( x_\alpha \), \( y_\beta \) be two distinct fuzzy singletons in \( X \). In order to prove that \((X,T_S)\) is fuzzy Urysohn, we have to consider the following two cases:

Case I When \( x \neq y \), \( x_\alpha \) and \( y_\beta \) have fuzzy open nbds \( U \) and \( V \) respectively in \((X,T)\) such that \( T\text{-cl} U \nsubseteq T\text{-cl} V \). Put \( T\text{-int} T\text{-cl} U = U_S \) and \( T\text{-int} T\text{-cl} V = V_S \). Then \( U_S \nsubseteq V_S \) and consequently, \( T_S\text{-cl} U_S = T\text{-cl} U \nsubseteq T\text{-cl} V_S = T_S\text{-cl} V_S \), where \( U_S \) and \( V_S \) are fuzzy open nbds of \( x_\alpha \) and \( y_\beta \) respectively in \((X,T_S)\).

Case II When \( x = y \) and \( \alpha \neq \beta \) (say), \( x_\alpha \) has a fuzzy open nbd \( U \) and \( y_\beta \) has a fuzzy open q-nbd \( V \) in \((X,T)\) such that \( T\text{-cl} U \nsubseteq T\text{-cl} V \). Then as in Case I, \( U_S = T\text{-int} T\text{-cl} U \) and \( V_S = T\text{-int} T\text{-cl} V \) are respectively \( T_S\text{-open nbd} \) of \( x_\alpha \) and \( T_S\text{-open q-nbd} \) of \( y_\beta \) such that \( T_S\text{-cl} U_S \nsubseteq T_S\text{-cl} V_S \). Hence \((X,T_S)\) is fuzzy Urysohn.
It is clear that fuzzy Urysohn property is an expansive property and hence the converse part follows.

A weaker form of covering properties than fuzzy compactness under the terminology fuzzy almost compactness has been defined by Concilio and Gerla in [20]. According to them an fts \((X,T)\) is fuzzy almost compact iff every fuzzy open cover of \(X\) has a finite fuzzy proximate subcover.

**Theorem 1.2.10.** Fuzzy almost compactness is a fuzzy semiregular property.

**Proof.** If \((X,T)\) is fuzzy almost compact then clearly so is \((X,T_S)\), since \(T_S \subseteq T\).

Conversely, suppose \((X,T_S)\) is fuzzy almost compact and let \(\{ G_\alpha : \alpha \in I \} \) be a \(T\)-open fuzzy cover of \(X\). Then \(\{ T\text{-int } T\text{-cl } G_\alpha : \alpha \in I \} \) is a \(T_S\)-open fuzzy cover of \(X\). By fuzzy almost compactness of \((X,T_S)\), there exists a finite subset \(I_0\) of \(I\) such that \(\bigcup \{ T_S\text{-cl } T\text{-int } T\text{-cl } G_\alpha : \alpha \in I_0 \} = I_X\). By Theorem 1.1.6, \(T_S\text{-cl } T\text{-int } T\text{-cl } G_\alpha = T\text{-cl } T\text{-int } T\text{-cl } G_\alpha = T\text{-cl } G_\alpha\) and hence \(\bigcup \{ T\text{-cl } G_\alpha : \alpha \in I_0 \} = I_X\).

**Remark 1.2.11.** It is clear that many fuzzy topological properties can be tested in the same way as to whether they are fuzzy semiregular ones. Some such properties will be treated in the subsequent chapters in course of the study of the corresponding concepts (e.g. see Corollary 2.2.11, Theorem 5.18, etc.).
§ 1.3. FUZZY SEMIREGULARIZATION SPACES AND MAPPINGS

As detailed in Chapter 0, after the introduction of fuzzy continuous functions by Chang [17] a large variety of continuous-like functions has been investigated by many authors. In this section our aim is to consider some of these functions and investigate their behaviours when the domain and/or co-domain spaces are replaced by their respective fuzzy semiregularization spaces. Obviously there are various other types of functions which could be treated along the same line and in fact, we will consider some of them later in Chapter VIII. We begin by quoting a few definitions as follows.

DEFINITIONS 1.3.1. [42] A function \( f : (X,T) \to (Y,R) \) is called fuzzy almost continuous (\( \delta \)-continuous) iff corresponding to any fuzzy singleton \( x_\alpha \) in \( X \) and any fuzzy regularly open q-nbd \( V \) of \( f(x_\alpha) \), there is a fuzzy open q-nbd (resp. fuzzy regularly open q-nbd) \( U \) of \( x_\alpha \) such that \( f(U) \triangleleft V \).

It is shown in [42] that the above definition of fuzzy almost continuity is equivalent to that given by Azad [6], viz. \( f : X \to Y \) is fuzzy almost continuous iff for every fuzzy regularly open (regularly closed) set \( V \) in \( Y \), \( f^{-1}(V) \) is fuzzy open (resp. fuzzy closed) in \( X \).

Clearly, a function which is either fuzzy continuous or fuzzy \( \delta \)-continuous is fuzzy almost continuous. These implications can not be reversed, in general (see [6] and [42]). Moreover, it is shown in [42] that fuzzy \( \delta \)-continuity and fuzzy continuity are independent notions.

THEOREM 1.3.2. Let \( f : (X,T) \to (Y,R) \) be a function. Then

(a) \( f : (X,T) \to (Y,R) \) is fuzzy almost continuous \( \Rightarrow \) \( f : (X,T) \to (Y,R) \) is fuzzy continuous.
(b) \( f : (X,T) \rightarrow (Y,R_S) \) is fuzzy almost continuous \( \Rightarrow \) \( f : (X,T) \rightarrow (Y,R) \) is fuzzy almost continuous.

(c) \( f : (X,T_S) \rightarrow (Y,R_S) \) is fuzzy almost continuous \( \Rightarrow \) \( f : (X,T) \rightarrow (Y,R) \) is fuzzy \( \delta \)-continuous.

**PROOF.** The proofs being straightforward, are omitted.

It is clear that many more results of the type as found in Theorem 1.3.2 can be achieved if the spaces \( X \) and/or \( Y \) are themselves fuzzy semi-regular or fuzzy almost regular or fuzzy regular. We consider here a few such cases. Similar study involving other types functions will be taken up in Chapter VIII.

**DEFINITION 1.3.3.** [6] A function \( f : X \rightarrow Y \) is called fuzzy weakly continuous iff for each fuzzy open set \( B \) in \( Y \), \( f^{-1}(B) \subseteq \text{int} f^{-1}(\text{cl}B) \).

The above type of functions was further studied in [103] and it was proved that a function \( f : X \rightarrow Y \) is fuzzy weakly continuous iff for any fuzzy singleton \( x_a \) in \( X \) and any fuzzy open set \( V \) containing \( f(x_a) \), there is a fuzzy open set \( U \) containing \( x_a \) such that \( f(U) \subseteq \text{cl}V \).

It is shown in [6] that a fuzzy weakly continuous function may not be fuzzy almost continuous, though the converse is obviously true. The following lemma gives a sufficient condition for the equivalence of these two types of functions.

**LEMMA 1.3.4.** If \( f : (X,T) \rightarrow (Y,R) \) is fuzzy weakly continuous and \( Y \) is fuzzy almost regular, then \( f \) is fuzzy almost continuous.
PROOF. Let \( x_\alpha \) be any fuzzy singleton in \( X \) and \( V \) any fuzzy regularly open q-nbd of \( f(x_\alpha) \). By fuzzy almost regularity of \( Y \), there exists a fuzzy regularly open q-nbd \( W \) of \( f(x_\alpha) \) such that \( \text{cl}W \not\subseteq V \). If \( y = f(x) \), then \( W(y) > 1 - \alpha \), we choose \( \varepsilon > 0 \) such that \( W(y) > 1 - \alpha + \varepsilon = \beta \). Then \( W \) is a fuzzy regularly open set in \( Y \) containing \( x_\beta \). By fuzzy weak continuity of \( f \), there exists a fuzzy open set \( U \) in \( X \) containing \( x_\alpha \) such that \( f(U) \not\subseteq \text{cl}W \not\subseteq V \). Now, \( x_\beta \not\subseteq U \) is a fuzzy open q-nbd of \( x_\alpha \). Thus \( f \) is fuzzy almost continuous.

**Theorem 1.3.5.** Let \( f : (X,T) \to (Y,R) \) be a function, where \( (Y,R) \) is fuzzy almost regular. Then \( f : (X,T) \to (Y,R) \) is fuzzy \( \delta \)-continuous whenever \( f : (X,T_S^\delta) \to (Y,R_S^\delta) \) is fuzzy weakly continuous.

**Proof.** Follows from Lemma 1.3.4, Corollary 1.2.7 and Theorem 1.3.2 (c).

**Theorem 1.3.6.** (a) If \( (Y,R) \) is fuzzy almost regular and \( f : (X,T) \to (Y,R) \) is fuzzy weakly continuous then \( f : (X,T_S^\delta) \to (Y,R) \) is fuzzy \( \delta \)-continuous.

(b) If \( f : (X,T_S^\delta) \to (Y,R) \) is fuzzy almost continuous then \( f : (X,T) \to (Y,R) \) is fuzzy almost continuous.

**Proof.** Let \( x_\alpha \) be any fuzzy singleton in \( X \) and \( V \) any fuzzy regularly open q-nbd of \( f(x_\alpha) \) in \( (Y,R) \). By fuzzy almost regularity of \( (Y,R) \), there is a fuzzy regularly open q-nbd \( W \) of \( f(x_\alpha) \) such that \( \text{cl}W \not\subseteq V \). Since \( f : (X,T) \to (Y,R) \) is fuzzy almost continuous (by Lemma 1.3.4), \( f^{-1}(W) \) and \( f^{-1}(V) \) are fuzzy open and \( f^{-1}(\text{cl}W) \) is fuzzy closed in \( (X,T) \). Put \( U = T\text{-int } T\text{-cl}f^{-1}(W) \).
Then $U$ is a $T_\mathcal{S}$-regularly open q-nbd of $x_\mathcal{A}$. By Corollary 1.1.7, $U$ is a $T_\mathcal{S}$-regularly open q-nbd of $x_\mathcal{A}$. Furthermore, $U \subseteq \text{cl}^{-1}(W) \subseteq \text{cl}^{-1}(\text{cl}W) = f^{-1}(\text{cl}W) \subseteq f^{-1}(V)$ and hence $f(U) \subseteq V$. Thus $f : (X,T_\mathcal{S}) \rightarrow (Y,R)$ is fuzzy $\&$-continuous.

(b) It is obvious, since $T_\mathcal{S} \subseteq T$.

NOTE 1.3.7. The above Theorems 1.3.5 and 1.3.6 establish that the results in Theorems 1 and 2 of Noiri [111] can be strengthened if the fts's $X$ and $Y$ of the above theorems are treated in particular, as ordinary set topological spaces.