CHAPTER-VII
CHAPTER VII

PERFECT FUNCTIONS BETWEEN FUZZY TOPOLOGICAL SPACES

This chapter is mainly a study of fuzzy perfect functions from a standpoint different from those taken by others. As already mentioned in § 0.2, Azad [8] took one of the two equivalent approaches as shown by Bourbaki [12] while Srivastava and Lai [143] and Christoph [18] have followed the another approach. But none of them could prove the equivalence of these two definitions in fuzzy topological context. We have opted to deal with the fuzzy perfect maps in terms of fuzzy directed families and some associated concepts. Although we have not been able to achieve the desired equivalence completely in fuzzy setting, one way of the equivalence and a partial converse have been established. In the place of compactness which is unavoidably an underlying concept for the study of perfect maps between classical set-topological spaces, we have proposed a new definition viz. q-compactness. Such a definition suits our purpose rather satisfactorily enabling us to apply fuzzy perfect maps to different fuzzy topological properties. In Section 7.1 we develop the concept of fuzzy directed families to some extent and study it vis-a-vis fuzzy perfect maps which are defined in the same section. In the concluding part we take up some fuzzy topological properties for their study under such a map. For instance, it is shown that it keeps fuzzy regularity invariant, while preimage of a fuzzy compact space under such a map is fuzzy compact. Perfectness of mappings along with some others have been found helpful in keeping some more properties invariant. From the investigations it becomes apparent that many other such properties can be treated along the same line.
§ 7.1. FUZZY DIRECTED FAMILY AND FUZZY PERFECT FUNCTIONS

DEFINITION 7.1.1. A fuzzy singleton \( x_\alpha \) in an fts \( X \) is called a fuzzy cluster point of a fuzzy directed family (henceforth abbreviated as f.d.f.) \( \mathcal{F} \) iff \( x_\alpha \in \bigcap \{ c : F \in \mathcal{F} \} \).

DEFINITION 7.1.2. An f.d.f. \( \mathcal{F} \) in an fts \( X \) is called subordinate to an f.d.f. \( \mathcal{G} \) in \( X \) iff for every \( F \in \mathcal{G} \), there exists \( G \in \mathcal{F} \) such that \( G \subseteq F \).

DEFINITION 7.1.3. An f.d.f. \( \mathcal{F} \) in \( (X,T) \) is said to be directed towards a fuzzy set \( A \) (written as \( \mathcal{F} \rightarrow A \)) iff for every f.d.f. \( \mathcal{G} \) subordinate to \( \mathcal{G} \), there exists \( x \in \text{supp} A \) such that \( x_A(x) \) is a fuzzy cluster point of \( \mathcal{F} \).

THEOREM 7.1.4. Let \( A \) be a fuzzy set and \( x_\alpha \) a fuzzy singleton in an fts \( X \).

(a) For an f.d.f. \( \mathcal{F} \) in \( X \), if \( \mathcal{F} \rightarrow A \) (\( \mathcal{F} \rightarrow x_\alpha \)) then for every f.d.f. \( \mathcal{G} \) subordinate to \( \mathcal{G} \), \( \mathcal{G} \rightarrow A \) (resp. \( \mathcal{G} \rightarrow x_\alpha \)).

(b) If \( x_\alpha \) is a fuzzy cluster point of an f.d.f. \( \mathcal{F} \) in \( X \), then \( \mathcal{U} = \{ F \cap U : F \in \mathcal{F} \} \) and \( U \) is a fuzzy open q-nbd of \( x_\alpha \) is an f.d.f. subordinate to \( \mathcal{F} \), converging to \( x_\alpha \).

(c) The collection of all fuzzy open q-nbds of \( x_\alpha \) is an f.d.f. in \( X \) converging to \( x_\alpha \).

(d) For an f.d.f. \( \mathcal{F} \) in \( X \), \( \mathcal{F} \rightarrow x_\alpha \) is f.d. \( \mathcal{F} \rightarrow x_\alpha \) and \( x_\alpha \) is a fuzzy cluster point of \( \mathcal{F} \).

PROOF. (a) Let \( \mathcal{F} \rightarrow A \). If \( \mathcal{H} \) is an f.d.f. subordinate to \( \mathcal{G} \), then it is also subordinate to \( \mathcal{F} \) and hence for some \( y \in \text{supp} A \), \( y_A(y) \) is a fuzzy...
Next let \( y \rightarrow x_\alpha \). For every fuzzy open q-nbd \( U \) of \( x_\alpha \), there exists \( F \in \mathcal{F} \) such that \( F \nsubseteq U \). Then there exists \( G \in \mathcal{F} \) such that \( G \nsubseteq F \). Thus \( G \nsubseteq U \) and \( y \rightarrow x_\alpha \).

(b) Let \( x_\alpha \in \bigcap \{ \text{cl} F : F \in \mathcal{F} \} \). Then for each \( F \in \mathcal{F} \) and each fuzzy open q-nbd \( U \) of \( x_\alpha \), \( F \nsubseteq U \). This implies that \( F \cap U \neq 0 \), for all \( F \cap U \in \mathcal{U} \).

Again let \( F_1 \cap U_1, F_2 \cap U_2 \in \mathcal{U} \). Then there exists \( F_3 \in \mathcal{F} \) such that \( F_3 \nsubseteq F_1 \cap F_2 \) and \( U_1 \cap U_2 = V \) (say) is a fuzzy open q-nbd of \( x_\alpha \). Thus \( F_3 \cap V \in \mathcal{U} \) and \( F_3 \cap V \nsubseteq (F_1 \cap U_1) \cap (F_2 \cap U_2) \). Hence \( \mathcal{U} \) is an f.d.f. subordinate to \( \mathcal{F} \).

Again, for any fuzzy open q-nbd \( U \) of \( x_\alpha \), \( F \cap U \in \mathcal{U} \) (for any \( F \in \mathcal{F} \)) such that \( F \cap U \nsubseteq U \). Thus \( y \rightarrow x_\alpha \).

(c) Clear.

(d) If \( \mathcal{F} \dashv x_\alpha \), there exists a fuzzy open q-nbd \( U \) of \( x_\alpha \) such that \( F \nsubseteq U \), for all \( F \in \mathcal{F} \), so that \( F \cap (1-U) \), for all \( F \in \mathcal{F} \). Let \( F^* = F \cap (1-U) \) \((\# 0 \alpha)\). Consider \( \mathcal{F}^* = \{ F^* : F \in \mathcal{F} \} \). For \( F_1^*, F_2^* \in \mathcal{F}^* \), \( F_1^* = F_1 \cap (1-U) \) and \( F_2^* = F_2 \cap (1-U) \), where \( F_1, F_2 \in \mathcal{F} \), then there exists \( F \in \mathcal{F} \) such that \( F \nsubseteq F_1 \cap F_2 \) so that \( F^* \nsubseteq (F_1 \cap F_2) \cap (1-U) = [F_1 \cap (1-U)] \cap [F_2 \cap (1-U)] = F_1^* \cap F_2^* \). Hence \( \mathcal{F}^* \) is an f.d.f. and is clearly subordinate to \( \mathcal{F} \). Again \( U \nsubseteq F^* \), for all \( F \in \mathcal{F} \). Thus \( x_\alpha \) is not a fuzzy cluster point of \( \mathcal{F} \), i.e., \( \mathcal{F} \not\vdash x_\alpha \). Hence \( \mathcal{F} \vdash x_\alpha \Rightarrow \mathcal{F} \vdash x_\alpha \). But \( \mathcal{F} \vdash x_\alpha \Rightarrow x_\alpha \) is a fuzzy cluster point of \( \mathcal{F} \).

**THEOREM 7.1.5.** Let \( f : X \rightarrow Y \) be a function.

(a) If \( \mathcal{F} \) is an f.d.f. in \( X \), then so is \( f(\mathcal{F}) = \{ f(F) : F \in \mathcal{F} \} \) in \( Y \).

(b) If \( \mathcal{G} \) is an f.d.f. in \( Y \) and \( f \) is onto, then \( f^{-1}(\mathcal{G}) = \{ f^{-1}(B) : B \in \mathcal{G} \} \) is an f.d.f. in \( X \).

(c) Let \( \mathcal{F} \) be an f.d.f. in \( X \) and \( \mathcal{G} \) be an f.d.f in \( Y \) subordinate to
If $f$ is onto and $\mathcal{F}' = f^{-1}(\mathcal{G})$, then $\mathcal{F} \cap \mathcal{F}' = \{ F \cap F' : F \in \mathcal{F} \text{ and } F' \in \mathcal{F}' \}$ is an f.d.f. subordinate to both $\mathcal{F}$ and $\mathcal{F}'$.

**PROOF.** We only prove (c), considering the proofs of (a) and (b) to be straightforward.

Let $F \in \mathcal{F}$ and $f^{-1}(B) \in \mathcal{F}'$, where $B \in \mathcal{B}$. Then there exists $B_1 \in \mathcal{B}$ such that $B_1 \subseteq f(F)$. Also, $B_1 \cap f^{-1}(B) \supseteq \text{a member } B \in \mathcal{B}$. Thus $f^{-1}(B') \cap F \neq \emptyset$. Moreover, $f^{-1}(B') \subseteq f^{-1}(B)$ and hence $f^{-1}(B) \cap F \neq \emptyset$. Again, for $F_1 \cap F_1'$, $F_2 \cap F_2'$, there exist $F_1$, $F_2 \in \mathcal{F}$ such that $F_1 \subseteq F_1 \cap F_2$ and $F_2 \subseteq F_1 \cap F_2'$. Thus $F_2 \cap F_2' \subseteq f^{-1}(B) \cap f^{-1}(B')$. Hence $\mathcal{F} \cap \mathcal{F}'$ is an f.d.f. That it is subordinate to both $\mathcal{F}$ and $\mathcal{F}'$ is obvious.

**THEOREM 7.1.6.** Let $f : X \rightarrow Y$ be onto and $B$ be a fuzzy set in $Y$. If for each f.d.f. $\mathcal{G}$ in $Y$, $\mathcal{G}' \rightarrow y_B(y)$ for some $y \in \text{supp}B$ implies $f^{-1}(\mathcal{G}') \rightarrow f^{-1}(y_B(y))$, then for any f.d.f. $\mathcal{F}$ in $Y$, $\mathcal{F} \rightarrow B$ implies $f^{-1}(\mathcal{F}) \rightarrow f^{-1}(B)$.

**PROOF.** Let $\mathcal{U}$ be an f.d.f. in $X$ subordinate to $f^{-1}(\mathcal{F})$, where $\mathcal{F}$ is any f.d.f. in $Y$. Then $\mathcal{G}' = f(\mathcal{U})$ is an f.d.f. in $Y$ subordinate to $\mathcal{F}$. Thus there exists $y \in \text{supp}B$ such that $y_B(y)$ is a fuzzy cluster point of $\mathcal{G}'$. By Theorem 7.1.4 (b), there exists an f.d.f. $\mathcal{U}'$ subordinate to $\mathcal{G}'$ such that $\mathcal{U}' \rightarrow \mathcal{U}'$ where $\alpha = B(y)$. By hypothesis, $\mathcal{U}' = f^{-1}(\mathcal{G}') \rightarrow f^{-1}(y_B(y))$. Also, by Theorem 7.1.5(c), $\mathcal{U}$ and $\mathcal{U}'$ have a common subordinate f.d.f. $\mathcal{U}''$. Then for some $x \in f^{-1}(y)$, $x_\alpha$ is a fuzzy cluster point of $\mathcal{U}''$. It is clear that $x_\alpha$ is also a fuzzy cluster point of $\mathcal{U}$. Since $f^{-1}(B)(x) = B(f(x)) = B(y) = \alpha$, it follows that $f^{-1}(\mathcal{F}) \rightarrow f^{-1}(B)$. 


DEFINITION 7.1.7. A collection $\mathcal{U}$ of fuzzy sets in an fts $X$ is called a q-cover of a fuzzy set $A$ iff for each $x \in \text{supp} A$, $\chi_{A(x)} (\cup \mathcal{U})$. If each member of $\mathcal{U}$ is fuzzy open, then $\mathcal{U}$ is called an open q-cover.

DEFINITION 7.1.8. An f.d.f. $\mathcal{F}$ is said to converge to a fuzzy set $A$, written as $\mathcal{F} \rightarrow A$, iff for every open q-cover $\mathcal{U}$ of $A$, there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ and exists an $F \in \mathcal{F}$ such that $F \notin \cup \mathcal{U}_0$.

NOTE 7.1.9. The above definition agrees with the definition 5.14(i) when the fuzzy set $A$ in the above definition is replaced by a fuzzy singleton.

THEOREM 7.1.10. Let $f : X \rightarrow Y$ be an onto function. If for every f.d.f. $\mathcal{F}$ in $Y$ converging to $y_\alpha$, $f^{-1}(y_\alpha)$ converges to $f^{-1}(y_\alpha)$, then for every f.d.f. $\mathcal{F}$ in $X$, $\cap \{\text{clf}(F) : F \in \mathcal{F}\} \subseteq f(\cap \{\text{clf} : F \in \mathcal{F}\})$.

PROOF. If possible, let there exist a fuzzy singleton $y_\alpha$ in $Y$ such that $y_\alpha \notin \cap \{\text{clf}(F) : F \in \mathcal{F}\}$ but $y_\alpha \notin f(\cap \{\text{clf} : F \in \mathcal{F}\})$. Then for each $x \in f^{-1}(y)$, $x_\alpha \notin \cap \{\text{clf} : F \in \mathcal{F}\}$, and hence there exist a fuzzy open q-nbd $U_x$ of $x_\alpha$ and an $F \in \mathcal{F}$ such that $U_x \notin F$. Then $\mathcal{U} = \{U_x : x \in f^{-1}(y)\}$ is an open q-cover of $f^{-1}(y_\alpha)$. Now $\mathcal{G} = \{V : V$ is a fuzzy open q-nbd of $y_\alpha\}$ is clearly an f.d.f. in $Y$ converging to $y_\alpha$. By hypothesis, $f^{-1}(\mathcal{G})$ converges to $f^{-1}(y_\alpha)$. Since $\mathcal{U}$ is an open q-cover of $f^{-1}(y_\alpha)$, there exist a finite subset $B$ of $f^{-1}(y_\alpha)$ and a $V \in \mathcal{G}$ such that $f^{-1}(V) \notin \cup \{U_x : x \in B\}$. Let $F \in \mathcal{F}$ such that $F \notin \cap \{F_x : x \in B\}$. Then $F \notin f^{-1}(V)$ and hence $f(F) \notin V$. Thus $y_\alpha \notin \text{clf}(F)$ which is a contradiction.

THEOREM 7.1.11. Let $\mathcal{F}$ be an f.d.f. in $X$ and $A$ any fuzzy set in $X$. If
PROOF. Let \( q \rightarrow A \) and \( \mathcal{U} \) an open \( q \)-cover of \( A \). If for some \( x \in \text{supp}A \), there exist a \( U_x \in \mathcal{U} \) and a \( F \in \mathcal{F} \) such that \( F_x \not\subseteq U_x \), then the requirement is obviously met. If this is not the case, then for each \( x \in \text{supp}A \), there exists a \( V_x \in \mathcal{U} \) with \( x_{A(x)} \cap V_x \) such that \( F \not\subseteq V_x \), for all \( F \in \mathcal{F} \). Thus \( F \cap (1-V_x) \), for all \( F \in \mathcal{F} \). Suppose \( \mathcal{Q}_x = \{ F \cap (1-V_x) : F \in \mathcal{F} \} \) and let \( \mathcal{Q} = \bigcup_{x \in \text{supp}A} \mathcal{Q}_x \). If \( \mathcal{Q} \) has finite intersection property, then all finite intersections of members of \( \mathcal{Q} \) is an f.d.f. subordinate to \( \mathcal{F} \), and clearly \( x_{A(x)} \not\in \text{cl}F \). Let \( \mathcal{F} = \mathcal{F} \), and \( \mathcal{F} \rightarrow A \). Thus there exist finitely many sets \( F \cap (1-V_x) \), \( F \cap (1-V_x) \), such that \( \bigcap_{i=1}^{n} F_i \cap (1-V_x) = 0_x \). Let \( F \in \mathcal{F} \) such that \( F \not\subseteq F_1 \) \( \bigcap_{i=1}^{n} F_i \cap (1-V_x) = 0_x \). Then \( F \cap [ \bigcap_{i=1}^{n} (1-V_x) ] = 0_x \) and hence \( F \not\subseteq \bigcap_{i=1}^{n} (1-V_x) \). Consequently, \( \mathcal{F} \rightarrow A \).

It then follows clearly from the Definition 0.9.34 that

**Theorem 7.1.12.** A function \( f : X \rightarrow Y \) is fuzzy closed iff for any fuzzy set \( A \) in \( X \), \( \text{cl}(A) \not\subseteq f(\text{cl}A) \).

**Lemma 7.1.13.** Let \( f : X \rightarrow Y \) be a function. Then for any fuzzy set \( B \) in \( Y \), \( f[1 - f^{-1}(B)] \not\subseteq 1-B \), where equality holds if \( f \) is onto.

**Proof.** Simple and omitted.

**Theorem 7.1.14.** If \( f : X \rightarrow Y \) is a fuzzy closed function, then for any fuzzy set \( S \) in \( Y \) and any fuzzy open set \( U \) with \( U \not\subseteq f^{-1}(S) \), there exists
a fuzzy open set \( V \) in \( Y \) such that \( V \supset S \) and \( f^{-1}(V) \subseteq U \).

**PROOF.** Let us put \( V = 1 - f(1 - U) \). Then \( V \) is fuzzy open in \( Y \). Now, \( f^{-1}(S) \subseteq U \iff 1 - f^{-1}(S) \supset 1 - U \iff 1 - f'[1 - f^{-1}(S)] \subseteq 1 - f'(1 - U) = V \iff V \supset 1 - (1 - S) = S \) (by Lemma 7.1.13). Again, \( f^{-1}(V) = f^{-1}[1 - f(1 - U)] = 1 - f^{-1}(1 - U) \subseteq 1 - (1 - U) = U \).

**THEOREM 7.1.15.** Let \( f : X \to Y \) be a fuzzy closed mapping. Let \( S \) be a fuzzy set in \( Y \) and \( U \) any fuzzy open set in \( X \) such that there exists \( y \in \text{supp}S \) satisfying \( f(1 - U)(y) < S(y) \). Then there exists a fuzzy open set \( V \) in \( Y \) such that \( V \supset S \) and \( f^{-1}(V) \subseteq U \).

**PROOF.** As in the proof of Theorem 7.1.14, we put \( V = 1 - f(1 - U) \). Then \( V \) is a fuzzy open set such that \( f^{-1}(V) \subseteq U \). Again, there exists \( y \in \text{supp}S \) such that \( f(1 - U)(y) < S(y) \). Then \( V(y) + S(y) > 1 \) and hence \( V \supset S \).

**DEFINITION 7.1.16.** A fuzzy set \( A \) in an ft's \( X \) is called \( q \)-compact iff for every open \( q \)-cover \( \mathcal{U} \) of \( A \), there exists a finite subcollection \( \mathcal{U}_0 \) of \( \mathcal{U} \) such that \( \sup \{ (1 - \bigcup_{U_i} U_0) (x) : x \in \text{supp}A \setminus A(x) \} \), for every \( x \in \text{supp}A \).

**THEOREM 7.1.17.** A fuzzy set \( A \) in an ft's \( X \) is \( q \)-compact if for every f.d.f. \( \mathcal{Y} \) in \( A \), there exists \( x \in \text{supp}A \) such that \( x_{\mathcal{Y}}(x) \) is a fuzzy cluster point of \( \mathcal{Y} \).

**PROOF.** Let \( A \) be not \( q \)-compact. Then there is an open \( q \)-cover \( \mathcal{U} \) of \( A \) such that for every finite subcollection \( \{ U_1, U_2, \ldots, U_n \} \) (say) of \( \mathcal{U} \), there exists \( x \in \text{supp}A \) such that \( \sup \{ (1 - \bigcup_{i=1}^n U_i) (x) : x \in \text{supp}A \setminus A(x) \} > 0 \).
i.e., at some point \( x_0 \in \text{supp}A \), \( (1 - \bigcup_{i=1}^{n} U_i) (x_0) > 0 \). Then \( \mathcal{F} = \{ A \cap (1 - \bigcup_{i=1}^{n} U_i) : \mathcal{U}_0 \text{ is a finite subset of } \mathcal{U} \} \) is an f.d.f. in \( A \). By hypothesis, there exists \( x \in \text{supp}A \) such that \( x A(x) \) is a fuzzy cluster point of \( \mathcal{F} \).

Now, \( x A(x) \cap (U \mathcal{U}) = \) there is some \( U \in \mathcal{U} \) such that \( x A(x) \cap U \). But \( F = (1 - U) \cap A \in \mathcal{F} \) and \( F \notin U \) so that \( x A(x) \notin \text{cl}F \). Hence \( x A(x) \) cannot be a fuzzy cluster point of \( \mathcal{F} \) which is a contradiction.

**DEFINITION 7.1.18.** A function \( f : X \to Y \) is called fuzzy perfect iff \( f \) is onto, fuzzy closed, fuzzy continuous and \( f^{-1}(y_{\alpha}) \) is \( q \)-compact, for each fuzzy singleton \( y_{\alpha} \) in \( Y \).

**THEOREM 7.1.19.** If an onto function \( f : X \to Y \) is fuzzy closed and inverse image under \( f \) of every fuzzy singleton in \( Y \) is \( q \)-compact in \( X \), then for each f.d.f. \( \mathcal{F} \) in \( Y \), whenever \( \mathcal{F}_{\alpha} \) (where \( y_{\alpha} \) is some fuzzy singleton in \( Y \)) then \( \mathcal{F} = f^{-1}(\mathcal{F}_{\alpha}) \).

**PROOF.** If possible, let \( \mathcal{F}' \) be an f.d.f. in \( X \) subordinate to \( \mathcal{F} \) such that for each \( x \in f^{-1}(y_{\alpha}) \), \( x_{\alpha} \) is not a fuzzy cluster point of \( \mathcal{F}' \). For each \( x \in f^{-1}(y_{\alpha}) \), there exist a fuzzy open \( q \)-nbd \( U_x \) of \( x_{\alpha} \) and \( F_x \in \mathcal{F}' \) such that \( U_x \subseteq F_x' \). Then \( \mathcal{V} = \{ U_x : x \in f^{-1}(y_{\alpha}) \} \) forms an open \( q \)-cover of \( f^{-1}(y_{\alpha}) \). By \( q \)-compactness of \( f^{-1}(y_{\alpha}) \), there exist finitely many members \( U_{x_1}, U_{x_2}, \ldots, U_{x_n} \) of \( \mathcal{V} \) such that \( \sup \{ (1 - U)(x) : x \in f^{-1}(y_{\alpha}) \} \) is \( q \)-compact in \( Y \). Also, \( V \) is fuzzy open in \( Y \) since \( f \) is a fuzzy closed map. Thus \( V \) is a fuzzy open \( q \)-nbd of \( y_{\alpha} \). Let \( F' \in \mathcal{F}' \) such that \( F' \subseteq f^{-1}(y_{\alpha}) \) and \( f(F') \subseteq \mathcal{V} \). By Theorem 7.1.5(a) since \( f(\mathcal{F}') = \mathcal{F} \) (say) is an f.d.f. subordinate to \( \mathcal{F} \) and \( f(F') \subseteq \mathcal{F} \), it follows that \( y_{\alpha} \) is not a fuzzy cluster point.
point of \( y' \). Hence \( y' \) cannot be directed towards \( y \), which is a contradiction.

As a partial converse of the above theorem, we have:

**Theorem 7.1.20.** Let \( f \) be an onto function. If for every f.d.f. \( \xi \) in \( Y \), whenever \( \xi \) converges to some fuzzy singleton \( y_a \) in \( Y \), \( f^{-1}(y_a) \rightarrow f^{-1}(y_a) \), then \( f \) is a fuzzy closed map such that \( f^{-1}(z_B) \) is \( q \)-compact for each fuzzy singleton \( z_B \) in \( Y \).

**Proof.** First suppose that under the given condition, \( f \) is not fuzzy closed.

Then there exists a fuzzy closed set \( A \) in \( X \) and there exists a fuzzy singleton \( y_a \) in \( Y \) such that \( y_a \notin \text{cl}(A) \) but \( y_a \notin f(A) \). Let \( \mathcal{U} = \{ f(A) \cap U : U \text{ is a fuzzy open q-nbd of } y_a \} \). Then \( \mathcal{U} \) is an f.d.f. in \( Y \) such that \( \mathcal{U} \rightarrow y_a \). Let \( U = f^{-1}(y_a) \) and \( U' = \{ A \cap E : E \in \mathcal{U} \} \). Then \( U' \) is an f.d.f. subordinate to \( U \).

Again, \( y_a \notin f(A) \rightarrow y_a q [1 - f(A)] \rightarrow f^{-1}(y_a) \). Therefore, \( f^{-1}(1-f(A)) = 1 - f^{-1}(f(A) \setminus 1 - A) \rightarrow f^{-1}(y_a) q (1-A) \).

For any \( x \in f^{-1}(y_a) \), consider the fuzzy singleton \( x_a \) in \( X \). Now, \( y_a \notin f(A) \rightarrow a \rightarrow [f(A)] \) \( (y_a) = \sup \{ A(z) : z \in f^{-1}(y_a) \} \rightarrow A(x) = x \rightarrow q (1-A) \).

Thus, \( (1-A) \) is a fuzzy open q-nbd of \( x_a \) and clearly \( (1-A) \notin U' \) for each \( U \in U' \). Thus for each \( x \in f^{-1}(y_a) \), \( x \neq f^{-1}(y_a) \) is not a fuzzy cluster point of \( U' \) so that \( U \) is not directed towards \( f^{-1}(y_a) \) which is a contradiction to the hypothesis. Hence \( f \) is fuzzy closed.

Now we show that for each fuzzy singleton \( y_a \) in \( Y \), \( f^{-1}(y_a) \) is \( q \)-compact in \( X \). For this, it suffices by virtue of Theorem 7.1.17 to show that for every f.d.f. \( \xi \) in \( f^{-1}(y_a) \), there exists \( x \in f^{-1}(y_a) \) such that \( x_a \) is a fuzzy cluster point of \( \xi \). It is clear that \( \xi = \{ y_B : 0 < B \leq a \} \) is an f.d.f. in \( Y \) such that \( \xi = y_a \). Then by hypothesis, \( U = f^{-1}(y_a) \). Let \( \mathcal{U} \) be an f.d.f. in \( f^{-1}(y_a) \). Consider \( \mathcal{U}^* = \).
RESULT 7.1.21. [98] If $f : X \rightarrow Y$ is fuzzy $\Theta$-continuous, then the following are true:

(a) $f([A]_\Theta) \subseteq [f(A)]_\Theta$, for every fuzzy set $A$ in $X$.
(b) For each fuzzy $\Theta$-closed set $A$ in $Y$, $f^{-1}(A)$ is fuzzy $\Theta$-closed in $X$.

THEOREM 7.1.22. If $f : X \rightarrow Y$ is a fuzzy perfect function then for a fuzzy set $B$ in $Y$,

(a) $[B]_\Theta = f([f^{-1}(B)]_\Theta)$,
(b) $B$ is fuzzy $\Theta$-closed iff $f^{-1}(B)$ is fuzzy $\Theta$-closed.

PROOF. (a) Let $y_\alpha$ be any fuzzy singleton in $Y$ such that $x_\alpha \not\in f([f^{-1}(B)]_\Theta)$. Let us denote the support of $f^{-1}(y_\alpha)$ by $S$. Now, for each $x \in S$, $x_\alpha \not\in [f^{-1}(B)]_\Theta$. Thus there exists a fuzzy open q-nbd $V_x$ of $x_\alpha$ such that $\text{cl}V_x \not\subseteq f^{-1}(B)$. Then $\{V_x : x \in S\}$ is an open q-cover of $f^{-1}(y_\alpha)$. Since $f^{-1}(y_\alpha)$ is q-compact, there exists a finite subset $\mathcal{V}_0$ of $\{V_x : x \in S\}$ such that $\text{sup} \{ f^{-1}(y_\alpha) (x) : x \in f^{-1}(y) \} \subseteq \{ f^{-1}(y_\alpha) (x) : x \in f^{-1}(y) \}$, for each $x \in f^{-1}(y)$. Then $f(1-V) (y) \subseteq (y_\alpha) (f(x)) = y_\alpha (y) = \alpha$, where $V = \bigcup \mathcal{V}_0$, and also $\text{cl}V \not\subseteq f^{-1}(B)$. By Theorem 7.1.15, there exists a fuzzy open set $W$ in $Y$ such that $W \cap y_\alpha$ and $f^{-1}(W) \not\subseteq V$. From $\text{cl}V \not\subseteq f^{-1}(B)$ and $f^{-1}(W) \not\subseteq V$, we obtain $\text{cl}f^{-1}(W) \not\subseteq f^{-1}(B)$. Then $f(\text{cl}f^{-1}(W)) \not\subseteq f(1-f^{-1}(B)) = 1-B$ (by Lemma 7.1.13). It then follows from Theorem 7.1.12 that $\text{cl}W \not\subseteq B$. Consequently, $y_\alpha \not\in [B]_\Theta$. Hence $[B]_\Theta \subseteq f([f^{-1}(B)]_\Theta)$.
Conversely, $f$ being fuzzy continuous, it is $\alpha$-continuous. Hence by Result 7.1.21 (a), $f([f^{-1}(B)]_{\alpha}) \subset [f^{-1}(B)]_{\alpha} = [B]_{\alpha}$. Thus (a) is established.

(b) Let $f^{-1}(B)$ be fuzzy $\theta$-closed. Then $[B]_{\theta} = f([f^{-1}(B)]_{\theta})$ (by (a)) = $ff^{-1}(B) = B$, and hence $B$ is fuzzy $\theta$-closed.

Conversely, if $B$ is fuzzy $\theta$-closed in $Y$, by Result 7.1.21(b), $f^{-1}(B)$ is fuzzy $\theta$-closed in $X$ as $f$ is fuzzy $\theta$-continuous (it being fuzzy continuous).

**Remark 7.1.23.** In the above theorem, the condition of fuzzy continuity of $f$ could be replaced by that of fuzzy $\theta$-continuity.

§ 7.2. **Fuzzy Topological Properties and Fuzzy Perfect Maps**

**Theorem 7.2.1.** Let $f : X \rightarrow Y$ be fuzzy perfect. If $X$ is fuzzy regular, then so is $Y$.

**Proof.** Let $y_\alpha$ be a fuzzy singleton in $Y$ and $U$ a fuzzy open $q$-nbdd of $y_\alpha$. Since $f$ is fuzzy continuous, $f^{-1}(U)$ is a fuzzy open $q$-nbdd of $x_\alpha$, for each $x \in f^{-1}(y)$. $X$ being fuzzy regular, there exists a fuzzy open $q$-nbdd $V_x$ of $x_\alpha$, for each $x \in f^{-1}(y)$, such that $cl V_x \subset f^{-1}(U)$. Then $\{ V_x : x \in f^{-1}(y) \}$ is an open $q$-cover of $f^{-1}(y_\alpha)$. By $q$-compactness of $f^{-1}(y_\alpha)$, there exists, as in the proof of Theorem 7.1.22(a), a finite subset $\{ V_{x_1}, ..., V_{x_n} \}$ (say) of $\{ V_x : x \in f^{-1}(y) \}$ such that $f(1-V)(y_\alpha) \subset y_\alpha(y) = \alpha$, where $V = \bigcup_{i=1}^{n} V_{x_i}$, and moreover, $cl V \subset f^{-1}(U)$. By Theorem 7.1.15, there exists a fuzzy open $q$-nbdd $W$ of $y_\alpha$ such that $f^{-1}(W) \subset V$. Now, $W = ff^{-1}(W) \subset f(V) \subset f(cl V) \subset ff^{-1}(U) = U$, and hence $cl W \subset cl f(cl V) \subset f(cl V)$ (using Theorem 7.1.12) $\subset U$. Consequently, in view of Result 0.9.37, $Y$ is fuzzy regular.
THEOREM 7.2.2. Let \( f : X \rightarrow Y \) be fuzzy perfect and \( Y \) be fuzzy compact. Then \( X \) is also fuzzy compact.

PROOF. Let \( \mathcal{U} = \{ U_\alpha : \alpha \in A \} \) be a fuzzy open cover of \( X \). Consider any \( y \in Y \) and an \( n \in \mathbb{N} \), where \( \mathbb{N} \) denotes as usual, the set of naturals. Now, \( \mathcal{U} \) is an open q-cover of \( f^{-1}(y) \). By q-compactness of \( f^{-1}(y) \) there exists a finite subset \( \mathcal{V}^X_y \) of \( \mathcal{U} \) such that \( \sup \{ (1-U)(x) : x \in f^{-1}(y) \} < (f^{-1}(y)) \) for each \( x \in f^{-1}(y) \), where \( U = \bigcup \mathcal{V}^X_y \). Then \( [f(1-U)](y) < (y)(f(x)) = (y)(y) = (1/n) \). By Theorem 7.1.15, there exists a fuzzy open q-nbd \( V^X_y \) of \( y \) such that \( f^{-1}(V^X_y) \ll \cup \mathcal{U} \). Since \( V^X_y \cap Y_1 \) is a fuzzy open cover of \( Y \). By fuzzy compactness of \( Y \), there exist finitely many points \( y_1, y_2, \ldots, y_k \in Y \) and \( x_1, x_2, \ldots, x_k \in f^{-1}(Y) \) such that \( \bigcup \mathcal{V}^Y_{y_1} \). Then \( I_X = f^{-1}(I_Y) = f^{-1}\left[ \bigcup \mathcal{V}^X_{y_i} \right] \ll \bigcup \mathcal{V}^Y_{y_i} \). Hence \( X \) is fuzzy compact.

Ganguly and Saha [40] defined almost fuzzy open function in the way as given below. Mukherjee and Sinha [100], calling it a.f.o.G. function proved that this concept is independent of that of Nanda [105] as already given in Definition 0.9.35.

DEFINITION 7.2.3. A function \( f : X \rightarrow Y \) is said to be almost fuzzy open in the sense of Ganguly and Saha, or simply a.f.o.G. iff for each fuzzy open set \( B \) in \( Y \), \( f^{-1}(\text{cl}B) \ll \text{cl}f^{-1}(B) \).

THEOREM 7.2.4. Let \( f : X \rightarrow Y \) be fuzzy perfect.

(a) If \( f \) is a.f.o.G. and \( Y \) is fuzzy almost compact, then \( X \) is also fuzzy.
almost compact.

(b) If $Y$ is fuzzy nearly compact, then so is $X$.

**PROOF.** The proof is similar to that of Theorem 7.2.2 and is omitted.

**THEOREM 7.2.5.** [102] An fts $X$ is fuzzy almost regular iff for any fuzzy open set $A$ in $X$, $[[A]_\theta]_\theta = [A]_\theta$.

**THEOREM 7.2.6.** If $f : X \rightarrow Y$ is a function which is a.f.o.G. and fuzzy perfect, and $X$ is fuzzy almost regular, then $Y$ is also fuzzy almost regular.

**PROOF.** Let $V$ be a fuzzy open set in $Y$. Then by Theorem 7.1.22(a), $[c_1V]_\theta = f([f^{-1}(c_1V)]_\theta) = f([c_1f^{-1}(V)]_\theta)$. As $f$ is a.f.o.G. and fuzzy continuous, $f^{-1}(c_1V) = c_1f^{-1}(V)$. Also, $f^{-1}(V)$ being fuzzy open, $c_1f^{-1}(V) = [f^{-1}(V)]_\theta$. Thus $[c_1V]_\theta = f([f^{-1}(V)]_\theta)$. Since $f^{-1}(V)$ is fuzzy open, by Theorem 7.2.5, we have $[[f^{-1}(V)]_\theta]_\theta = [f^{-1}(V)]_\theta = c_1f^{-1}(V)$. Thus $[c_1V]_\theta = f(c_1f^{-1}(V)) \subseteq c_1(f^{-1}(V)) = c_1V$, and consequently, $[c_1V]_\theta = c_1V$ (since for any fuzzy set $A$, $c_1A \not\subseteq [A]_\theta$). Since $V$ is fuzzy open, $[[V]_\theta]_\theta = [c_1V]_\theta = [V]_\theta$. Hence $Y$ is fuzzy almost regular (by Theorem 7.2.5).

Since a fuzzy S-closed space is fuzzy almost compact, in view of Theorem 7.2.4(a) we immediately have:

**THEOREM 7.2.7.** Let $f : X \rightarrow Y$ be fuzzy perfect and a.f.o.G. If $Y$ is fuzzy S-closed, then $X$ is fuzzy almost compact.

We shall shortly improve the above theorem (see Theorem 7.2.9) for which we need the following Lemma:
LEMMA 7.2.8. Let \( f : X \rightarrow Y \) be fuzzy semi-closed and \( S \) be any fuzzy set in \( Y \). Also, let \( U \) be a fuzzy semi-open set in \( Y \) such that for some \( y \in \text{supp}S \), \( f(U) (y) < S(y) \). Then there exists a fuzzy semi-open set \( V \) in \( Y \) such that \( V \cap U \neq \emptyset \) and \( f^{-1}(V) \subseteq U \).

PROOF. The proof being similar to that of Theorem 7.1.15 is omitted.

THEOREM 7.2.9. Let \( f : X \rightarrow Y \) be a.f.o.G., fuzzy semi-closed and be such that \( f^{-1}(y_{\alpha}) \) is \( q \)-compact for each fuzzy singleton \( y_{\alpha} \) in \( Y \). If \( Y \) is fuzzy \( S \)-closed, then \( X \) is fuzzy almost compact.

PROOF. We omit the proof which goes in a similar line to that of Theorem 7.2.2. with certain straightforward modifications.