CHAPTER - IV

SEMI-CONTINUOUS AND SEMI-CLOSED MAPPINGS AND

SEMI-CONNECTEDNESS IN FUZZY SETTING

This chapter visualises, at the very outset, the advent of fuzzy semi-continuous and semi-closed mappings. Such mappings were first introduced by Azad [6] with the initiation of fuzzy semi-open and semi-closed sets. Here in the first section of this chapter our endeavour is to produce further characterizing theorems for these maps. Unlike the case in general set-topology, semi-cluster points of a fuzzy set cannot be characterized in terms of fuzzy semi-open nbds. So it is natural to define semi-closure of fuzzy set in terms of semi-q-nbds, and this we do in the same section and ultimately show, as one should expect, that semi-closure of a fuzzy set $A$ is nothing but the intersection of all fuzzy semi-closed sets containing $A$. Such a study has facilitated us to characterize fuzzy semi-continuous and semi-closed maps.

In the sequel, fuzzy semi $T_{\frac{1}{2}}$-axioms ($i = 0, 1, 2$) for an fts have been introduced by using semi-q-nbds. Some characterization theorems are there to envisage the axioms; the interrelations have also been studied. The exact mappings, rather the properties of the mappings have been found which keep such semi $T_{\frac{1}{2}}$-properties invariant. Our intention here is not to study fuzzy semi $T_{\frac{1}{2}}$-spaces in detail, but to offer some glimpses as to how such
spaces can be studied in terms of fuzzy semi-continuous functions.

As a natural follow-up study with the fuzzy semi-open sets, we introduce and investigate to some extent the concept of fuzzy semi-connectedness in the sequent part, i.e., in Section 4.3.

§ 4.1. FUZZY SEMI-CONTINUOUS AND SEMI-CLOSED MAPPINGS

**DEFINITION 4.1.1.** [6] A mapping \( f : X \rightarrow Y \) from an fts \( X \) to an fts \( Y \) is said to be

(a) fuzzy semi-open (semi-closed) iff \( f(A) \) is fuzzy semi-open (semi-closed) in \( Y \) for each fuzzy open (closed) set \( A \) in \( X \),

(b) fuzzy semi-continuous iff \( f^{-1}(B) \) is fuzzy semi-open in \( X \), for each fuzzy open set \( B \) in \( Y \).

Obviously, \( f \) is fuzzy semi-continuous iff \( f^{-1}(B) \) is fuzzy semi-closed for each fuzzy closed set \( B \) in \( Y \). It is shown by Azad [6] that any union of fuzzy semi-open sets is fuzzy semi-open but the intersection of two fuzzy semi-open sets or even the intersection of a fuzzy open set with a fuzzy semi-open set may not be fuzzy semi-open.

**THEOREM 4.1.2.** A mapping \( f : X \rightarrow Y \) is fuzzy semi-continuous iff for any fuzzy singleton \( x_\alpha \) in \( X \) and any fuzzy open set \( V \) in \( Y \) with \( f(x_\alpha) \not\subseteq V \), there exists \( U \in \mathrm{FSO}(X) \) such that \( x_\alpha \not\subseteq U \) and \( f(U) \not\subseteq V \).

**PROOF.** Let \( f : X \rightarrow Y \) be fuzzy semi-continuous. Suppose \( x_\alpha \) is any fuzzy singleton in \( X \) and \( V \) is any fuzzy open set containing \( f(x_\alpha) \). Then \( U=f^{-1}(V) \)
is a fuzzy semi-open set in X containing \( x_\alpha \) such that \( f(U) \not\subseteq V \).

Conversely, let the given condition hold and let \( V \) be any fuzzy open set in \( Y \). Now if \( f^{-1}(V) = O_X \), then obviously \( f^{-1}(V) \) is a fuzzy semi-open set in \( X \). Otherwise, there exists a fuzzy singleton \( x_\alpha \) in \( f^{-1}(V) \). This implies that \( f(x_\alpha) \not\subseteq V \). So \( V \) is a fuzzy open set in \( Y \) containing \( f(x_\alpha) \).

By hypothesis, there exists a fuzzy semi-open set \( U_\alpha \) in \( X \) containing \( x_\alpha \) such that \( f(U_\alpha) \not\subseteq V \), and thus \( x_\alpha \not\subseteq U_\alpha \not\subseteq f^{-1}(V) \) \( \ldots \ldots \ldots \) (1). Doing the same thing for all fuzzy singletons \( x_\alpha \) in \( f^{-1}(V) \) and taking the union of all such relations like (1) we get \( f^{-1}(V) = \bigcup \{ x_\alpha : \alpha \not\subseteq f^{-1}(V) \} \not\subseteq \bigcup \{ U_\alpha : \alpha \not\subseteq f^{-1}(V) \} \) and consequently, \( f^{-1}(V) \) is fuzzy semi-open. Hence \( f \) is fuzzy semi-continuous.

**THEOREM 4.1.3.** A mapping \( f : X \rightarrow Y \) is fuzzy semi-continuous iff for each fuzzy singleton \( x_\alpha \) in \( X \) and for any fuzzy open q-nbd \( V \) of \( y_\beta \) in \( Y \) (where \( y = f(x) \)), there exists a fuzzy semi-open q-nbd \( U \) of \( x_\alpha \) in \( X \) such that \( f(U) \not\subseteq V \).

**PROOF.** Let \( f \) be fuzzy semi-continuous and let \( V \) be a fuzzy open q-nbd of \( y_\beta \) in \( Y \). Then \( V(y) + \alpha \geq 1 \) and hence there exists a positive real number \( \delta \) such that \( V(y) \geq \beta \geq 1 - \alpha \), so that \( V \) is a fuzzy open nbd of \( y_\beta \). By Theorem 4.1.2 there exists a fuzzy semi-nbd \( W \) of \( x_\alpha \) such that \( f(W) \not\subseteq V \). Now, \( W(x) \geq \beta \implies W(x) \geq 1 - \alpha \implies W \) is a fuzzy semi-open q-nbd of \( x_\alpha \).
Conversely, let the given condition hold. Let $V$ be a fuzzy open set in $Y$. Let us put $W = f^{-1}(V)$. If $W = O_x$, then it is fuzzy semi-open in $X$. If $\text{supp } W \neq \emptyset$, then for any $x \in \text{supp } W$, put $f(x) = y$ so that $W(x) = V(y)$. There exists a positive integer $m$ such that $\frac{1}{m} \leq W(x)$. We put $a_n = 1 - \frac{1}{m}$ for each positive integer $n \geq m$. Then $0 < a_n < 1$, for all $n \geq m$. Thus $V(y) + a_n = 1 - \frac{1}{m}$, for each $n \geq m$, and by the given condition, there exists a fuzzy semi-open set $U_n$ in $X$ such that $x_{q_n} \notin U_n$ and $f(U_n) \ll V$, for all $n \geq m$. Let us set $U_x = \bigcup\{U_n : n \geq m\}$; then $f(U_x) \ll V$. Also $n \geq m \Rightarrow U_n(x) + a_n \geq U_n(x) - \frac{1}{m} \Rightarrow U_x(x) \geq W(x) - \frac{1}{n}$. Thus $U_x(x) \gg W(x)$ which in turn, implies that $x_{\frac{1}{W(x)}} \ll U_x$. Again, $f(U_x) \ll V \Rightarrow U_x \ll f^{-1}(V) = W$. Hence $x_{\frac{1}{W(x)}} \ll U_x \ll W$, for all $x \in \text{supp } W$. Then $W = \bigcup\{U_x : x \in \text{supp } W\}$ and $W$ becomes fuzzy semi-open in $X$. Hence $f$ is fuzzy semi-continuous.

For further characterizations of fuzzy semi-continuous mappings and investigation of fuzzy semi-closed mappings we need the concept of fuzzy semi-closure of fuzzy sets which we wish to define by the introduction of the notion of semi-$q$-nbds as follows.

**DEFINITION 4.1.4.** A fuzzy set $A$ is called a semi-$q$-nbds of a fuzzy singleton $x$ in an fts $(X,T)$ iff there exists a fuzzy semi-open set $V$ in $X$ such that $x_\alpha \notin V$.

It is clear that every $q$-nbds of a fuzzy singleton is always a semi-$q$-nbds of the fuzzy singleton, though not conversely.

**DEFINITION 4.1.5.** A fuzzy singleton $x_\alpha$ in an fts $X$ is called a fuzzy semi-cluster point of a fuzzy set $A$ in $X$ iff every fuzzy semi-$q$-nbds of $x_\alpha$ is
q-coincident with \( A \). The union of all fuzzy semi-cluster points of \( A \) will be called the fuzzy semi-closure of \( A \) and will be denoted by \( \text{scl} A \).

The union of all fuzzy semi-open sets contained in a fuzzy set \( A \) in an fts \( X \) is called the fuzzy semi-interior of \( A \), to be denoted by \( \text{sint} A \). It is obvious that \( \text{scl} (1-A) = 1 - \text{sint} A \) and \( \text{sint} (1-A) = 1 - \text{scl} A \), for every fuzzy set \( A \) in \( X \).

**THEOREM 4.1.6.** For a fuzzy set \( A \) in an fts \( X \), \( \text{scl} A \) is the intersection of all fuzzy semi-closed sets, each containing \( A \).

**PROOF.** Let \( B \) denote the intersection of all fuzzy semi-closed sets containing \( A \). Suppose \( x_\alpha \not\in B \) and if possible, let there exist a semi-q-nbd \( N \) of \( x_\alpha \) such that \( N \notin A \). Then there exists a fuzzy semi-open set \( V \) in \( X \) such that \( x_\alpha \not\in V \not\subseteq N \), which shows that \( V \notin A \) and hence \( A \not\subseteq 1-V \). As \( (1-V) \) is fuzzy semi-closed, \( B \not\subseteq 1-V \). Since \( x_\alpha \notin 1-V \), we obtain \( x_\alpha \notin B \) which is a contradiction.

Conversely, suppose \( x_\alpha \notin B \). Then there exists a fuzzy semi-closed set \( F \supset A \) such that \( x_\alpha \notin F \). We then have \( x_\alpha \notin (1-F) \in \text{FSO}(X) \) and \( A \notin (1-F) \). Thus \( x_\alpha \notin \text{scl} A \).

**COROLLARY 4.1.7.** A fuzzy set \( A \) in an fts \( X \) is fuzzy semi-closed iff \( A = \text{scl} A \).

**THEOREM 4.1.8.** A mapping \( f : X \rightarrow Y \) is fuzzy semi-continuous iff for every fuzzy set \( A \) in \( X \), \( f(\text{scl} A) \subseteq \text{clf}(A) \).

**PROOF.** Suppose \( f \) is fuzzy semi-continuous. For a fuzzy set \( A \) in \( X \),
\( f^{-1}(\text{clf}(A)) \) is fuzzy semi-closed in \( X \). Furthermore, \( A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\text{clf}(A)) \). Therefore \( \text{scl}A \subseteq f^{-1}(\text{clf}(A)) \) and then \( f(\text{scl}A) \subseteq f(f^{-1}(\text{clf}(A))) \) \( \subseteq \text{clf}(A) \).

Conversely, let \( B \) be any fuzzy closed set in \( Y \). By the given condition we have, \( f(\text{scl}(f^{-1}(B))) \subseteq \text{clf}(f^{-1}(B)) \subseteq c\text{l}B \), and then \( \text{scl } f^{-1}(B) \subseteq f^{-1}(\text{clf}(B)) = f^{-1}(B) \). Hence \( f^{-1}(B) \) is fuzzy semi-closed in \( X \) and consequently, \( f \) is fuzzy semi-continuous.

**Theorem 4.1.9.** A mapping \( f : X \rightarrow Y \) is fuzzy semi-continuous iff for every fuzzy set \( B \) in \( Y \), \( \text{scl } f^{-1}(B) \subseteq f^{-1}(\text{clf}(B)) \).

**Proof.** Let \( f \) be fuzzy semi-continuous and \( B \) be any fuzzy set in \( Y \). Then \( f^{-1}(\text{clf}(B)) \) is a fuzzy semi-closed set in \( X \). Since \( f^{-1}(B) \subseteq f^{-1}(\text{clf}(B)) \), we have \( \text{scl } f^{-1}(B) \subseteq f^{-1}(\text{clf}(B)) \).

Conversely, let \( B \) be any fuzzy closed set in \( Y \). By the given condition, \( f^{-1}(B) \subseteq \text{scl } f^{-1}(B) \subseteq f^{-1}(\text{clf}(B)) = f^{-1}(B) \), i.e. \( f^{-1}(B) = \text{clf } f^{-1}(B) \). Hence \( f^{-1}(B) \) is fuzzy semi-closed in \( X \) and consequently, \( f \) is fuzzy semi-continuous.

**Theorem 4.1.10.** For a mapping \( f : X \rightarrow Y \) the following statements are equivalent:

(a) \( f \) is fuzzy semi-closed.

(b) \( f(c1A) \supseteq \text{int cl } f(A) \), for any fuzzy set \( A \) in \( X \).

(c) \( \text{scl } f(A) \subseteq f(c1A) \), for any fuzzy set \( A \) in \( X \).

**Proof.** (a) \( \Rightarrow \) (b) : For a fuzzy set \( A \) in \( X \), \( f(c1A) \) is fuzzy semi-closed in \( Y \), and hence by Result 0.9.28 ((a) \( \Leftrightarrow \) (c)), \( f(c1A) \supseteq \text{int cl } f(c1A) \) \( \supseteq \text{int cl } f(A) \).
For a fuzzy closed set $A$ in $X$ we have, by hypothesis, $\text{int} \ cl f(A) = f(A)$. So by Result 0.9.28 ((a) $\Rightarrow$ (c)), $f(A)$ is fuzzy semi-closed in $Y$ and hence $f$ is a fuzzy semi-closed mapping.

For a fuzzy set $A$ in $X$, $f(clA)$ is fuzzy semi-closed in $Y$. Since $f(A) \subseteq f(clA)$, we have $\text{scl} f(A) \subseteq f(clA)$.

For any fuzzy closed set $A$ in $X$, $f(A) \subseteq \text{scl}(f(A)) \subseteq f(clA) = f(A)$ $\Rightarrow$ $f(A) = \text{scl} f(A)$. Hence $f(A)$ is fuzzy semi-closed in $Y$ and consequently, $f$ is fuzzy semi-closed.

DEFINITION 4.1.11. [100] A mapping $f : X \rightarrow Y$ is said to be fuzzy irresolute iff $f^{-1}(B)$ is fuzzy semi-open in $X$, for any fuzzy semi-open set $B$ in $Y$.

Clearly $f : X \rightarrow Y$ is fuzzy irresolute iff $f^{-1}(B)$ is fuzzy semi-closed in $X$, for any fuzzy semi-closed set $B$ in $Y$. Also obviously every fuzzy irresolute mapping is fuzzy semi-continuous; that the converse is false is shwon in [100] in which paper it is also proved that if $f : X \rightarrow Y$ is fuzzy semi-continuous and almost fuzzy open, then $f$ is fuzzy irresolute. It then follows that

COROLLARY 4.1.12. If $f : X \rightarrow Y$ is fuzzy semi-continuous and almost fuzzy open, then for each fuzzy semi-closed set $B$ in $Y$, $f^{-1}(B)$ is fuzzy semi-closed in $X$.

THEOREM 4.1.13. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two mappings and let $g \circ f : X \rightarrow Z$ be a fuzzy semi-closed mapping.

(a) If $f$ is fuzzy continuous and onto, then $g$ is fuzzy semi-closed.

(b) If $g$ is almost fuzzy open, fuzzy semi-continuous and injective, then $f$ is fuzzy semi-closed.
PROOF. (a) For a fuzzy closed set \( B \) in \( Y \), \( f^{-1}(B) \) is fuzzy closed in \( X \), since \( f \) is continuous. Since \( g \circ f \) is fuzzy semi-closed and \( f \) is onto, \((g \circ f) \circ f^{-1}(B) = g(B) \) is fuzzy semi-closed in \( Z \). Hence \( g \) is a fuzzy semi-closed mapping.

(b) For a fuzzy closed set \( A \) in \( X \), \((g \circ f)(A)\) is fuzzy semi-closed in \( Z \). Since \( g \) is injective, we have \( g^{-1}[(g \circ f)(A)] = f(A) \). It follows immediately from Corollary 4.1.12 that \( f(A) \) is a fuzzy semi-closed set in \( Y \) and hence \( f \) is fuzzy semi-closed.

§ 4.2. FUZZY SEMI-\( T \_i \) AXIOMS (\( i = 0,1,2 \))

We introduce fuzzy semi-\( T \_i \)-spaces (\( i = 0,1,2 \)) in terms of fuzzy semi-open sets as follows.

**DEFINITION 4.2.1.** An fts \((X,T)\) is called fuzzy semi-\( T \_0 \) iff for every pair of distinct fuzzy singletons \( x\alpha \) and \( y\beta \), the following conditions are satisfied:

(a) When \( x \neq y \), either \( x\alpha \) has a fuzzy semi-nbd which is not \( q \)-coincident with \( y\beta \), or \( y\beta \) has a fuzzy semi-nbd which is not \( q \)-coincident with \( x\alpha \).

(b) When \( x = y \) and \( \alpha < \beta \) (say), there is a semi-\( q \)-nbd of \( y\beta \) which is not \( q \)-coincident with \( x\alpha \).

**DEFINITION 4.2.2.** An fts \( X \) is fuzzy semi-\( T \_1 \) iff for every pair of distinct fuzzy singletons \( x\alpha \) and \( y\beta \) the following conditions hold:

(a) When \( x \neq y \), \( x\alpha \) has a fuzzy semi-nbd \( U \) and \( y\beta \) has a fuzzy semi-nbd \( V \) such that \( U \not\subset y\beta \) and \( V \not\subset x\alpha \).

(b) When \( x = y \) and \( \alpha < \beta \) (say), \( y\beta \) has a semi-\( q \)-nbd \( U \) such that \( x\alpha \not\subset U \).
DEFINITION 4.2.3. An fts $X$ is fuzzy semi-$T_2^*$ iff for every pair of distinct fuzzy singletons $x_\alpha$ and $y_\beta$, the following conditions are satisfied:

(a) When $x \neq y$, $x_\alpha$ and $y_\beta$ have fuzzy semi-nbds which are not q-coincident;
(b) When $x = y$ and $\alpha < \beta$ (say), then $x_\alpha$ has a fuzzy semi-nbd $U$ and $y_\beta$ has a semi-q-nbd $V$ such that $U \nsubseteq V$.

REMARK 4.2.4. Obviously, fuzzy semi-$T_2^* \Rightarrow$ fuzzy semi-$T_1^*$ and fuzzy-$T_0^*$ axiom $\Rightarrow$ fuzzy semi-$T_1^*$ axiom, for $i = 0, 1, 2$.

THEOREM 4.2.5. An fts $X$ is fuzzy semi-$T_0^*$ iff for every pair of distinct fuzzy singletons $x_\alpha$ and $y_\beta$, either $x_\alpha \nsubseteq \text{scl}(y_\beta)$ or $y_\beta \nsubseteq \text{scl}(x_\alpha)$.

PROOF. Let $X$ be fuzzy semi-$T_0^*$ and $x_\alpha$ and $y_\beta$ be two distinct fuzzy singletons in $X$.

Case I. When $x \neq y$, $x_1$ has a fuzzy semi-nbd $U$ such that $U \nsubseteq y_\beta$, or $y_1$ has a fuzzy semi-nbd $V$ such that $V \nsubseteq x_\alpha$. Suppose $x_1$ has a fuzzy semi-nbd $U$ which is not q-coincident with $y_\beta$. Then $U$ is a semi q-nbd of $x_\alpha$ and $y_\beta \nsubseteq U$. Hence $x_\alpha \nsubseteq \text{scl}(y_\beta)$.

Case II. When $x = y$ and $\alpha < \beta$ (say), then $y_\beta$ has a semi-q-nbd which is not q-coincident with $x_\alpha$ and so in this case also $y_\beta \nsubseteq \text{scl}(x_\alpha)$.

Conversely, let $x_\alpha$ and $y_\beta$ be two distinct fuzzy singletons in $X$. We suppose, without loss of generality, that $x_\alpha \nsubseteq \text{scl}(y_\beta)$. When $x \neq y$, since $x_\alpha \nsubseteq \text{scl}(y_\beta)$, $x_1 \nsubseteq \text{scl}(y_\beta)$ and hence $1-[\text{scl}(y_\beta)](x_1) = 1$. Thus $(1 - \text{scl}(y_\beta))$ is a fuzzy semi-nbd of $x_\alpha$ such that $(1 - \text{scl}(y_\beta)) \nsubseteq y_\beta$. Also, in case when $x = y$ we must have $\alpha > \beta$ and then $x_\alpha$ has a semi-q-nbd which is not q-coincident with $y_\beta$. 
THEOREM 4.2.6. An fts $X$ is fuzzy semi- $T_1$ iff every fuzzy singleton $x_\alpha$ is fuzzy semi-closed in $X$.

PROOF. Straightforward and hence omitted.

THEOREM 4.2.7. An fts $(X,T)$ is fuzzy semi- $T_2$ iff for every fuzzy singleton $x_\alpha$ in $X$, $x_\alpha = \bigcap \{\text{scl}V : V$ is a fuzzy semi- nbd of $x_\alpha\}$ and for every $x,y \in X$ with $x \neq y$, there is a fuzzy semi-nbd $U$ of $x_1$ such that $y \notin \text{supp} (\text{scl} U)$.

PROOF. Let $(X,T)$ be fuzzy semi- $T_2$. Let $x_\alpha$ and $y_\beta$ be fuzzy singletons in $X$ such that $y_\beta \notin (x_\alpha)$. To establish the required equality, it suffices to show the existence of a fuzzy semi-nbd $V$ of $x_\alpha$ such that $y_\beta \notin \text{scl} V$. If $x \neq y$, then there are fuzzy semi-open sets $U$ and $V$ containing $y_1$ and $x_\alpha$ respectively such that $U \nsubseteq V$. Then $V$ is a fuzzy semi-nbd of $x_\alpha$ and $U$ is a semi-q-nbd of $y_\beta$ such that $U \nsubseteq V$. Hence $y_\beta \notin \text{scl} V$. If $x = y$, then $\beta > \alpha$, and hence there exist a semi q-nbd $U$ of $y_\beta$ and a fuzzy semi-nbd $V$ of $x_\alpha$ such that $U \nsubseteq V$. Then $y_\beta \notin \text{scl} V$.

Finally, for two distinct points $x,y$ of $X$, since $X$ is fuzzy semi-$T_2$, there exist fuzzy semi-open sets $U$ and $V$ such that $x_1 \nsubseteq U$, $y_1 \nsubseteq V$, and $U \nsubseteq V$. Then $1-V (y) = 0$ and $U \nsubseteq 1 - V$. Since $(1-V)$ is fuzzy semi-closed, $\text{scl} U \nsubseteq 1-V$. Then $(\text{scl}U) (y) = 0$ i.e., $y \notin \text{supp} (\text{scl} U)$.

Conversely, let $x_\alpha$ and $y_\beta$ be two distinct fuzzy singletons in $X$.

Case 1. Let $x \neq y$. We first suppose that at least one of $\alpha$ and $\beta$ is less than 1, say $0 < \alpha < 1$. Then there exists a positive real number $\epsilon$ with $0 < \alpha + \epsilon < 1$. By hypothesis, there exists a fuzzy semi-nbd $U$ of $y_\beta$ such that $x_\alpha \notin \text{scl} U$. Then $x_\alpha$ has a fuzzy semi-open semi-q-nbd $V$ such that
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V \notin U. Now, \varepsilon + V(x) > 1 so that V(x) > 1 - \varepsilon > \alpha and hence V is a fuzzy semi-nbd of x \alpha such that U \notin V, where U is a fuzzy semi-nbd of y \beta.

In case \alpha = \beta = 1, by hypothesis there is a fuzzy semi-nbd U of x_1 such that (scU)(y) = 0. Then (1 - scU) (=V_1) is a fuzzy semi-nbd of y_1 such that U \notin V.

Case II. Let x = y and \alpha < \beta (say), then there is a fuzzy semi-nbd U of x_\alpha such that y_\beta \notin scU. Consequently, there exists a semi-q-nbd V of y_\beta such that U \notin V.

Considering the above cases, we conclude that X is fuzzy semi-T_2.

THEOREM 4.2.8. Let f : X \rightarrow Y be injective and either fuzzy irresolute, or fuzzy semi-continuous and almost fuzzy open. If Y is fuzzy semi-T_i then so is X, for i = 0,1,2.

PROOF. We give a proof for i = 1 only; the other cases being similar, are omitted.

Let x_\alpha and y_\beta be two distinct fuzzy singletons in X.

When x \neq y, we have f(x) \neq f(y), and by the fuzzy semi-T_1 property of Y, (f(x))_\alpha and (f(y))_\beta have fuzzy semi-open semi-nbds U and V respectively such that (f(x))_\alpha \notin V and (f(y))_\beta \notin U. Then f^{-1}(U) and f^{-1}(V) are fuzzy semi-nbds of x_\alpha and y_\beta respectively such that x_\alpha \notin f^{-1}(V) and y_\beta \notin f^{-1}(U).

When x = y and \alpha < \beta (say), then f(x) = f(y). Y being fuzzy semi-T_1, there is a fuzzy semi-open semi-q-nbd V of (f(y))_\beta such that V \notin f(x)_\alpha. Then f^{-1}(V) is a semi-q-nbd of y_\beta in X such that x_\alpha \notin f^{-1}(V). Hence X is fuzzy semi-T_1.

THEOREM 4.2.9. An fts (X,T) is fuzzy semi-T_i iff for each pair of fuzzy singletons x_\alpha and y_\beta with distinct supports, there exists a fuzzy semi-
continuous and almost fuzzy open mapping \( f \) from \( X \) into a fuzzy semi-\( T_{1} \)-space \((Y,R)\) such that \( f(x) \neq f(y) \), where \( i = 0,1,2 \).

**Proof.** Let us prove the theorem for \( i=2 \). The other cases, i.e. when \( i=0 \) or \( 1 \), can similarly be tackled with proper modifications.

For the necessity part, it suffices to consider the identity mapping on \( X \).

Conversely, let \( x_\alpha \) and \( y_\beta \) be two distinct fuzzy singletons in \( X \). Then the following two cases may arise.

**Case I.** When \( x \neq y \), by hypothesis there exists a fuzzy semi-continuous and almost fuzzy open mapping \( f \) from \( X \) into a fuzzy semi-\( T_2 \) fts \( Y \) such that \( f(x) \neq f(y) \). \( Y \) being fuzzy semi-\( T_2 \), the fuzzy singletons \((f(x))_\alpha \) and \((f(y))_\beta \) have fuzzy semi-open semi-nbds \( U \) and \( V \) respectively such that \( U \notin V \). Since \( f \) is fuzzy semi-continuous and almost fuzzy open, \( f^{-1}(U) \) and \( f^{-1}(V) \) are fuzzy semi-nbds of \( x_\alpha \) and \( y_\beta \) respectively in \( X \) such that \( f^{-1}(U) \notin f^{-1}(V) \).

**Case II.** When \( x = y \) and \( \alpha < \beta \) (say), \((f(x))_\alpha \) and \((f(y))_\beta \) are fuzzy singletons in \( Y \) such that \( f(x) = f(y) \). Thus \((f(y))_\beta \) has a fuzzy semi-open semi-q-nbd \( V \) and \((f(x))_\alpha \) has a fuzzy semi-open semi-nbd \( U \) such that \( U \notin V \). Then \( f^{-1}(V) \) and \( f^{-1}(U) \) are fuzzy semi-open sets in \( X \) such that \( x_\alpha \in f^{-1}(U) \), \( y_\beta \notin f^{-1}(V) \) and \( f^{-1}(U) \notin f^{-1}(V) \). Hence \( X \) is fuzzy semi-\( T_2 \).

**Remark 4.2.10.** It is easy to observe that Theorem 4.2.8 can immediately be obtained from Theorem 4.2.9 as a corollary.

§ 4.3. **Fuzzy Semi-Connectedness**

**Definition 4.3.1.** Two non null fuzzy sets \( A \) and \( B \) in an fts \( X \) are said to be fuzzy semi-separated iff \( A \notin sc1B \) and \( B \notin sc1A \).
In [38], it is defined that two non-null fuzzy sets $A$ and $B$ in an fts $X$ are fuzzy separated iff $A \in cIB$ and $B \in cIA$. Since $sclA \subseteq cIA$, for any fuzzy set $A$, it follows that if $A$ and $B$ are fuzzy separated then they are fuzzy semi-separated. That the converse is false can be seen from Example 4.3.6.

**Theorem 4.3.2.** Let $A, B$ be non-null fuzzy sets in an fts $(X,T)$.

(a) If $A, B$ are fuzzy semi-separated, and $A^1, B^1$ are non-null fuzzy sets such that $A^1 \leq A$ and $B^1 \leq B$, then $A^1$ and $B^1$ are also fuzzy semi-separated.

(b) If $A \notin B$ and either both are fuzzy semi-open or both are fuzzy semi-closed, then $A, B$ are fuzzy semi-separated.

(c) If $A, B$ are either both fuzzy semi-open or both fuzzy semi-closed and if $C^A_B(B) = A \cap (1-B)$, $C^B_A(A) = B \cap (1-A)$, then $C^A_B(B)$ and $C^B_A(A)$ are fuzzy semi-separated.

**Proof.** The simple proofs of (a) and (b) are omitted.

(c) Since $C^A_B(B) \leq 1-B$, $scl(C^A_B(B)) \notin 1-B$ and hence $scl(C^A_B(B)) \notin B$. Then $C^B_A(A) \notin scl(C^A_B(B))$. Similarly, $C^A_B(B) \notin scl(C^B_A(A))$. Hence $C^A_B(B)$ and $C^B_A(A)$ are fuzzy semi-separated.

**Theorem 4.3.3.** Two non-empty fuzzy sets $A$ and $B$ are fuzzy semi-separated iff there exist two fuzzy semi-open sets $U$ and $V$ such that $A \leq U$, $B \leq V$, $A \notin V$ and $B \notin U$.

**Proof.** For two fuzzy semi-separated sets $A$ and $B$, $B \leq 1 - sclA = V$ (say) and $A \leq 1 - sclB = U$ (say), where $V$ and $U$ are clearly fuzzy semi-open, and $A \notin V$, $B \notin U$. 

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Conversely, let $U$ and $V$ be fuzzy semi-open sets such that $A \vartriangleleft U$, $B \vartriangleleft V$, $A \nsubseteq V$ and $B \nsubseteq U$. Then $A \vartriangleleft 1-V$ and $B \vartriangleleft 1-U$. Thus $\text{scl}A \vartriangleleft 1-V$ and $\text{scl}B \vartriangleleft 1-U$, which in turn imply that $\text{scl}A \nsubseteq B$ and $\text{scl}B \nsubseteq A$. Thus $A$ and $B$ are fuzzy semi-separated.

DEFINITION 4.3.4. A fuzzy set which cannot be expressed as the union of two fuzzy semi-separated sets is said to be a fuzzy semi-connected set.

REMARK 4.3.5. According to Ganguly and Saha [38], a fuzzy set $A$ in an fts $X$ is said to be fuzzy connected iff $A$ cannot be expressed as the join of two fuzzy separated sets. It is thus clear that every fuzzy semi-connected set is fuzzy connected; but that the converse is not necessarily true is seen from the following example.

EXAMPLE 4.3.6. Let $X = [0,1]$ and $T = \{0, 1_X, A\}$, where $A(0) = 0.3$ and $A(x) = 0$, for all $x \in (0,1]$. Then $(X,T)$ is an fts. We consider the fuzzy sets $B$ and $C$ in $X$, given by $B(0) = 0.5$, $C(0) = 0.4$ and $B(x) = C(x) = 0$, for all $x \in (0,1]$. Now $\text{scl}B = B$ and $\text{scl}C = C$. Then $B \nsubseteq \text{scl}C$ and $C \nsubseteq \text{scl}B$. Hence $B$ and $C$ are fuzzy semi-separated. But $\text{cl}B = \text{cl}C = 1-A$, so that $B \nsubseteq \text{cl}C$ and $C \nsubseteq \text{cl}B$. Hence $B$ and $C$ are not fuzzy separated.

Again, $B = B \cup C$ and $B$, $C$ are fuzzy semi-separated implies that $B$ is not fuzzy semi-connected. We show that $B$ is fuzzy connected. In fact, let $B = D \cup E$, where $D$ and $E$ are fuzzy sets in $X$. Then either $D(0) = 0.5$ or $E(0) = 0.5$. Suppose $D(0) = 0.5$, then $\text{cl}D = 1-A$. Since $E(0) \not< 0.5$, we have $\text{cl}E = 1-A$. Then $D \nsubseteq \text{cl}E$. Thus $D$ and $E$ cannot be fuzzy separated. Hence $B$ is fuzzy connected.

THEOREM 4.3.7. Let $A$ be a non-null fuzzy semi-connected set in $X$. Then
whenever $A$ is contained in the join of two fuzzy semi-separated sets $P$ and $Q$, exactly one of the following possibilities (a) and (b) holds:

(a) $A \not\subseteq P$ and $A \cap Q = 0^X$\
(b) $A \not\subseteq 0^X$ and $A \cap P = 0^X$.

**Proof.** We first note that when $A \cap Q = 0^X$, then $A \not\subseteq P$, since $A \not\subseteq P \cup Q$. Similarly, when $A \cap P = 0^X$, we must have $A \not\subseteq Q$.

Now, since $A \not\subseteq P \cup Q$, both of $A \cap Q = 0^X$ and $A \cap P = 0^X$ can not hold simultaneously. Again, if $A \cap Q \neq 0^X$ and $A \cap P \neq 0^X$ then $A \cap P$ and $A \cap Q$ become fuzzy semi-separated sets such that $A = (A \cap P) \cup (A \cap Q)$, contradicting the fuzzy semi-connectedness of $A$. Hence exactly one of the cases (a) and (b) must hold.

**Remark 4.3.8.** Clearly, a much stronger version of the converse of the above theorem is true, viz. a non-null fuzzy set $A$ in $X$ is semi-connected provided whenever $A$ is contained in the join of two fuzzy semi-separated sets $P$ and $Q$ then either $A \cap P = 0^X$ or $A \cap Q = 0^X$ is true. In fact, if $A$ is not fuzzy semi-connected and $A = P \cup Q$, where $P$ and $Q$ are fuzzy semi-separated, then $A \cap P = 0^X$ implies that $P = 0^X$, and $A \cap Q = 0^X$ implies that $Q = 0^X$, either of which leads to a contradiction.

**Theorem 4.3.9.** Let $\{ A_\alpha : \alpha \in \Lambda \}$ be a collection of fuzzy semi-connected sets in an fts $X$. Then $A = \bigcup \{ A_\alpha : \alpha \in \Lambda \}$ is fuzzy semi-connected provided there exists an $\alpha_0 \in \Lambda$ such that either (i) $A_\alpha$ and $A_{\alpha_0}$ are not fuzzy semi-separated, for each $\alpha \in \Lambda$, or (ii) $A_\alpha \cap A_{\alpha_0} \neq 0^X$ for each $\alpha \in \Lambda$.
PROOF. If A is not fuzzy semi-connected, then A = P ∪ Q, where P and Q are fuzzy semi-separated in X. For an arbitrary α ∈ Λ, either (a) Aα ⊈ P with Aα ∩ Q = 0_X or (b) Aα ⊈ Q with Aα ∩ P = 0_X (by Theorem 4.3.7). Similarly, either (c) Aα_o ⊈ P with Aα_o ∩ Q = 0_X, or (d) Aα_o ⊈ Q with Aα_o ∩ P = 0_X. Without loss of generality we can assume each Aα (α ∈ Λ) to be non-null, and hence exactly one of the possibilities (a) and (b), and exactly one of (c) and (d) will hold.

For case (ii), the possibilities (a) and (d) cannot happen, and similarly (b) and (c) cannot hold simultaneously. For case (i), if (a) and (d) hold, then A = Aα ∩ P and Aα_o = Aα_o ∩ Q are fuzzy-semi-separated, P and Q being so. This is a contradiction. Similarly, for case (ii) the possibilities (b) and (c) together are to be ruled out.

Thus in any case, either Aα ⊈ P with Aα ∩ Q = 0_X, or Aα ⊈ Q with Aα ∩ P = 0_X (but not both simultaneously), for each α ∈ Λ. Now, Aα ⊈ P and Aα ∩ Q = 0_X, for all α ∈ Λ implies that A ⊈ P and A ∩ Q = 0_X and thus Q = 0_X, a contradiction. Similarly, Aα ⊈ Q and Aα ∩ P = 0_X for all α ∈ Λ implies P = 0_X, again a contradiction.

THEOREM 4.3.10. Let A be a fuzzy set in an fts X such that there exists at least one point x ∈ X with A(x) > 1/2. Then A is fuzzy semi-connected iff any two fuzzy singletons contained in A are contained in a fuzzy semi-connected set contained in A.

PROOF. If A is fuzzy semi-connected, then the condition is clearly true, irrespective of whether A(x) > 1/2 at some point x ∈ X.

Conversely, let x ∈ X such that A(x) > 1/2. For each y ∈ suppA - {x}, there exists, by hypothesis, a fuzzy semi-connected set B_y ⊈ A such that x_A(x)
Clearly \( \cup \{ B_y : y \in \text{supp}A \setminus \{ x \} \} = A \), and \( \cup \{ B_y : y \in \text{supp}A \setminus \{ x \} \} \) is fuzzy semi-connected by Theorem 4.3.9(i).

**Corollary 4.3.11.** An fts \( X \) is fuzzy semi-connected iff every pair of fuzzy singletons is contained in a fuzzy semi-connected set.

**Theorem 4.3.12.** Let \( f \) be a fuzzy irresolute mapping from an fts \( X \) onto an fts \( Y \). If \( A \) is a fuzzy semi-connected set in \( X \), then so is \( f(A) \) in \( Y \).

**Proof.** If possible, let \( f(A) \) be not fuzzy semi-connected in \( Y \). Then there exist fuzzy semi-separated sets \( B \) and \( C \) in \( Y \) such that \( f(A) = B \cup C \). Since \( B \) and \( C \) are fuzzy semi-separated, there exist two fuzzy semi-open sets \( U \) and \( V \) such that \( B \preceq U \), \( C \preceq V \), \( B \not \preceq V \), \( C \not \preceq U \). Now, \( f \) being fuzzy irresolute, \( f^{-1}(B) \) and \( f^{-1}(C) \) are fuzzy semi-open sets in \( X \), and \( A = f^{-1}(f(A)) = f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C) \). Also it can be easily seen that \( f^{-1}(B) \) and \( f^{-1}(C) \) are fuzzy semi-separated in \( X \). Thus we arrive at a contradiction.

Proceeding in a manner similar to the above proof, we can establish the following theorem.

**Theorem 4.3.13.** Let \( f : X \rightarrow Y \) be onto and semi-continuous. If \( A \) is a fuzzy semi-connected set in \( X \), then \( f(A) \) is fuzzy connected set in \( Y \).