CHAPTER V

FUZZY EXTREMALLY DISCONNECTED SPACES

Our sole purpose of this chapter is to introduce extremally disconnected fts's, and to analyse the concept from variegated directions by means of a large number of characterization theorems. In the process, come the appliances like fuzzy semi-open and semi-closed sets, fuzzy closure, semi-closure, $\delta$-and $\theta$-closure etc. which explain effectively the introduced concept from different standpoints. We have also introduced the definitions of convergence, $\delta$-convergence, $\theta$-convergence and $rc$-convergence of directed families in the fuzzy setting to formulate the characterizations of fuzzy extremally disconnected spaces from an altogether new angle. It has been possible to show that fuzzy extremally disconnectedness is a fuzzy semi-regular property; the mapping which keeps this property invariant has also been found out. As to the applicational aspect of such a concept, we postpone till the next chapter, some such applications in connection with the study of fuzzy $S$-closed spaces. Nevertheless, in the concluding part of the present chapter we observe how the presence of fuzzy extremally disconnectedness on the domain spaces of certain types of mappings makes the properties of the maps to get strengthened.

DEFINITION 5.1. An fts $(X,T)$ is said to be fuzzy extremally disconnected (FED, for short) iff closure of every fuzzy open set is fuzzy open in $X$; equivalently, every fuzzy regularly closed set is fuzzy open.

THEOREM 5.2. An fts $X$ is fuzzy extremally disconnected iff every non $q$-coincident fuzzy open sets in $X$ have non $q$-coincident closures.
PROOF. Let $U$ and $V$ be two fuzzy open sets in a fuzzy extremally disconnected space $X$ such that $U \nsubseteq V$. Then by Result 0.9.16., $clU \nsubseteq V$. Since $clU$ is fuzzy open, applying the same result once again we have $clU \nsubseteq clV$.

Conversely, let $U$ be a fuzzy open set in $X$. Then $1 - clU$ is also fuzzy open. Now,

$$U \nsubseteq (1 - clU) \implies clU \nsubseteq cl(1 - clU)$$

$$\implies cl(1 - clU) \nsubseteq 1 - clU$$

$$\implies cl(1 - clU) = 1 - clU$$

$$\implies (1 - clU) \text{ is fuzzy closed}$$

$$\implies clU \text{ is fuzzy open.}$$

Hence $X$ is fuzzy extremally disconnected.

LEMMA 5.3. If $X$ is fuzzy extremally disconnected, then $clV = sclV$, for every fuzzy semi-open set $V$ in $X$.

PROOF. Since $sclV \nsubseteq c1V$, for every fuzzy set $V$ in $X$, if suffices to show that $sclV \nsubseteq clV$, for any fuzzy semi-open set $V$ in $X$. Let $x_\alpha \notin sclV$. Then there exists a fuzzy semi-open semi-q-nbd $U$ of $x_\alpha$ such that $U \nsubseteq V$. Then $intU \nsubseteq intV$. Since $X$ is fuzzy extremally disconnected, we have by Theorem 5.2 that $clintU \nsubseteq clint V$. Then $x_\alpha \notin clint V = c1V$. Hence $c1V \nsubseteq sclV$.

LEMMA 5.4. [102] In an fts $X$, $[A] = c1A$, for any $A \in FSO(X)$.

THEOREM 5.5. The following statements are equivalent for an fts $X$:

(a) $X$ is FED.

(b) The fuzzy closure of every fuzzy semi-open set in $X$ is fuzzy open.
(c) The fuzzy semi-closure of every fuzzy semi-open set in $X$ is fuzzy open.

(d) The fuzzy $\delta$-closure of every fuzzy semi-open set in $X$ is fuzzy open.

(e) Any two non $q$-coincident fuzzy semi-open sets in $X$ have non $q$-coincident fuzzy closures.

(f) $\text{cl}A = \text{scl}A$, for every fuzzy semi-open set $A$ in $X$.

(g) The fuzzy semi-closure of every fuzzy semi-open set in $X$ is fuzzy closed.

(h) $\text{int} A = \text{sint} A$, for every fuzzy semi-closed set $A$ in $X$.

(i) The semi-interior of every fuzzy semi-closed set in $X$ is fuzzy open.

PROOF. (a) and (b) are equivalent in view of the fact that the fuzzy closure of a fuzzy semi-open set is same as the fuzzy closure of a fuzzy open set, namely its fuzzy interior. Also "(f) $\Rightarrow$ (g)" and "(h) $\Rightarrow$ (i)" are obvious.

(a) $\Rightarrow$ (c): Let $A$ be any fuzzy semi-open set in $X$. They by Lemma 5.3, $\text{scl}A = \text{cl}A$. Hence $\text{scl}A$ is fuzzy open in $X$.

(c) $\Rightarrow$ (a): Let $G$ be any fuzzy open set in $X$. To prove this it is sufficient to prove that $\text{cl}G = \text{scl}G$. Obviously, $\text{scl}G \subseteq \text{cl}G$. Let $x_\alpha \notin \text{sc}1G$. Then there exists a fuzzy semi-open semi-$q$-nbhd $U$ of $x_\alpha$ such that $U \notin G \Rightarrow U \subseteq 1 - G \Rightarrow \text{scl}U \subseteq \text{sc}1 (1 - G) = 1 - G \Rightarrow \text{scl}U \notin G$. Since $\text{scl}U$ is a fuzzy open $q$-nbhd of $x_\alpha$, $x_\alpha \notin \text{cl}G$. Hence it follows that $\text{cl}G = \text{scl}G$.

(b) $\Longleftrightarrow$ (d): Follows from Lemma 5.4.

(a) $\Rightarrow$ (f): Follows from Lemma 5.3.

(q) $\Rightarrow$ (f): For any fuzzy set $A$ in $X$, since $A \not\subseteq \text{sc}1A \not\subseteq \text{cl}A$, we have $\text{cl}A = \text{cl} (\text{sc}1A)$. If $A$ is fuzzy semi-open, by (g), $\text{sc}1A$ is fuzzy closed, and so $\text{sc}1A = \text{cl}A$. 


(i) $\Rightarrow$ (h): For any fuzzy set $A$ in $X$, $\text{int} A \subseteq \text{sint} A \subseteq A$, and hence $\text{int} A = \text{int} (\text{sint} A)$. If $A$ is fuzzy semi-closed, $\text{sint} A$ is fuzzy open (by (i)), and then $\text{int} A = \text{sint} A$.

(q) $\Rightarrow$ (i): Follows from the fact that $\text{sint} A = 1 - \text{scl} (1 - A)$ and $\text{scl} A = 1 - \text{sint} (1 - A)$, for any fuzzy set $A$.

(a) $\Rightarrow$ (e): Let $A$ and $B$ be fuzzy semi-open sets in $X$ such that $A \nsubseteq B$. Then $\text{int} A \nsubseteq \text{int} B$, and hence $\text{cl} \text{int} A \nsubseteq \text{cl} \text{int} B$ (by Theorem 5.2). Consequently, $\text{cl} A \nsubseteq \text{cl} B$.

(e) $\Rightarrow$ (b): If $A \in \text{FSO}(X)$, then $A \nsubseteq (1 - \text{cl} A)$ and $1 - \text{cl} A \in \text{FSO}(X)$. Thus by (e), $\text{cl} A \nsubseteq \text{cl} (1 - \text{cl} A) = 1 - \text{int} \text{cl} A$, and hence $\text{cl} A \nsubseteq \text{int} \text{cl} A$. Thus $\text{cl} A = \text{int} \text{cl} A$ which means that $\text{cl} A$ is fuzzy open in $X$.

(f) $\Rightarrow$ (e): If $A$ and $B$ are non q-coincident fuzzy semi-open sets in $X$, then $\text{scl} A$ and $\text{scl} B$ are fuzzy semi-open and so $\text{scl} A \nsubseteq \text{scl} B$. By (f), $\text{cl} A \nsubseteq \text{cl} B$.

In [87], a fuzzy set $A$ in $X$ is defined to be fuzzy pre-open iff $A \nsubseteq \text{int} \text{cl} A$. Let us now define as follows.

**DEFINITION 5.6.** A fuzzy set $A$ in $X$ is said to be fuzzy semi pre-open iff $A \nsubseteq \text{cl} (\text{int} \text{cl} A)$.

We shall use the notations $\text{FPO}(X)$ and $\text{FSPO}(X)$ to represent the sets of all fuzzy pre-open sets and fuzzy semi pre-open sets in $(X,T)$ respectively. Obviously $T \subseteq \text{FPO}(X) \subseteq \text{FSPO}(X)$, although the reverse inclusions are false.

For example, on a non-empty set $X$, consider the fuzzy sets $A,B,C,D$ given by $A(x) = \frac{1}{4}$, $B(x) = 0.6$, $C(x) = 0.2$ and $D(x) = 0.7$, for all $x \in X$. Then for the fts $(X,T)$, where $T = \{ 0_X, 1_X, A, B \}$, $C \in \text{FPO}(X)$, $C \notin T$, $D \in \text{FSPO}(X)$ and $D \notin \text{FPO}(X)$. 
The following Lemma improves Lemma 5.4.

**Lemma 5.7.** (a) For any $A \in \text{FPO}(X)$, $\text{cl}A = [A]_{\delta} = [A]_{\Theta}$.

(b) For any $A \in \text{FSPO}(X)$, $\text{cl}A = [A]_{\delta}$.

**Proof.** (a) It is obvious that $\text{cl}A \subseteq [A]_{\delta} \subseteq [A]_{\Theta}$, for every fuzzy set $A$ in $X$. Thus it remains to show that $[A]_{\Theta} \subseteq \text{cl}A$. Now, $\forall x \alpha \notin \text{cl}A \Rightarrow$ there exists a fuzzy open q-nbd $U$ of $x_{\alpha}$ such that $U \notin A \Rightarrow U \notin \text{cl}A \Rightarrow U \notin \text{int} \text{cl}A \Rightarrow c1U \notin \text{int} \text{cl}A \Rightarrow c1U \notin A$ (since $A \in \text{FPO}(X)$) $\Rightarrow x_{\alpha} \notin [A]_{\Theta}$. Thus $[A]_{\Theta} \subseteq \text{cl}A$.

(b) To prove this, it is sufficient to prove that $[A]_{\delta} \subseteq \text{cl}A$. Now, $\forall x \alpha \notin \text{cl}A \Rightarrow$ there exists a fuzzy open q-nbd $U$ of $x_{\alpha}$ such that $U \notin A \Rightarrow U \notin \text{cl}A \Rightarrow U \notin \text{int} \text{cl}A \Rightarrow \text{int} \text{cl}U \notin \text{cl}(\text{int} \text{cl}A) \Rightarrow \text{int} \text{cl}U \notin A \Rightarrow x_{\alpha} \notin [A]_{\delta}$. Hence $[A]_{\delta} \subseteq \text{cl}A$.

**Theorem 5.8.** The following statements are equivalent for an fts $(X,T)$:

(a) $X$ is FED.

(b) The fuzzy closure of every fuzzy semi pre-open set in $X$ is fuzzy open.

(c) The fuzzy $\delta$-closure of every fuzzy semi pre-open set in $X$ is fuzzy open.

(d) The fuzzy $\delta$-closure of every fuzzy pre-open set in $X$ is fuzzy open.

(e) The fuzzy $\Theta$-closure of every fuzzy pre-open set in $X$ is fuzzy open.

(f) The fuzzy closure of every fuzzy pre-open set in $X$ is fuzzy open.

**Proof.** Follows from Lemma 5.7 by using the facts that $\text{FPO}(X) \subseteq \text{FSPO}(X)$, and $\text{cl}A = c1(\text{int} \text{cl}A)$, for every $A \in \text{FSPO}(X)$.
LEMMA 5.9. For any fuzzy set $A$ in $X$,
(a) $\text{int} \ clA \subseteq \text{sc} \clA$ and (b) $\text{int}(\text{sc} \clA) = \text{int} \ clA$.

PROOF. (a) Since $\text{sc} \clA$ is fuzzy semi-closed, there exists a fuzzy closed set $U$ in $X$ such that $\text{int} U \subseteq \text{sc} \clA \subseteq U$. Then $\text{int} U \subseteq \text{sc} \clA \subseteq \text{cl}A \subseteq U$ and consequently, $\text{int} U \subseteq \text{int} \ clA \subseteq \text{int} U$. Hence $\text{int} \ clA \subseteq \text{sc} \clA$.
(b) Follows easily by using (a).

PROPOSITION 5.10. Let $A$ be any fuzzy set in $X$. Then
(a) $A \in \text{FPO}(X)$ iff $\text{sc} \clA = \text{int} \ clA$
(b) $A \in \text{FPO}(X)$ iff $\text{sc} \clA \in \text{FRO}(X)$
(c) $\text{FRO}(X) = \text{FPO}(X) \cap \text{FSC}(X)$.

PROOF. (a) Let $A \in \text{FPO}(X)$, then $\text{sc} \clA \subseteq \text{sc} \text{int} \ clA$, and since $\text{int} \ clA \in \text{FSC}(X)$, $\text{sc} \clA \subseteq \text{int} \ clA$. From Lemma 5.9 (a) it follows that $\text{sc} \clA = \text{int} \ clA$. The converse is obvious.
(b) Let $\text{sc} \clA \in \text{FRO}(X)$. Then $\text{sc} \clA = \text{int} \ cl(\text{sc} \clA)$ and hence $\text{sc} \clA \subseteq \text{int} \ cl(\text{sc} \clA) = \text{int} \ clA$. By 5.9 (a) it follows that $\text{sc} \clA = \text{int} \ clA$. By (a), $A \in \text{FPO}(X)$.
The converse follows from (a).
(c) $A \in \text{FPO}(X) \cap \text{FSC}(X) \implies A \in \text{FRO}(X)$ (by (b)) $\implies \text{FRO}(X) \cap \text{FSC}(X) \subseteq \text{FRO}(X)$. Again $A \in \text{FRO}(X) \implies \text{int} \ clA = A \implies \text{int} \ clA = \text{sc} \clA = A \implies A \in \text{FPO}(X) \cap \text{FSC}(X)$ (by (a)). Thus $\text{FRO}(X) = \text{FPO}(X) \cap \text{FSC}(X)$.

THEOREM 5.11. The following statements are equivalent for an fts $(X,T)$:
(a) $X$ is FED.
(b) $\text{scl}A = [A]_g$, for every $A \in \text{FPO}(X) \cup \text{FSO}(X)$.
(c) $\text{scl}A = \text{cl}A$, for every $A \in \text{FSPO}(X)$.
(d) $\text{scl}A = [A]_d$, for every $A \in \text{FSPO}(X)$.

**Proof.** $(a) \implies (b)$: For any fuzzy set $A$ in $X$, $\text{scl}A \subseteq [A]_g$. Thus it is only required to prove $[A]_g \subseteq \text{scl}A$, for every $A \in \text{FPO}(X) \cup \text{FSO}(X)$. In fact, $x \in x$ such that $U \not\subseteq A \implies$ there exists a fuzzy semi-open semi-q-nbd $U$ of $x$ such that $U \not\subseteq A \implies$ there exists a fuzzy open set $V$ in $X$ such that $V \subseteq U \not\subseteq A \implies V \not\subseteq \text{cl}A \implies V \not\subseteq \text{int} \text{cl}A$.

Now, if $A \in \text{FPO}(X)$, then $A \not\subseteq \text{int} \text{cl}A$ and hence $\text{cl}V \not\subseteq A$. If $A \in \text{FSO}(X)$, since $X$ is FED, $\text{cl}V$ is fuzzy open, and thus $\text{cl}V \not\subseteq \text{cl} \text{cl}A = \text{cl}V \not\subseteq \text{cl} \text{int} \text{cl}A$. Thus in any case, $x \notin [A]_g$.

$(b) \implies (a)$: First let $A \in \text{FPO} (X)$. By proposition 5.10 and Lemma 5.7 (a) we have $\text{int cl}A = \text{scl}A = [A]_g = \text{cl}A$. Therefore, $\text{cl}A$ is fuzzy open, and hence it follows from Theorem 5.8 that $X$ is FED. Next, let $A \in \text{FSO} (X)$. We have $\text{scl}A \not\subseteq \text{cl}A \not\subseteq [A]_g = \text{cl}A$ and hence $\text{scl}A = \text{cl}A$. Therefore it follows from Theorem 5.5 (f) that $X$ is FED.

$(a) \implies (c)$: It follows from Lemma 5.9 that for every fuzzy set $A$ in $X$, $\text{int cl}A \not\subseteq \text{scl}A \not\subseteq \text{cl}A$. Since $X$ is FED, by Theorem 5.8, $\text{cl}A$ is fuzzy open in $X$ for every $A \in \text{FSPO} (X)$. Thus we have $\text{scl}A = \text{cl}A$, for every $A \in \text{FSPO}(X)$.

$(c) \implies (d)$: Follows from Lemma 5.7 (b).

$(d) \implies (a)$: Let $U$ and $V$ be fuzzy open sets such that $U \not\subseteq V$. Then $U \not\subseteq 1-V \implies \text{scl}U \not\subseteq \text{scl}(1-V) = 1-V \implies \text{scl}U \not\subseteq V$. Since $\text{scl}U \in \text{FSO}(X)$, $\text{scl}U \not\subseteq \text{scl}V$. By Lemma 5.7 (b) we obtain that $\text{cl}U \not\subseteq V$, since $T \subseteq \text{FSPO}(X)$. This shows that $X$ is FED, by Theorem 5.2.
THEOREM 5.12. The following statements are equivalent for an fts $X$:

(a) $X$ is FED.

(b) $A \in \text{FSPO}(X)$, $B \in \text{FSO}(X)$ and $A \not\subseteq B \implies \text{cl}A \not\subseteq \text{cl}B$.

(c) $A \in \text{FSPO}(X)$, $B \in \text{FSO}(X)$ and $A \not\subseteq B \implies \{A\}_\delta \not\subseteq \{B\}_\delta$.

(d) $A \in \text{FPO}(X)$, $B \in \text{FSO}(X)$ and $A \not\subseteq B \implies \{A\}_\Theta \not\subseteq \{B\}_\Theta$.

(e) $A \in \text{FPO}(X)$, $B \in \text{FSO}(X)$ and $A \not\subseteq B \implies \text{cl}A \not\subseteq \text{cl}B$.

PROOF. (a) $\implies$ (b): Suppose $A \in \text{FSPO}(X)$, $B \in \text{FSO}(X)$ and $A \not\subseteq B$. Then $A \not\subseteq \text{int}B$ and hence $\text{cl}A \not\subseteq \text{int}B$. By Theorem 4.8, $\text{cl}A$ is fuzzy open in $X$ and hence $\text{cl}A \not\subseteq \text{cl}\text{int}B$. Since $B \in \text{FSO}(X)$, $\text{cl}B = \text{cl}(\text{int}B)$. Thus $\text{cl}A \not\subseteq \text{cl}B$.

(b) $\implies$ (c)”, “(c) $\implies$ (d)” and “(d) $\implies$ (e)” follow from Lemma 5.7.

Also, “(e) $\implies$ (a)” follows from Theorem 5.2, since every fuzzy open set is fuzzy pre-open and fuzzy semi-open.

DEFINITION 5.13. A collection $\mathcal{J}$ of non null fuzzy sets in an fts $X$ is called a fuzzy directed family in $X$ iff every finite intersection of members of $\mathcal{J}$ contains a member of $\mathcal{J}$.

DEFINITION 5.14. A fuzzy directed family $\mathcal{J}$ in $X$ is said to

(i) Converge ($\delta$-converge, $\Theta$-converge) to a fuzzy singleton $x_\alpha$, written as $\mathcal{J} \rightarrow x_\alpha$ (resp. $\mathcal{J} \rightarrow \delta x_\alpha$, $\mathcal{J} \rightarrow \Theta x_\alpha$), iff for each fuzzy open $q$-nbd $U$ of $x_\alpha$, there exists an $F \in \mathcal{J}$ such that $F \not\subseteq U$ (resp. $F \not\subseteq \text{int} \text{cl}U$, $F \not\subseteq \text{cl}U$).

(ii) rc-converge to a fuzzy singleton $x_\alpha$ iff for each fuzzy open set $U$ with $x_\alpha \not\subseteq \text{cl}U$, there is $F \in \mathcal{J}$ such that $F \not\subseteq \text{cl}U$.

LEMMA 5.15. A fuzzy set $A$ in an fts $X$ is fuzzy open iff it is a $q$-nbd of every fuzzy singleton with which it is $q$-coincident.
PROOF. If $A$ is fuzzy open then the condition is obviously satisfied.

Conversely, for any fuzzy singleton $x$ with $x \in A$, there exists by hypothesis, a fuzzy open set $U_x$ such that $x \in U_x \subseteq A$. Let $U = \bigcup_{x \in A} U_x$. Then $U$ is fuzzy open and $U \subseteq A$. We show that $A \subseteq U$. Let $x \in \text{supp}A$ and put $A(x) = \alpha$. Choose $n \in \mathbb{N}$ such that $\frac{1}{m} < \alpha$. For any $n \in \mathbb{N}$ with $n > m$, we set $\alpha_n = 1 - \alpha + \frac{1}{n}$. Then $0 < \alpha_n < 1$ and $x_{\alpha_n} \in A$, and hence there is a fuzzy open set $U_{\alpha_n}$ such that $x_{\alpha_n} \in U_{\alpha_n} \subseteq A$ for all $n > m$. This shows that $1 - \alpha + \frac{1}{n} + U_{\alpha_n}$

$(x) > 1$, i.e., $\alpha \leq \bigcup_{n} U_{\alpha_n}(x) > \frac{1}{n}$. Then $\alpha \leq \bigcup_{n} U_{\alpha_n}(x) \subseteq U(x)$. Thus $A(x) \subseteq U(x)$. Since $x \in \text{supp}A$ is arbitrary, we have $A \subseteq U$. Hence $A = U$ and consequently, $A$ is fuzzy open.

**THEOREM 5.16.** For an fts $(X,T)$, the following statements are equivalent:

(a) $X$ is FED.

(b) If a fuzzy directed family on $X$ $\delta$-converges then it $\Theta$-converges.

(c) A fuzzy directed family on $X$ $\Theta$-converges iff it $\Theta$-converges.

(d) If a fuzzy directed family on $X$ converges then it $\Theta$-converges.

**PROOF.** (b) $\implies$ (a). Let $G \in T$ and let $x$ be any fuzzy singleton in $X$ such that $x \in c1G$. By virtue of Lemma 5.15, it is sufficient to show that $c1G$ is a $\delta$-nbd of $x$. We consider the collection $\mathcal{F}$ of all fuzzy open $\delta$-nbd of $x$. Obviously it is a fuzzy directed family and $\delta$-converges to $x$. By hypothesis, it $\Theta$-converges to $x$. Since $x \in c1G$ and $G \in T$, there exists $U \in \mathcal{F}$ such that $U \subseteq c1G$. Thus $c1G$ is a $\delta$-nbd of $x$. 


Let $\mathcal{F}$ be any fuzzy directed family which $\delta$-converges to a fuzzy singleton $x_\alpha$. Let $x_\alpha q \text{cl}G$, where $G \in T$. Then $\text{cl}G \in T$ and consequently there is $F \in \mathcal{F}$ such that $F \nsubseteq \text{int} \text{cl}(\text{cl}G) = \text{int} \text{cl}G = \text{cl}G$. Hence $\mathcal{F}$ is $\delta$-convergent to $x_\alpha$.

**(a) $\implies$ (c):** To prove the requirement it suffices to prove that a fuzzy directed family $\mathcal{F}$ is $\delta$-convergent to $x_\alpha$ whenever it is $\theta$-convergent to $x_\alpha$. Let $x_\alpha q \text{cl}U$, where $U \in T$. Since $X$ is FED, $\text{cl}U \in T$. Then $\text{cl}U$ is a fuzzy open $\delta$-nbhd of $x_\alpha$. So there exists $F \in \mathcal{F}$ such that $F \nsubseteq \text{cl} \text{cl}(\text{cl}U) = \text{cl}U$. Hence $\mathcal{F}$ is $\delta$-convergent to $x_\alpha$.

**(c) $\implies$ (d):** Obvious.

**(d) $\implies$ (a):** Let $G \in T$ and $x_\alpha$ be any fuzzy singleton in $X$ such that $x_\alpha q \text{cl}G$. In view of Lemma 5.15, we only show that $\text{cl}G$ is a $\delta$-nbhd of $x_\alpha$. Now, consider the collection $\mathcal{F}$ of all fuzzy open $\delta$-nbds of $x_\alpha$. Obviously it is a fuzzy directed family converging to $x_\alpha$. Then it $\delta$-converges to $x_\alpha$. Hence there exists $F \in \mathcal{F}$ such that $x_\alpha q F \subseteq \text{cl}G$. Thus $\text{cl}G$ is a $\delta$-nbhd of $x_\alpha$.

**Theorem 5.17.** An fts $(X,T)$ is FED iff $\text{FSO}(X) \subseteq \text{FPO}(X)$.

**Proof.** Suppose $X$ is FED, and $A \in \text{FSO}(X)$. Then there exists $U \in T$ such that $U \nsubseteq A \subseteq \text{cl}U$. Since $X$ is FED, $\text{cl}U \in T$. Then $U \nsubseteq A \subseteq \text{int} \text{cl}U$ and hence $A \in \text{FPO}(X)$.

Conversely, let $A \in \text{FRC}(X)$. Then $A \in \text{FSO}(X)$, and by hypothesis, $A \in \text{FPO}(X)$ so that $A \subseteq \text{int} \text{cl}A$. Since $A$ is fuzzy closed, it then follows that $A \in T$. Hence $X$ is FED.

Having made an extensive study as to the various characterizations of FED spaces from different angles, we now turn our attention to other properties of such a space.
THEOREM 5.18. The property of an fts being fuzzy extremally disconnected is a fuzzy semiregular property.

PROOF. Let \((X,T)\) be fuzzy extremally disconnected and let \(U \in T_S\). Then \(T_S -c1U = T-c1U = V\) (say). Since \((X,T)\) is fuzzy extremally disconnected, \(V \in T\) and consequently, \(T_S -c1U = T\)-int \(T-c1U = T\)-int \(V \in T_S\). Hence \((X,T_S)\) is fuzzy extremally disconnected.

Conversely, suppose \((X,T_S)\) is fuzzy extremally disconnected. For any \(U \in T\), \(T_S -c1U = T-c1U\) and \(T\)-int \(T-c1U \in T_S\). Put \(V = T\)-int \(T-c1U\). Then \(T-c1U = T - c1V = T_S - c1V \in T_S \subseteq T\). Hence \((X,T)\) is fuzzy extremally disconnected.

In order to investigate for the suitable function under which fuzzy extremally disconnectedness is preserved, we require the following lemma.

LEMMA 5.19. If \(f : (X,T) \rightarrow (Y,R)\) is almost fuzzy open and fuzzy semi-continuous then \(f(A) \in FPO(Y)\), for every \(A \in FPO(X)\).

PROOF. Let \(A \in FPO(X)\). Since \(f\) is fuzzy semi-continuous, \(f(A) \preceq f(sclA) \preceq c1f(A)\), by Theorem 4.18. By Proposition 5.10 (b), \(sclA \in FRO(X)\), and hence \(f(sclA) \in FPO(Y)\), because \(f\) is almost fuzzy open. By Proposition 5.10 (a), \(c1f(f(sclA)) = \text{int } c1(f(sclA))\). Hence \(sc1f(A) \preceq sc1f(f(sclA)) = \text{int } c1f(sclA) \preceq c1f(A)\). Since \(\text{int } c1f(A) = c1f(sc1A)\), we conclude that \(f(A) \preceq sc1f(A) \preceq \text{int } c1f(A)\), and consequently, \(f(A) \in FPO(Y)\).

THEOREM 5.20. Let \(f : (X,T) \rightarrow (Y,R)\) be a fuzzy semi-continuous and almost fuzzy open surjection. If \(X\) is FED then so is \(Y\).
PROOF. Let \( V \in \text{FSO}(Y) \). Since \( f \) is fuzzy semi-continuous and almost fuzzy open, \( f \) is fuzzy irresolute so that \( f^{-1}(V) \in \text{FSO}(X) \). By Theorem 5.17, \( f^{-1}(V) \in \text{FPO}(X) \), and hence by Lemma 5.19, \( V \in \text{FPO}(Y) \). Thus \( \text{FSO}(Y) \subseteq \text{FPO}(Y) \).

Hence by Theorem 5.17, \( Y \) is FED.

Certain applications of the concept of fuzzy extremally disconnectedness will be seen in the next chapter. For the time being, we observe in the following theorem how a property of a function gets strengthened when the domain space becomes FED. To this end, we recall the following definition from [98].

**Definition 5.21.** A function \( f : (X,T) \rightarrow (Y,R) \) is said to be

(a) fuzzy almost strongly \( \theta \)-continuous iff for each fuzzy singleton \( x_\alpha \) in \( X \) and each fuzzy open \( q \)-nbd \( V \) of \( f(x_\alpha) \), there exists a fuzzy open \( q \)-nbd \( U \) of \( x_\alpha \) such that \( f(\text{cl}(U)) \subseteq \text{int} \text{cl}(V) \),

(b) fuzzy weakly \( \theta \)-continuous iff for each fuzzy singleton \( x_\alpha \) in \( X \) and each fuzzy open \( q \)-nbd \( V \) of \( f(x_\alpha) \), there exists a fuzzy open \( q \)-nbd \( U \) of \( x_\alpha \) such that \( f(U) \subseteq \text{cl}(V) \).

It is shown in [98], that fuzzy almost strongly \( \theta \)-continuity \( \implies \) fuzzy \( \delta \)-continuity \( \implies \) fuzzy \( \theta \)-continuity \( \implies \) fuzzy weakly \( \theta \)-continuity, where the implications are not reversible, in general.

**Theorem 5.22.** Let \( X \) be an FED space. If a function \( f : X \rightarrow Y \) is fuzzy weakly \( \theta \)-continuous (fuzzy \( \delta \)-continuous) then \( f \) is fuzzy \( \theta \)-continuous (resp. fuzzy almost strongly \( \theta \)-continuous).
PROOF. We prove only the case when \( f \) is fuzzy weakly \( \theta \)-continuous, the proof of the other case being similar. Since \( f \) is fuzzy weakly \( \theta \)-continuous, for each fuzzy singleton \( x_\alpha \) in \( X \) and each fuzzy open \( q \text{-nbd} \) \( V \) of \( f(x_\alpha) \), there exists a fuzzy regularly open \( q \text{-nbd} \) \( U \) of \( x_\alpha \) such that \( f(U) \subseteq \text{cl} V \). Since \( X \) is FED, we have \( \text{cl} U = U \) and hence \( f(\text{cl} U) \subseteq \text{cl} V \). Hence \( f \) is fuzzy \( \theta \)-continuous.